Classical coherent risk measures and their application to the solution of problems of financial mathematics

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Motivation and definition of coherent risk measures

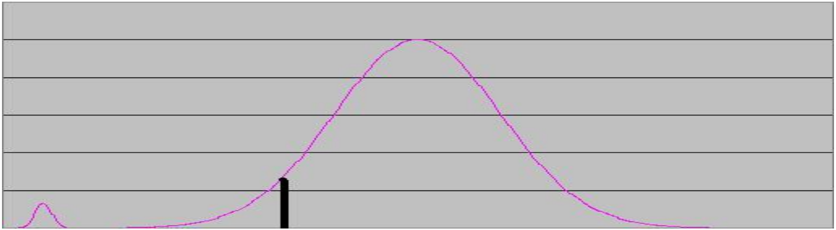
The risk measure which is widely used in practice is V@R.

Definition 1.1. Let $\lambda \in (0,1)$. Then

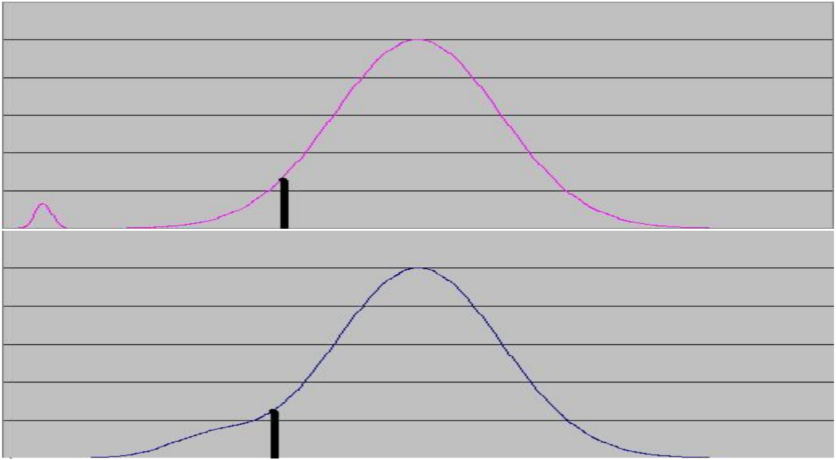
$$V@R_\lambda(X) = -q_\lambda(X),$$

where $q_\lambda(X)$ is quantile of the level λ:

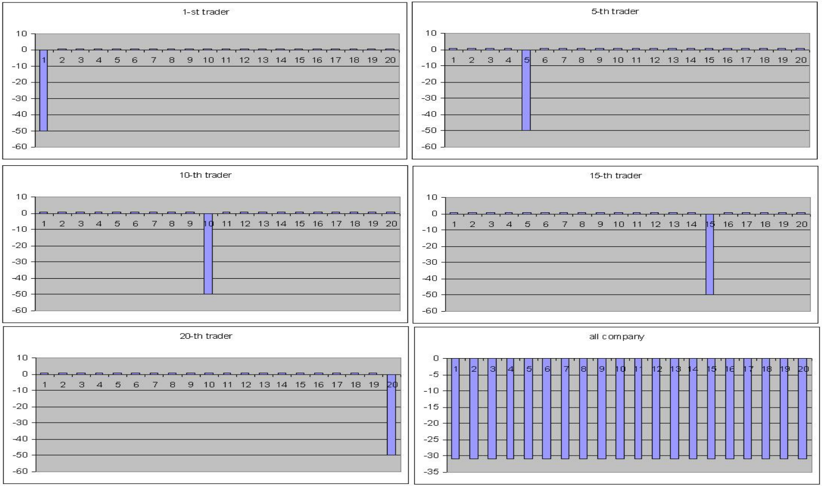
$$q_\lambda(X) = \inf\{x \in \mathbb{R} : \mathsf{P}(X \leq x) \geq \lambda\}$$



Picture 1. Representation of V@R.



Picture 2. Drawbacks of V@R.



Picture 3. Drawbacks of V@R.

Definition 1.2. ([ADEH97]) *Coherent utility function* — mapping $u : L^\infty \to \mathbb{R}$, satisfying the following properties:

(a) (diversification) $u(X+Y) \geq u(X) + u(Y)$;
(b) (partial ordering) if $X \leq Y$ P-a.s., then $u(X) \leq u(Y)$;
(c) (positive homogeneity) $u(\lambda X) = \lambda u(X)$ for all $\lambda \geq 0$;
(d) (translation invariance) $u(X+m) = u(X) + m$ for all $m \in \mathbb{R}$;
(e) (Fatou property) if $|X_n| \leq c$ and $X_n \xrightarrow{P} X$, then $u(X) \geq \overline{\lim}_n u(X_n)$.

The corresponding *coherent risk measure* is defined as $\rho(X) = -u(X)$.

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- (i) V@R does not satisfy diversification property.

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- (ii) Variance (semivariance) does not satisfy monotonicity property.

Representation theorem

Theorem 1.3. ([ADEH99]) A function $u : L^\infty \to \mathbb{R}$ is a coherent utility function if and only if there exists a nonempty set $\mathcal{D} \subseteq \mathcal{P}$ such that

$$u(X) = \inf_{\mathsf{Q} \in \mathcal{D}} \mathsf{E}_\mathsf{Q} X,$$

where $\mathcal{P} = \{\mathsf{Q} : \mathsf{Q} \ll \mathsf{P}\}$.

Definition 1.4. Let us the largest set, for which the representation is true, the *determining set* for coherent utility function u.

Definition 1.5. *Coherent utility function on* L^0 is a mapping $u : L^0 \to \mathbb{R} \cup \{\pm\infty\}$, defined as:

$$u(X) = \inf_{Q \in \mathcal{D}} E_Q X,$$

where $\mathcal{D}$ — set of probability measures Q absolutely continuous under measure P, and $E_Q X = E_Q X^+ - E_Q X^-$ with an agreement: $+\infty - \infty = -\infty$.

Remarks.

- (i) It is obvious, that the determining set is a convex set. If a coherent utility function is defined on L^∞, then its determining set is $\sigma(L^\infty, L^1)$-closed.

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- `(ii)` If $\mathcal{D} - \sigma(L^\infty, L^1)$-closed convex set and a coherent utility function u is defined by the representation, then $\mathcal{D}$ is a determining set for u.

Properties and examples of classical coherent risk measures

Definition 2.1. ([K01]) The utility function u is *law invariant* if for all X, Y such that $X \stackrel{Law}{=} Y$ it is true that

$$u(X) = u(Y).$$

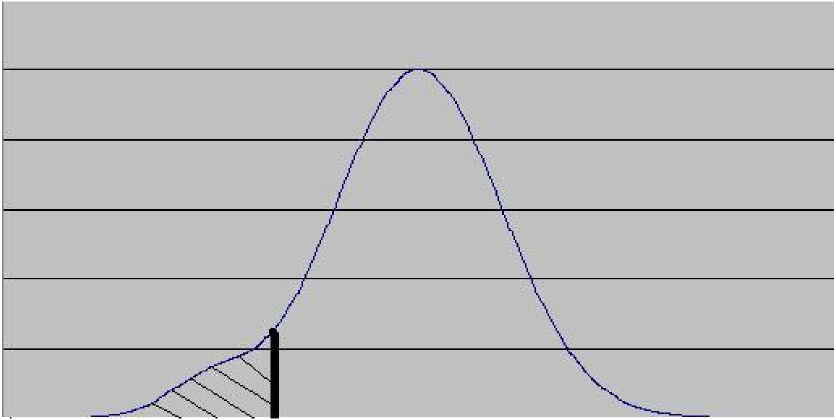
Definition 2.2. ([ADEH99]) Suppose $\lambda \in (0, 1]$. Consider the set

$$\mathcal{D}_\lambda = \left\{ \mathsf{Q} : \frac{d\mathsf{Q}}{d\mathsf{P}} \le 1/\lambda \right\}.$$

Let us construct the function

$$u_\lambda(X) = \inf_{\mathsf{Q} \in \mathcal{D}_\lambda} \mathsf{E}_\mathsf{Q} X, \quad X \in L^0.$$

This is a coherent utility function. The corresponding coherent risk measure is called *Tail V@R of level* λ.
Consider an atomless probability space.
Proposition 2.3. ([K01]) Tail V@R is the minimal law invariant coherent risk measure that dominates V@R.



Picture 4. Tail V@R.

Definition 2.4. ([K01]) Suppose μ is a probability measure on $(0,1]$. *Weighted V@R* on L^∞ is a coherent risk measure, corresponding to the coherent utility function

$$u_\mu(X) = \int_{(0,1]} u_\lambda(X)\mu(d\lambda), \quad X \in L^\infty.$$

The coherent utility function can be rewritten in the following way:

$$u_\mu(X) = \inf_{Q \in \mathcal{D}_\mu} E_Q X,$$

where $\mathcal{D}_\mu \subseteq L^1$. Using this formula we can extend a function on L^0. The corresponding coherent risk measure is called *Weighted V@R* on L^0.

Definition 2.5. **([CM05])** Take $\alpha \in \mathbb{N}$. Let us consider the following function

$$u_\alpha(X) = \mathsf{E} \min_{i=1,\dots,\alpha} X_i,$$

where $X_1, \dots, X_\alpha$ are independent copies of random variable X. Due to [CM05] this is a classical coherent utility function. It belongs to the class of Weighted V@R and has the probability measure μ of the following form:

$$\mu_\alpha(dx) = B(2, \alpha - 1)^{-1} x (1 - x)^{\alpha - 2} dx, x \in (0, 1], \qquad (1)$$

where B is Beta-function. The corresponding classical coherent risk measure ρ_α is called *Alpha V@R*.

Consider an atomless probability space. Then

Proposition 2.6. ([K01]) Coherent utility function u is law invariant if and only if it has the following form:

$$u(X) = \inf_{\mu \in \mathfrak{M}} u_\mu(X), \text{ where} \tag{2}$$

$$\mathfrak{M} \text{ is a set of probabilities measures on } (0, 1]. \tag{3}$$

Definition 2.7. Coherent utility function has *strictly diversification property* if for all $X, Y \in L^\infty : corr(X, Y) \neq 1$ it is valid that

$$u(X + Y) > u(X) + u(Y).$$

Proposition 2.8. ([K01]) Weighted V@R has strictly diversification property if and only if

$$supp(\mu) = [0, 1]. \tag{4}$$

Extreme measures, generators and their application for solution of some financial mathematical problems

Let us introduce the following spaces:

$$L^1_w(\mathcal{D}) = \left\{ X \in L^0 : \sup_{\mathsf{Q}\in\mathcal{D}} |\mathsf{E}_\mathsf{Q} X| < \infty \right\};$$

$$L^1_s(\mathcal{D}) = \left\{ X \in L^0 : \lim_{n\to\infty} \sup_{\mathsf{Q}\in\mathcal{D}} \mathsf{E}_\mathsf{Q} |X| I\{|X| > n\} = 0 \right\}.$$

Definition 3.1. Let u be a utility function with the determining set $\mathcal{D}$. Suppose $X \in L^0$. We will call a measure $\mathsf{Q} \in \mathcal{D}$ an *extreme measure* for X for coherent utility function u if

$$u(X) = \mathsf{E}_{\mathsf{Q}} X.$$

The set of extreme measures for X is denoted by $\mathcal{X}_{\mathcal{D}}(X)$.

Theorem 3.2. If $\mathcal{D}$ is weakly compact, $X \in L^1_s(\mathcal{D})$, then $\mathcal{X}_{\mathcal{D}}(X) \neq \varnothing$.

Definition 3.3. Let u be a utility function with the determining set $\mathcal{D}$. Suppose $X = (X_1, \ldots, X_d)$. We will call the set

$$G = cl\{\mathsf{E}_{\mathsf{Q}}X : \mathsf{Q} \in \mathcal{D}\}$$

a *generator* for X and u.

`Remark`. If every $X_i \in L^1_w(\mathcal{D})$ and $\mathcal{D}$ is weakly compact then G is convex compact.

Example 3.4. Let u be law invariant utility function which is finite on Gaussian random variables.
Then there exists $\gamma \geq 0$ such that for Gaussian random variable ξ with mean a and variance σ^2 it is valid that

$$u(X) = a - \gamma\sigma.$$

Let X have Gaussian distribution with mean a and covariance matrix C. Let L denote the image of $\mathbb{R}^d$ under the map $x \to Cx$. Then inverse image is correctly defined. So it is easy to see that the generator for $X = (X_1, \dots, X_d)$ has the following form:

$$G = a + \{C^{1/2}x : \|x| \leq \gamma\} = a + \{y \in L : \langle y, C^{-1}y\rangle \leq \gamma^2\}.$$

Problems of financial mathematics, for solution of which we use extreme measures and generators:

- (i) Capital allocation;

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- (i) Capital allocation;
- (ii) Risk contribution.

Definition 3.5. Let us call $x_1, \dots, x_d \in \mathbb{R}$ *a capital allocation between* $X_1, \dots, X_d$, *if*
(i) $\sum_{i=1}^d x_i = \rho(\sum_{i=1}^d X_i)$;
(ii) for all $h_1, \dots, h_d \geq 0$ it is true that
$\sum_{i=1}^d h_i x_i \leq \rho(\sum_{i=1}^d h_i X_i)$.

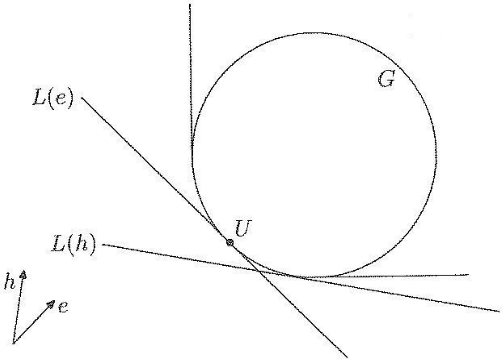
Definition 3.6. Let us call $x_1, \dots, x_d \in \mathbb{R}$ *a utility allocation between* $X_1, \dots, X_d$, *if*
(i) $\sum_{i=1}^d x_i = u(\sum_{i=1}^d X_i)$;
(ii) for all $h_1, \dots, h_d \geq 0$ it is true that
$\sum_{i=1}^d h_i x_i \geq u(\sum_{i=1}^d h_i X_i)$.

Theorem 3.7. The set U of utility allocation problem solutions between $X_1, \dots, X_d$ has the form

$$U = \operatorname{argmin}_{x \in G} \langle e, x \rangle,$$

where $e = (1, \dots, 1)$. Furthermore, for any utility allocation it is valid that

$$\sum_i h_i x_i \geq u\Big(\sum_i h_i x_i\Big) \; \forall h_1, \dots, h_d \in \mathbb{R}.$$



Picture 5. Geometric solution of utility allocation problem.

Theorem 3.8. **(i)** Suppose $X_1, \dots, X_d \in L^1_s(\mathcal{D})$ and $\mathcal{D}$ is weakly compact. Then there exists a collection $(x_1, \dots, x_d)$ such that the following conditions are satisfied:
(a) $\sum_{i=1}^d x_i = u(\sum_i X_i)$;
(b) there exists $\mathsf{Q} \in \mathcal{X}_{\mathcal{D}}(\sum_{i=1}^d X_i)$ such that

$$x_i = \mathsf{E}_\mathsf{Q} X_i. \tag{5}$$

Every such collection is a utility allocation between $X_1, \dots, X_d$.
(ii) All the solutions of utility allocation problem between $X_1, \dots, X_d$ are represented in the form given.

Example 3.9. Let us consider Example 3.4 for Gaussian vector X, i.e. for Gaussian random variable ξ with mean a and variance σ^2 it is valid that

$$u(X) = a - \gamma\sigma.$$

Let us assume that $\langle C, e\rangle \neq 0$. Then the utility allocation between $X_1, \dots, X_d$ is unique and has the following form:

$$x = a - \gamma\langle e, Ce\rangle^{-1/2} Ce.$$

Definition 3.10. The *risk contribution of* X *to* Y is

$$\rho^c(X;Y) = -\inf_{\mathsf{Q}\in\mathcal{X}_{\mathcal{D}(Y)}} \mathsf{E}_\mathsf{Q} X.$$

Theorem 3.11. If $\mathcal{D}$ is weakly compact and $X, Y \in L^1_s(\mathcal{D})$ then

$$\rho^c(X;Y) = \lim_{\varepsilon\downarrow 0} \frac{\rho(Y+\varepsilon X) - \rho(Y)}{\varepsilon}.$$

Corollary 3.12. In Gaussian case we have

$$\rho^c(X;Y) = -\mathsf{E}X + (\mathsf{E}X - u(X))corr(X;Y).$$

NGD pricing

Example 4.1. Let $S_1 \sim U[0, 100]$ is the price at moment 1. Then NA-condition in this model is equivalent to $S_0 \in (0, 100)$, which is not natural from financial point of view, because if S_0 is very small then all the participants will try to buy it and if it is closed to 100 then all the participants will try to sell it.

Let A be a convex closed subset in L^0.

Definition 4.2. A *risk-neutral measure* is a measure $\mathsf{Q} \ll \mathsf{P}$ such that $\mathsf{E}_\mathsf{Q} X \leqslant 0$ for all $X \in A$.
The set of risk-neutral vectors is denoted by $\mathcal{R}$ or $\mathcal{R}(A)$, if there is a risk of ambiguity.
Definition 4.3. ([C07]) We will call that the set A *is $\mathcal{D}$-consistent*, if there exists a subset $A' \subseteq A \cap L^1_s(\mathcal{D})$ such that $\mathcal{D} \cap \mathcal{R} = \mathcal{D} \cap \mathcal{R}(A')$.

Definition 4.4. ([C07], [D05]) The model satisfies *NGD condition* if there exist no $X \in A$ such that $u(X) > 0$.

Theorems of asset pricing

Theorem 4.5. ([C07], [D05]) The model satisfies NGD-condition iff $\mathcal{D} \cap \mathcal{R} \neq \varnothing$.

Definition 4.6. A *utility based NGD-price* of contingent claim F is a number $x \in \mathbb{R}$ such that the extended model $(\Omega, \mathcal{F}, \mathsf{P}, \mathcal{D}, A + \{h(F - x) : h \in \mathbb{R}\})$ satisfies NGD-condition. The interval of NGD-prices of contingent claim F will be denoted by $I_{NGD}(F)$.

Corollary 4.7. For $F \in L^1_s(\mathcal{D})$

$$I_{NGD}(F) = \{\mathsf{E}_\mathsf{Q} F : \mathsf{Q} \in \mathcal{D} \cap \mathcal{R}\}.$$

Proposition 4.8. If $A = \left\{\langle h, X\rangle : h \in \mathbb{R}^d\right\}$ then

$$\text{NGD} \Leftrightarrow \{0\} \in G^\circ$$

Example 4.9. Consider Example 4.1. If we use Tail V@R with level λ then NGD-condition in this model is equivalent to $S_0 \in (100\lambda/2, 100(1-\lambda/2))$.

Example 4.10. Consider Gaussian case, i.e. $(S_1^1, \dots, S_1^d, F)$ is Gaussian vector, where $a = \mathsf{E}S_1, c = cov(S_1, F)$. Let

$$\begin{aligned} F &= \langle b, S_1 - a\rangle + \mathsf{E}F + \tilde{F}, \mathsf{E}\tilde{F} = 0, \\ b \in \mathbb{R}^d &: Cb = c, \\ \sigma^2 &= \mathrm{var}\tilde{F} = \mathrm{var}F - \langle b, c\rangle, \\ \alpha &= \left(\sigma^2\gamma^2 - \sigma^2\langle S_0 - a, C^{-1}(S_0 - a)\rangle\right). \end{aligned}$$

Then

$$NGD \Leftrightarrow \langle S_0 - a, C^{-1}(S_0 - a)\rangle \le \gamma^2,$$

$$I_{NGD}(F) = \Big[\langle b, S_0 - a\rangle + \mathsf{E}F - \alpha, \langle b, S_0 - a\rangle + \mathsf{E}F + \alpha\Big].$$

NGD hedging

Definition 4.11. *The upper and lower NGD price* of a contingent claim F can be defined in the following form:

$$\overline{V}(F) = \inf\{x : \exists X \in A : u(X - F + x) \geq 0\},$$
$$\underline{V}(F) = \sup\{x : \exists X \in A : u(X + F - x) \geq 0\}.$$

Theorem 4.12. If A is a cone and $F \in L^1_s(\mathcal{D})$, then

$$\overline{V}(F) = \sup_{Q \in \mathcal{D} \cap \mathcal{R}} E_Q X,$$
$$\underline{V}(F) = \inf_{Q \in \mathcal{D} \cap \mathcal{R}} E_Q X, \ \overline{V}(F) = -\underline{V}(-F).$$

Let us now consider a sub- and superhedging problems for a static model with finite number of assets. Thus we are given a coherent utility function u with determining set $\mathcal{D}$ and $S_1, \dots, S_d \in L^1_s(\mathcal{D})$. Then consider the following definition:

Definition 4.13. *The sub- and superhedging strategies* of a contingent claim F can be defined in the following form:

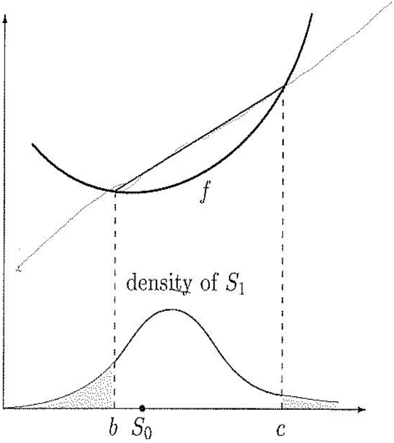
$$\overline{H}(F) = \{h \in \mathbb{R}^d : u(\langle h, S_1 - S_0 \rangle - F + \overline{V}(F) \geq 0\},$$
$$\underline{H}(F) = \{h \in \mathbb{R}^d : u(\langle h, S_1 - S_0 \rangle + F - \underline{V}(F) \geq 0\}.$$

Example 4.14. **([C07])** Let $S_0 \in (0, \infty)$ and $S_1 \in L^1$ such that supp$Law(S_1) = (0, \infty)$ and $Law(S_1)$ has no atoms. Let NGD condition be satisfied for Tail V@R u_λ $(u_\lambda(S_1) < S_0 < -u_\lambda(-S_1))$. Then there exists a unique pair of numbers $0 < b < c$ such that

$$\mathsf{P}(S_1 \notin (b, c)) = \lambda,$$
$$\mathsf{E}[S_1 | S_1 \notin (b, c)] = S_0.$$

Then if $F = f(S_1)$, where f is convex, we have that

$$\overline{V}(F) = \mathsf{E}[f(S_1) | S_1 \notin (b, c)],$$
$$\overline{H}(F) = \frac{f(b) - f(c)}{b - c}.$$



Picture 6. Geometric representation of Example 4.14.

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Thank you for your attention

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