Multidimensional coherent risk measures and their application to the solution of problems of financial mathematics

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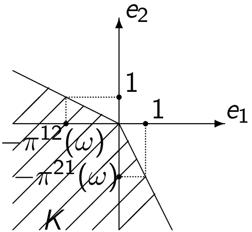
Definition and motivation of multidimensional coherent risk measures

Definition 1.1. ([ADEH97]) *Coherent utility function* — mapping $u : L^\infty \to \mathbb{R}$, satisfying the following properties:

(a) (diversification) $u(X+Y) \geq u(X) + u(Y)$;
(b) (partial ordering) if $X \leq Y$ P-a.s., then $u(X) \leq u(Y)$;
(c) (positive homogeneity) $u(\lambda X) = \lambda u(X)$ for all $\lambda \geq 0$;
(d) (translation invariance) $u(X+m) = u(X) + m$ for all $m \in \mathbb{R}$;
(e) (Fatou property) if $|X_n| \leq c$ and $X_n \xrightarrow{\mathsf{P}} X$, then $u(X) \geq \overline{\lim}_n u(X_n)$.

The corresponding *coherent risk measure* is defined as $\rho(X) = -u(X)$.



Picture 1. Example of cone K

$X \preceq Y$, if $X(\omega) - Y(\omega) \in K(\omega)$ for a.a. ω, where $K(\omega)$ is a cone of currency exchange rates.

Definition 1.2. ([K07]) *Multidimensional coherent utility function* — mapping $u : (L^\infty)^d \to \mathcal{C} \setminus \{\mathbb{R}^d\}$, satisfying the following properties:

(a) (diversification) $u(X+Y) \supseteq u(X) + u(Y)$;
(b) (partial ordering) if $X \preceq Y$ then $u(X) \subseteq u(Y)$;
(c) (positive homogeneity) $u(\lambda X) = \lambda u(X)$ for all $\lambda > 0$;
(d) (translation invariance) $u(X+m) = u(X) + m$ for all $m \in \mathbb{R}^d$;
(e) (Fatou property) if $|X_n| \le 1; X_n \xrightarrow{\mathsf{P}} X$, then $u(X) \supseteq \overline{\lim}_n u(X_n)$, i.e. if x belongs to infinitely many $u(X_n)$, then x belongs to $u(X)$.

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Remarks.

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- (i) If u is a coherent utility function, then $v(X) = (-\infty, u(X)]$ is a multidimensional coherent utility function with $d = 1$ and $K(\omega) = \mathbb{R}_-$.

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- (ii) If $d = 1$, $K(\omega) = \mathbb{R}_-$ and u is a multidimensional coherent utility function, then $v(X) = \sup\{x \in \mathbb{R} : x \in u(X)\}$ is a coherent utility function.

Representation theorem

Theorem 1.3. ([K07]) A function $u : (L^\infty)^d \to \mathcal{C} \setminus \{\mathbb{R}^d\}$ is a multidimensional coherent utility function if and only if there exists a nonempty set $\mathcal{D} \subseteq (L^1)^d$ such that $Z(\omega) \in K^*(\omega)$ P-a.s. and

$$u(X) = \left\{ x \in \mathbb{R}^d : \forall Z \in \mathcal{D} \sum_{i=1}^{d} \mathsf{E} x^i Z^i \le \sum_{i=1}^{d} \mathsf{E} X^i Z^i \right\},$$

where $K^*(\omega)$ — negative polar to $K(\omega)$, i. e. $K^*(\omega) = \{x \in \mathbb{R}^d : \forall z \in K(\omega)\ \langle x, z \rangle \le 0\}$.

Definition 1.4. We will call the largest set, for which the representation is true, the *determining set* for multidimensional coherent utility function u.

Definition 1.5. *Multidimensional coherent utility function on* $(L^0)^d$ is a mapping $u : (L^0)^d \to \mathcal{C} \cup \{\varnothing\}$, defined as:

$$u(X) = \left\{ x \in \mathbb{R}^d : \forall Z \in \mathcal{D} \sum_{i=1}^{d} \mathsf{E} x^i Z^i \leq \sum_{i=1}^{d} \mathsf{E} X^i Z^i \right\}, \qquad (1)$$

where $\mathcal{D}$ — set of d-dimensional random vectors $Z \in (L^1)^d$ such that $Z(\omega) \in K^*(\omega)$ P-a.s. and $\mathsf{E} X^i Z^i = \mathsf{E}(X^i Z^i)^+ - \mathsf{E}(X^i Z^i)^-$ with an agreement: $+\infty - \infty = -\infty$.

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- `(ii)` It is obvious, that the determining set is a convex cone. If a multidimensional coherent utility function is defined on $(L^\infty)^d$, then its determining set is $(L^1)^d$-closed.
- `(iii)` If $\mathcal{D} - (L^1)^d$-closed convex cone and a multidimensional coherent utility function u is defined by the representation, then $\mathcal{D}$ is a determining set for u.

Properties and examples of multidimensional coherent risk measures

Law invariance

Definition 2.1. ([K01]) The one-dimensional utility function u is *law invariant* if for all X, Y such that $X \overset{Law}{=} Y$ it is true that

$$u(X) = u(Y).$$

Definition 2.2. The multidimensional utility function u is *law invariant* if for all X, Y such that $(X, K) \overset{Law}{=} (Y, K)$ it is true that

$$u(X) = u(Y).$$

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- (i) It is easily seen that this definition coincides with the definition of law invariance given in [K01] in case $d = 1$.

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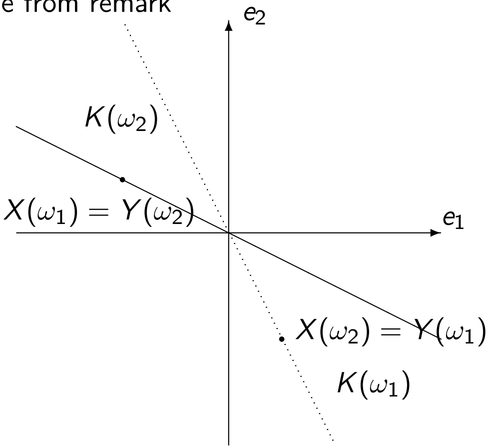
- (ii) The fact that not only $X \overset{Law}{=} Y$ but $(X, K) \overset{Law}{=} (Y, K)$ is very important. Let us introduce an example.
  Let $\Omega = (\omega_1, \omega_2), \mathcal{F} = 2^\Omega, \mathsf{P}(\omega_1) = \mathsf{P}(\omega_2) = \frac{1}{2}$, 2 is the dimension of the space,

$$X(\omega_1) = Y(\omega_2) = (-2, 1), X(\omega_2) = Y(\omega_1) = (1, -2).$$

  Let $K(\omega_1)$ be the lower half-plane, orthogonal to vector $(1, 2)$,
  $K(\omega_2)$ be the lower half-plane, orthogonal to vector $(2, 1)$.
  Consequently, $X \overset{Law}{=} Y$ but $(X, K) \overset{Law}{\neq} (Y, K)$. It is easily seen that the portfolio $X \approx (0, 0)$ but the portfolio Y is unacceptable from financial point of view.

Consider example from remark



Picture 2. Example.

Space consistency

Let change the basic units along each datum line by multiplying them on vector $\gamma = (\gamma^1, \dots, \gamma^d) \in \mathbb{R}^d_{++}$. Using the obtained determining set let us construct the multidimensional utility function u^γ in new datum lines.

Definition 2.3. The multidimensional utility function u on $(L^0)^d$ is *space consistent* if for all $X \in (L^0)^d, \gamma \in R^d_{++}$

$$u^\gamma(X^\gamma) \cdot \gamma = u(X),$$

which means that for all $X \in (L^0)^d, \gamma \in \mathbb{R}^d_{++}$

$$x \in u(X) \text{ iff } x^\gamma \in u^\gamma(X^\gamma).$$

Lemma 2.4. Multidimensional coherent utility function u on $(L^0)^d$ is space consistent if and only if one of the following properties takes place:
(i) A random vector $X \in A_u$ ($X \in (L^0)^d$) iff $X^\gamma \in A_{u^\gamma}$ for all $\gamma \in \mathbb{R}^d_{++}$, where A_u and A_{u^γ} are the acceptance sets of the multidimensional coherent utility function u (resp., u^γ).
(ii) A random vector $Z \in \mathcal{D}$ ($Z \in (L^1)^d$) iff $Z^{1/\gamma} \in \mathcal{D}^\gamma$ for all $\gamma \in \mathbb{R}^d_{++}$, where $\mathcal{D}$ and $\mathcal{D}^\gamma$ are the determining sets of the multidimensional coherent utility function u (resp., u^γ).

`Remark`. One-dimensional coherent risk measure is always space consistent due to positive homogeneity property.

Multidimensional analogues of Tail V@R

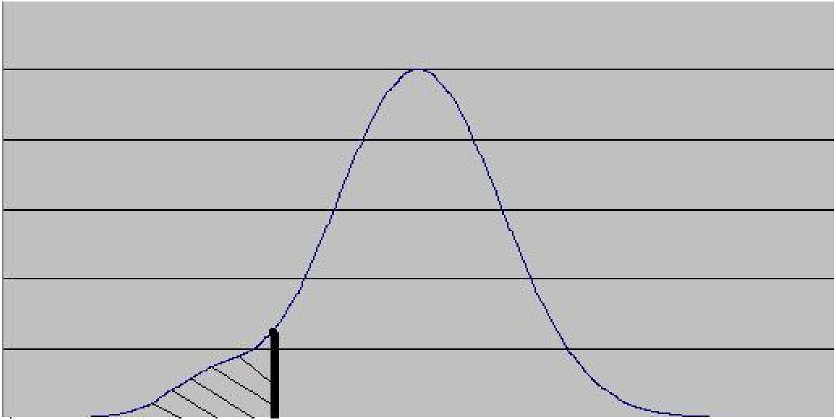
Definition 2.5. ([ADEH99]) Suppose $\lambda \in (0, 1]$. Consider the set

$$\mathcal{D}_\lambda = \{Z \in L^1_+ : Z \le 1/\lambda, \mathsf{E}Z = 1\}.$$

Let us construct the function

$$u_\lambda(X) = \inf_{Z \in \mathcal{D}_\lambda} \mathsf{E}ZX, \quad X \in L^0.$$

This is a coherent utility function. The corresponding coherent risk measure is called *Tail V@R of level λ*.



Picture 3. Tail V@R.

Definition 2.6. ([JMT04]) Suppose $\lambda \in (0,1]$. Consider the set

$$A_{WCE_\lambda} = \{X \in (L^\infty)^d : \mathsf{E}(X|B) \in -K \ \forall B \in \mathcal{F}^\lambda\},$$

where

$$\mathcal{F}^\lambda = \{B \in \mathcal{F} : \mathsf{P}(B) > \lambda\}.$$

Then *Worst Conditional Expectation of level* λ *(*WCE_λ*)* in multidimensional case is defined as follows:

$$WCE_\lambda(X) = \{x \in \mathbb{R}^d : X + x \in A_{WCE_\lambda}\}.$$

`Remark`. WCE_λ is not law invariant risk measure but it is not space consistent. It is also can not be correctly defined in the case of random cone of currency exchange rates.

Definition 2.7. ([Ham07]) Consider

$$\mathcal{Z}_\lambda = \mathrm{cone}\Big\{ Z \in (L^1)^d : \mathsf{E}\sum_{i=1}^d Z^i = 1, Z(\omega) \in K^* \text{ P-a.s.},$$

$$\exists v \in \mathbb{R}^d_+ : \sum_{i=1}^d v^i = \frac{1}{\lambda}, Z^i \le v^i \, \forall i = 1, \dots, d \Big\}.$$

Consider the corresponding multidimensional coherent risk measure:

$$AV@R_\lambda(X) = -u_\lambda(X) = -\Big\{ x \in \mathbb{R}^d : \forall Z \in \mathcal{Z}_\lambda \sum_{i=1}^d \mathsf{E}x^i Z^i \le \sum_{i=1}^d \mathsf{E}X^i Z^i \Big\}.$$

It is called *Average V@R.*

`Remark`. It is easily seen that this risk measure is law invariant but it is not space consistent. It can be defined in the case of random cone of currency exchange rates if $\lambda \le 1/d$.

Definition 2.8. ([K07]) Suppose $\lambda \in (0, 1]$ and determining set

$$\widetilde{\mathcal{D}}_\lambda = \text{cone}\{\eta\xi \in (L^1)^d : \eta \in \mathcal{D}_\lambda,\ \xi \in (L^0)^d,\ \xi(\omega) \in K^*(\omega) \text{ P-a.s.},\ \xi^1 \cdot \ldots \cdot \xi^d = 1 \text{ P-a.s.}\},$$

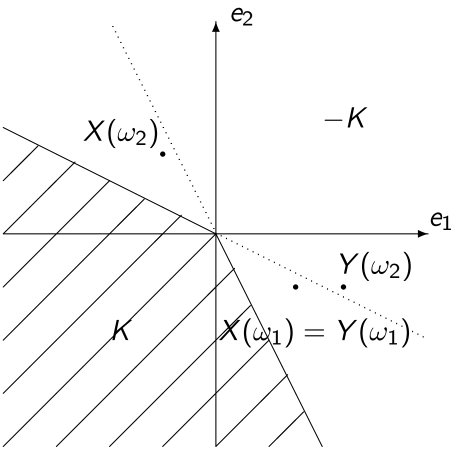
We can consider the corresponding multidimensional coherent utility function

$$\widetilde{u}_\lambda(X) = \left\{x \in \mathbb{R}^d : \forall Z \in \widetilde{\mathcal{D}}_\lambda\ \sum_{i=1}^d \mathsf{E} x^i Z^i \le \sum_{i=1}^d \mathsf{E} X^i Z^i\right\}.$$

Let us call the corresponding multidimensional coherent risk measure *Tail V@R of level* λ

`Remarks`. `(i)` It is well defined if the cone K of currency exchange rates is constructed by using a matrix $(\pi^{ij})_{1\le i,j\le d}$ of currency exchange rates such that $\pi^{ij} \in L^1$ for all $1 \le i, j \le d$.
`(ii)` It is easily seen that this risk measure is law invariant and space consistent.

Consider example



Picture 4. Example.

Measures/ Properties	Tail V@R	Average V@R	WCE
Law invariance	+	+	−
Space consistency	+	−	+
Random cone	+	±	−
Economical sense	+	+	−

Table 1. Multidimensional analogues of Tail V@R and their properties.

NGD pricing

Let $X \stackrel{Law}{=} 1/X$ is a currency exchange rate.
It means that these 2 currencies have the similar behavior.
But for Tail V@R if X has a continuous distribution then for $0 < \lambda < 1$

$$u_\lambda(X) > \frac{1}{-u_\lambda(-X)},$$

which means that the interval of fair currency exchange rates is not symmetric.

Let $\mathcal{L} = \left\{ Z \in (L^1)^d : \mathsf{E} \sum_{i=1}^d Z^i = 1 \right\}$. Then let us introduce the following space:

$$L_s^1(\mathcal{D}) = \left\{ X \in (L^0)^d : \lim_{n \to \infty} \sup_{Z \in \mathcal{D} \cap \mathcal{L}} \sum_{i=1}^d \mathsf{E} |Z^i X^i| I \left\{ \sum_{j=1}^d |X^j| > n \right\} = 0 \right\}.$$

Let A be a convex closed subset in $(L^0)^d$.

Definition 4.1. A *risk-neutral vector* is a nonzero vector $Z \in (L^1_+)^d$ such that $\mathsf{E}\sum_{i=1}^d Z^i X^i \leqslant 0$ for all $X \in A$.
The set of risk-neutral vectors is denoted by $\mathcal{R}$ or $\mathcal{R}(A)$, if there is a risk of ambiguity.
Definition 4.2. ([C07]) We will call that the set A *is $\mathcal{D}$-consistent*, if there exists a subset $A' \subseteq A \cap L^1_s(\mathcal{D})$ such that $\mathcal{D} \cap \mathcal{R} = \mathcal{D} \cap \mathcal{R}(A')$.

Definition 4.3. ([K08]) The model satisfies *NGD condition* if there exist no $X \in A$ such that $u(X) \cap (\mathbb{R}^d_+ \setminus \{0\}) \neq \varnothing$.

Theorems of asset pricing

Theorem 4.4. ([K08]) The model satisfies NGD-condition iff $\mathcal{D} \cap \mathcal{R} \neq \varnothing$.

Definition 4.5. A *utility based NGD-price* of contingent claim F is a vector $x \in \mathbb{R}^d$ such that the extended model $(\Omega, \mathcal{F}, \mathsf{P}, \mathcal{D}, A + \{h(F - x) : h \in \mathbb{R}\})$ satisfies NGD-condition. The set of NGD-prices of contingent claim F will be denoted by $I_{NGD}(F)$.

Corollary 4.6. For $F \in L^1_s(\mathcal{D})$

$$I_{NGD}(F) = \{x : \mathsf{E}\langle Z, x\rangle = \mathsf{E}\langle Z, F\rangle \text{ for some } Z \in \mathcal{D} \cap \mathcal{R}\}.$$

Consider the following model. Let

$$A = \left\{ \sum_{n=0}^{N} \xi_n : \xi_n(\omega) \in K_{u_n}(\omega)(\mathsf{P}\text{-a.s.}), N \in \mathbb{N}, \right. \\ \left. u_0 \leqslant \ldots \leqslant u_N \text{ — } (\mathcal{F}_t)\text{-stopping times} \right\}. \tag{2}$$

Definition 4.7. **(K99)** An adapted $\mathbb{R}^d_+$-valued process $Z = (Z_t)_{0 \le t \le T}$ is called *price-process* adapted with the set of cones of currency exchange rates $(K_t(\omega))_{0 \le t \le T}$, if Z is a martingale under measure P and for each t it is valid that $Z_t(\omega) \in K^*_t(\omega)(\mathsf{P}\text{-a.s.})$.

Denote by

$$\mathcal{R}' = \{Z_T : \mathsf{P}(Z_T \neq 0) > 0, Z - \text{price process}\}.$$
$$\mathcal{R}'' = \{Z_T : \mathsf{P}(Z_T \neq 0) = 1, Z - \text{price process}\}.$$

Proposition 4.8. (K99) Model satisfies NA-condition iff $\mathcal{R}'' \neq \varnothing$.

Theorem 4.9. We have $\mathcal{D} \cap \mathcal{R} = \mathcal{D} \cap \mathcal{R}(A') = \mathcal{D} \cap \mathcal{R}'$, where

$$\begin{aligned} A' =& \{\xi_u(\omega) : \xi_u(\omega) \in K_u(\omega)(\mathsf{P}\text{-a.s.}), \xi_u(\omega) \in L^1_s(\mathcal{D}), \\ & u \in [0, T] - \text{ simple } \mathcal{F}_t\text{-stopping time}\}, \end{aligned}$$

Let $\pi^{ij}_t \in L^1_s(\mathcal{D})$ for each $1 \le i, j \le d$ and $t \in [0, T]$. Then model satisfies NGD condition iff $\mathcal{D} \cap \mathcal{R}' \neq \varnothing$.

Corollary 4.10. For $F \in L^1_s(\mathcal{D})$ it is valid that

$$I_{NGD}(F) = \{x : \mathsf{E}\langle Z, x\rangle = \mathsf{E}\langle Z, F\rangle \text{ for some } Z \in \mathcal{D} \cap \mathcal{R}'\}.$$

NGD hedging

Definition 4.11. *The upper and lower NGD price in the direction* $z \in \mathbb{R}_+^d \setminus \{0\}$ of a contingent claim F can be defined in the following form:

$$\overline{V}_z(F) = \inf\{x : \exists X \in A : u(X - F + xz) \cap \mathbb{R}_+^d \neq \varnothing\},$$
$$\underline{V}_z(F) = \sup\{x : \exists X \in A : u(X + F - xz) \cap \mathbb{R}_+^d \neq \varnothing\}.$$

Theorem 4.12. If A is a cone and $F \in L^1_s(\mathcal{D})$, then

$$\overline{V}_z(F) = \sup\left\{\frac{\sum_{i=1}^d \mathsf{E} Z^i F^i}{\sum_{i=1}^d \mathsf{E} Z^i z^i} : Z \in \mathcal{D} \cap \mathcal{R}\right\},$$

$$\underline{V}_z(F) = \inf\left\{\frac{\sum_{i=1}^d \mathsf{E} Z^i F^i}{\sum_{i=1}^d \mathsf{E} Z^i z^i} : Z \in \mathcal{D} \cap \mathcal{R}\right\}.$$

Example 4.13. Let $d = 2, \sqrt{\pi_1^{12}}, \sqrt{\pi_1^{21}} \in L^1$. Let $z = (1, 0), F = (0, 1)$. If we use Tail V@R and $\widetilde{\mathcal{D}}_\lambda \cap \mathcal{R} \neq \varnothing$, then

$$\overline{V}_z(F) = \frac{\rho_\lambda(-(\pi_1^{12})^{1/2})}{-\rho_\lambda((\pi_1^{12})^{-1/2})} \wedge \pi_0^{12},$$

$$\underline{V}_z(F) = \frac{-\rho_\lambda((\pi_1^{21})^{-1/2})}{\rho_\lambda(-(\pi_1^{21})^{1/2})} \vee \frac{1}{\pi_0^{21}}.$$

Example 4.14. Let $\mathsf{Law}(\pi_1^{12})$ has the following density

$$\rho^{12}(x) = \begin{cases} \frac{(1/2+\alpha)x^{-1/2+\alpha}}{2c_1^{1/2+\alpha}}, & x \in (0, c_1]; \\ \frac{(1/2+\alpha)c_1^{1/2+\alpha}}{2x^{3/2+\alpha}}, & x \in [c_1, \infty), \end{cases}$$

and $\pi_1^{21} = \frac{c_1 c_2}{\pi_1^{12}}$. Then $\mathsf{Law}(\pi_1^{21})$ has the following density

$$\rho^{21}(x) = \begin{cases} \frac{(1/2+\alpha)x^{-1/2+\alpha}}{2c_2^{1/2+\alpha}}, & x \in (0, c_2]; \\ \frac{(1/2+\alpha)c_2^{1/2+\alpha}}{2x^{3/2+\alpha}}, & x \in [c_2, \infty), \end{cases}$$

where $c_1 c_2 \geq 1$ and $\alpha > 0$.

Let $\lambda \le 1/2$. Then

$$\overline{V}_z(F) = \frac{b_1(\alpha+1)}{\alpha} \wedge \pi_0^{12}, \underline{V}_z(F) = \frac{\alpha}{b_2(\alpha+1)} \vee \frac{1}{\pi_0^{21}},$$

where $b_i^{1/2+\alpha} = \frac{c_i^{1/2+\alpha}}{2\lambda}, i = 1, 2.$
Let $\lambda \ge 1/2$. Then

$$\overline{V}_z(F) = \frac{\frac{c_1^{1/2}}{\alpha} + \frac{c_1^{1/2}}{\alpha+1} - \frac{c_1^{-1/2-\alpha} b_1^{\alpha+1}}{\alpha+1}}{\frac{c_1^{-1/2}}{\alpha} + \frac{c_1^{-1/2}}{\alpha+1} - \frac{c_1^{-1/2-\alpha} b_1^{\alpha}}{\alpha}} \wedge \pi_0^{12},$$

$$\underline{V}_z(F) = \frac{\frac{c_2^{-1/2}}{\alpha} + \frac{c_2^{-1/2}}{\alpha+1} - \frac{c_2^{-1/2-\alpha} b_2^{\alpha}}{\alpha}}{\frac{c_2^{1/2}}{\alpha} + \frac{c_2^{1/2}}{\alpha+1} - \frac{c_2^{-1/2-\alpha} b_2^{\alpha+1}}{\alpha+1}} \vee \frac{1}{\pi_0^{21}},$$

where $b_i^{1/2+\alpha} = 2c_i^{1/2+\alpha}(1-\lambda)$.

Remarks. (i) Let $\alpha = 1/2$. Consider 2 situations.
Let $\lambda \le 1/2$. Then $b_1 = \frac{c_1}{2\lambda}$. Then

$$\overline{V}_z(F) = \frac{3c_1}{2\lambda} \wedge \pi_0^{12}, \underline{V}_z(F) = \frac{2\lambda}{3c_2} \vee \frac{1}{\pi_0^{21}}.$$

Let $\lambda \ge 1/2$. Then $b_1 = 2c_1(1-\lambda)$. Then

$$\overline{V}_z(F) = c_1 \frac{4-(1-\lambda)^{3/2}2^{3/2}}{4-3(1-\lambda)^{1/2}2^{1/2}} \wedge \pi_0^{12},$$

$$\underline{V}_z(F) = c_2^{-1} \frac{4-3(1-\lambda)^{1/2}2^{1/2}}{4-(1-\lambda)^{3/2}2^{3/2}} \vee \frac{1}{\pi_0^{21}}.$$

(ii) If $\alpha \to \infty$ or $\lambda \to 1$ then $\underline{V}_z(F) \to \frac{1}{c_2} \vee \frac{1}{\pi_0^{21}}$ and $\overline{V}_z(F) \to c_1 \wedge \pi_0^{12}$. If we use NA condition then $\underline{V}_z(F) = \frac{1}{\pi_0^{21}}$, $\overline{V}_z(F) = \pi_0^{12}$.

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Thank you for your attention

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