Stochastic Volatility Models with Jumps
Exotic Derivatives in Financial Markets

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Summer School in Stochastic Finance 2010
Ulm University, September 2010

Based on joint work with Martijn Pistorius
Overview of the Course

Part I: The Models (SVJ)

Part II: Exotic Derivatives (volatility derivatives, forward-starting options, asymptotics of the implied volatility smile)

Part III: Fluctuation Theory and Barrier Options
Part I

The Models
The Models

What are models used for?

Understanding the risk of portfolios of derivative securities:
- Pricing
- Hedging
- Risk Management

Features they must possess:
- Jumps (Gamma Regime)
- Stochasticity of Volatility (Vega Regime, Volatility Clustering)
- Analytical Tractability (Calibration, Hedging and Risk Management)
Regime switching Lévy processes: the volatility chain

- State-space $E^0 := \{1, \ldots, N_0\}$, $N_0 \in \mathbb{N}$, of a continuous-time Markov chain $Z = (Z_t)_{t \geq 0}$.
- Generator of $Z$ is $Q \in \mathbb{R}^{N_0 \times N_0}$.
- Notation: $M \in \mathbb{C}^{N_0 \times N_0}$, $m \in \mathbb{C}^{N_0}$ are identified with functions

\[
M : E^0 \times E^0 \to \mathbb{C}, \quad M(i, j) = M_{ij} = e_i^t M e_j, \\
m : E^0 \to \mathbb{C}, \quad m(j) = m_j = m^t e_j,
\]

where $i, j = 1, \ldots, N_0$, and $e_i$ are the standard basis of $\mathbb{C}^{N_0}$. 
Regime switching Lévy processes: the volatility chain

- Let $B : E^0 \to \mathbb{C}$.
- Let $\Lambda_B$ be a diagonal matrix such that $\Lambda_B(i, i) = B(i), i = 1, \ldots, N_0$. Then it holds that
  \[
  \mathbb{P}_i [Z_t = j] = \exp(tQ)(i, j) \]
  \[
  \mathbb{E}_i \left[ \exp \left( \int_0^t B(Z_s) ds \right) I_{\{Z_t = j\}} \right] = \exp(t(Q + \Lambda_B))(i, j)
  \]
  for any $i, j \in E^0$, $t \geq 0$.
- We denote $\mathbb{E}_i[\cdot] = \mathbb{E}[\cdot | Z_0 = i]$, $\mathbb{P}_i[\cdot] = \mathbb{P}[\cdot | Z_0 = i]$, and $I_{\{\cdot\}}$ is the indicator of the set $\{\cdot\}$.
- Note that the former expression is a special case of the latter.
Regime switching Lévy processes

- Let \( i \in E^0 \) and \( X^i = (X^i_t)_{t \geq 0} \) Lévy process with characteristic exponent \( \psi_i : \mathbb{R} \to \mathbb{R} \),

\[
\mathbb{E}[e^{iuX^i_t}] = e^{t\psi_i(u)},
\]

with the Lévy-Khintchine representation

\[
\psi_i(u) = i\mu_i u - \frac{\sigma^2_i}{2} u^2 + \int_{-\infty}^{\infty} [e^{iux} - 1 - iuxI_{\{|x| \leq 1\}}] \nu_i(dx),
\]

where \( \sigma_i, \mu_i \in \mathbb{R} \) are constants and \( \nu_i \) is the Lévy measure.

- Hence \( \nu_i \) satisfies the integrability condition

\[
\int_{\mathbb{R}} (1 \wedge x^2) \nu_i(dx) < \infty.
\]

- \( (\mu_i, \sigma_i^2, \nu_i) \) is the characteristic triplet of \( X^i \).
Regime switching Lévy processes

- Vanilla option prices must be finite!
- Hence exponential moments must be finite: assume $\exists p_i > 1$

$$\int_1^\infty e^{p_i x} \nu_i(dx) < \infty.$$  

- This is equivalent to

$$\mathbb{E}\left[e^{p_i X^i_t}\right] < \infty \quad \text{for all} \quad t \geq 0.$$  

- Then identity $\mathbb{E}[e^{iuX^i_t}] = e^{t\psi_i(u)}$ remains valid for all $u$ in strip

$$\{u \in \mathbb{C} : \Im(u) \in (-p_i, 0]\} \subset \mathbb{C}$$  

where the function $\psi_i$ is analytically extended to this strip.
Regime switching Lévy model

Model for the foreign exchange rate \( S = (S_t)_{t \geq 0} \) is given by

\[
S_t := S_0 \exp(X_t) \quad \text{where} \quad S_0 \in (0, \infty) \quad \text{and} \quad X_t := \sum_{i \in E^0} \int_0^t I\{Z_s=i\} dX^i_s.
\]

Domestic and foreign money market accounts (MMA)

\( B^D = (B^D_t)_{t \geq 0} \) and \( B^F = (B^F_t)_{t \geq 0} \):

\[
B^D_t := \exp \left( \int_0^t R_D(Z_s) ds \right), \quad B^F_t := \exp \left( \int_0^t R_F(Z_s) ds \right).
\]

Functions \( R_D, R_F, \mu, \sigma : E^0 \to \mathbb{R} \) and Lévy measures \( \nu_i, i \in E^0 \), are given and \( R_D, R_F \geq 0 \) and \( \sigma > 0 \).

\( X^i \) are independent Lévy processes with triplets \( (\mu(i), \sigma(i)^2, \nu_i) \) for \( i \in E^0 \).
Regime switching Lévy model: basic observations

- The process $X$ is not Markovian!
- The pair $(X, Z)$, is Markov and task is to understand its law!
- Let $J^i$, $i \in E^0$, be independent pure jump Lévy processes (i.e. with characteristic triplets $(0, 0, \nu_i)$ and $W = (W)_{t \geq 0}$ standard Brownian motion. Then the process $\tilde{X}$, defined by

$$\tilde{X}_t := \int_0^t \mu(Z_s)ds + \int_0^t \sigma(Z_s)dW_s + \sum_{i \in E^0} \int_0^t I_{\{Z_s=i\}}dJ^i_s,$$

has the same law as $X$. 
The characteristic matrix exponent

The characteristic matrix exponent \( K : \mathbb{R} \to \mathbb{C}^{N_0 \times N_0} \) of \((X, Z)\) is

\[
K(u) := Q + \Lambda(u), \quad \text{where} \quad \Lambda(u)(i, i) = \psi_i(u), \quad i \in E^0,
\]

\( \Lambda(u) \) is a diagonal matrix and \( Q \) the generator of \( Z \).

Define diagonal matrices \( \Lambda_D \) and \( \Lambda_F \) by

\[
\Lambda_D(i, i) := R_D(i), \quad \Lambda_F(i, i) := R_F(i).
\]

**Theorem 1** The discounted characteristic function of Markov process \((X, Z)\):

\[
\mathbb{E}_{x,i}\left[ \frac{\exp(iuX_t)}{B_t^D} I\{Z_t=j\} \right] = \exp(iux) \cdot \exp(t(K(u) - \Lambda_D))(i, j), \quad u \in \mathbb{R},
\]

where \( \mathbb{E}_{x,i}[\cdot] \) denotes the conditional expectation \( \mathbb{E}[\cdot | X_0 = x, Z_0 = i] \).
The characteristic matrix exponent

**Proof.** Define $\Psi(i, u) := \psi_i(u)$, $i \in E^0$, and condition on $\mathcal{F}_t^Z := \sigma(Z_s : s \in [0,t])$:

$$\mathbb{E}_{x,i} \left[ \frac{\exp(i u X_t)}{B_t^D} I_{\{Z_t = j\}} \bigg| \mathcal{F}_t^Z \right] = \exp \left( iux + \int_0^t (\Psi(Z_s, u) - R_D(Z_s)) \, ds \right) (i, j)$$

Recall that

$$\mathbb{E}_i \left[ \exp \left( \int_0^t B(Z_s) \, ds \right) I_{\{Z_t = j\}} \right] = \exp \left( t(Q + \Lambda_B) \right) (i, j)$$

for any function $B : E^0 \to \mathbb{C}$, with $\Lambda_B$ diagonal, $\Lambda_B(i, i) = B(i)$.

□
Regime switching Lévy model

- A risk-neutral measure for $S$ makes $(S_t B_t^F / B_t^D)_{t \geq 0}$ into a positive martingale.
- Pricing measure is non-unique (the market is incomplete).
- Natural choice is given by
  \[ \Lambda(-i) = \Lambda_D - \Lambda_F \implies \mathbb{E}_{i,x}[S_t B_t^F / B_t^D] = e^x [\exp(tQ)] 1(i) = S_0 B_0^F / B_0^D \]
  for all $S_0 = e^x \in (0, \infty)$. This, together with Markov property of $(X, Z)$, implies that $(S_t B_t^F / B_t^D)_{t \geq 0}$ is a martingale.
- Here we are implicitly using the assumption $p_i > 1$ for $i \in E^0$. 
Regime switching Lévy model

- The price at time \( s \) of a zero coupon bond maturing at \( t \geq s \)

\[
\mathbb{E}_i \left[ \frac{1}{B_t^D} \bigg| \mathcal{F}_s^{(X,Z)} \right] = \frac{1}{B_s^D} \cdot (\exp((t-s)(Q - \Lambda_D)))1)(Z_s),
\]

where \( \mathcal{F}_s^{(X,Z)} = \sigma((X_u, Z_u) : u \in [0, s]) \).

- Infinitesimal generator \( \mathcal{L} \) of Markov process \((X, Z)\) is for sufficiently smooth functions \( f : \mathbb{R} \times E^0 \rightarrow \mathbb{R} \) as

\[
\mathcal{L} f(x, i) = \frac{\sigma^2(i)}{2} f''(x, i) + \mu(i) f'(x, i)
\]
\[
+ \int_{\mathbb{R}} \left[ f(x + z, i) - f(x, i) - f'(x, i) z I_{\{|z| \leq 1\}} \right] \nu_i(dz),
\]
\[
+ \sum_{j \in E^0} Q(i, j) [f(x, j) - f(x, i)].
\]
Markov additive process \((X, Z)\)

An important subclass of regime switching Lévy processes:

\[
X_t := x + \int_0^t \mu(Z_s)ds + \int_0^t \sigma(Z_s)dW_s + \sum_{i \in E^0} \int_0^t I\{Z_s = i\}dJ^i_s.
\]

- \(J^i = (J^i_t)_{t \geq 0}\) are independent compound Poisson processes with Lévy exponents
  \[
  \psi_i(u) = \lambda_i \left(\Phi_i(u) - 1\right), \quad u \in \mathbb{R}, \ i \in E^0,
  \]
  where jump intensity \(\lambda_i \geq 0\) and \(\Phi_i(u)\) is the characteristic function of the jump distribution in regime \(i\)
  with:
  \[
  \Phi_i(-i) < \infty \iff p_i \geq 1 \iff \text{jump distrib has exp moment}.
  \]

- \(Z = (Z_t)_{t \geq 0}\) a continuous-time MC on \(E^0 = \{1, \ldots, N\}\).
Phase-type distributions

**Definition** A cdf \( F : \mathbb{R}_+ \to [0, 1] \) is *phase-type* if it is a cdf of the absorption time of a continuous-time MC on \( m + 1 \in \mathbb{N} \) states, with one state absorbing and the remaining states transient.

- \( F \sim PH(\alpha, A) \): vector \( \alpha \in [0, 1]^m \) satisfies \( 0 \leq \alpha' \mathbf{1} \leq 1 \) and \( A \in \mathbb{R}^{m \times m} \) is a sub-generator matrix, i.e. a generator of the chain restricted to the transient states.
- \( F \) is uniquely determined by vector \( \alpha \) and matrix \( A \in \mathbb{R}^{m \times m} \).
- The initial distribution and generator of the original chain are

\[
\begin{pmatrix}
\alpha \\
1 - \alpha' \mathbf{1}
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
A & (-A) \mathbf{1} \\
0 & 0
\end{pmatrix}.
\]
Phase-type distributions: properties and examples

If $F \sim PH(\alpha, A)$ then

- cdf and pdf take the following form

$$F(t) = 1 - \alpha' e^{tA} \mathbf{1} \quad \text{and} \quad f(t) = -\alpha' e^{tA} A \mathbf{1} \quad \text{for any} \quad t \in \mathbb{R}_+.$$ 

- Characteristic function

$$\Phi(-iu) = \mathbb{E}[\exp(uX)] = \alpha'(A + uI)^{-1} A \mathbf{1} + (1 - \alpha' \mathbf{1}),$$

exists and is finite if and only if $\Re(u) < -\Re(\lambda_0)$, where $\lambda_0$ is the eigenvalue of $A$ with the largest real part.

Examples: Hyper-exponential, Erlang
Double phase-type distributions

\( F \sim \text{DPH}(p, \beta^+, B^+, \beta^-, B^-) \) is double phase-type if its pdf is

\[
f(x) = pf^+(x)I_{(0,\infty)}(x) + (1 - p)f^-(x)I_{(-\infty,0)}(x) \quad \text{such that}
\]

\[
p \in [0, 1], \quad f^\pm(x) = -\left(\beta^\pm\right)'e^{xB^\pm}B^\pm 1 \quad \text{and} \quad 1'\beta^\pm = 1.
\]

- Condition \( 1'\beta^\pm = 1 \) ensures that the distribution of jump sizes has no atom at zero.
- The DPH contains double exponential,

\[
f(x) := p\alpha^+ e^{-x\alpha^+}I_{(0,\infty)}(x) + (1 - p)\alpha^- e^{x\alpha^-}I_{(-\infty,0)}(x),
\]

where \( \alpha^\pm > 0 \) and \( p \in [0, 1] \), and double Erlang distributions.
Markov additive process \((X, Z)\)

**Proposition 1**  Let \(F\) be a probability distribution function on \(\mathbb{R}\). Then there exists a sequence \((F_n)_{n \in \mathbb{N}}\) of double-phase-type distributions \(F_n\) such that \(F_n \Rightarrow F\) as \(n \to \infty\).

- Class of Markov additive process \((X, Z)\) where

\[
X_t = x + \int_0^t \mu(Z_s)ds + \int_0^t \sigma(Z_s)dW_s + \sum_{i \in E_0} \int_0^t I\{Z_s = i\}dJ_s^i,
\]

and jumps of \(J^i\) are DPH, is dense in regime-switching Lévy.

- When generalised appropriately, the lack-of-memory property holds for phase-type distributions.

- Wiener-Hopf theory can be developed for \((X, Z)\).
Regime switching Lévy models

How are regime switching Lévy models used in practice?

• As approximations to general stochastic volatility models with jumps (the chain $Z$ has many states).
• As parsimonious descriptions of risk-neutral probability laws implied by the markets (the chain $Z$ has two or three states).
• $v = \{v_t\}_{t \geq 0}$ a Markov process in $\mathbb{R}_+$ (stochastic variance).

• $X$ be a Lévy process (possibly Brownian motion) with characteristic exponent $\psi(u)$, independent of $v$.

A class of stochastic volatility models in a time interval $[0, T]$

$$S_t := S_0 \exp \left( (r - d)t + \int_0^t \sqrt{v_u}dX_u - \int_0^t \psi(-i\sqrt{v_s})ds \right),$$

where

$$\int_0^T |\psi(-i\sqrt{v_s})|ds < \infty \quad \text{a.s.}$$

• If $X$ is BM and $v$ indep. square-root process, then $S$ Heston.

• $v$ scales the jump-size distribution of $S$ and does NOT affect the jump-intensity!
Stochastic volatility models with jumps

\[ S_t := S_0 \exp((r - d)t + X_{V_t} - \psi(-i)V_t), \quad \text{where} \]

\[ V_t := \int_0^t v_u du < \infty \quad \text{a.s.} \]

- Stochasticity of volatility is achieved by randomly changing the time-scale.
- If \( X \) Brownian motion with drift: the scaling property of BM implies both SV models are the same.
- \( v \) modulates jump-intensity not jump-size.

**HOMEWORK:** Prove that in both cases \( (e^{-(r-d)t} S_t)_{t \in [0,T]} \) is a martingale.
Two step approximation of SVJ

(i) Approximate variance process $v$ by a finite-state continuous-time Markov chain.

(ii) Approximation of the Lévy process $X$ by a Lévy process with double-phase-type jumps.

Basic idea: approximate the respective generators of $v$ and $X$ and define a Markov additive process that approximates $S$.

- In (i) fix a state-space and approximate the generator of $v$ locally at every state by a generator matrix.
- In (ii) approximate the Lévy triplet.
European options in regime switching Lévy model

A call option struck at $K$ with expiry $T$ is defined as

$$C_T(K) := C(S_0, i, K, T) := \mathbb{E}_{x,i} \left[ (B_T^D)^{-1}(S_T - K)^+ \right].$$

- Fourier transform $c_T^*$ in log-strike $k = \log K$ of $C_T(K)$ is

$$c_T^*(\xi) = \int_{\mathbb{R}} e^{i \xi k} C_T(e^k) dk \quad \text{where} \quad \Im(\xi) < 0.$$

- Let $\xi \in \mathbb{C}\backslash\{0, i\}$, $x \in \mathbb{R}$, $j \in E^0$. Define

$$D(\xi, x, j) := \frac{e^{(1+i\xi)x}}{i\xi - \xi^2} \cdot \left[ \exp \left\{ T(K(1+i\xi) - \Lambda_D) \right\} 1 \right](j).$$

- If $\Im(\xi) < 0$, then for $x = \log S_0$ and $Z_0 = j$, it holds

$$c_T^*(\xi) = D(\xi, x, j) \quad \text{since} \ldots$$
European options in regime switching Lévy model

\[ c^*_T(\xi) = \int_{\mathbb{R}} \exp((iv + \alpha)k) \mathbb{E}_{x,j} \left[ (B^D_T)^{-1} (S_T - \exp(k))^+ \right] \, dk \]

\[ = \mathbb{E}_{x,j} \left[ (B^D_T)^{-1} \int_{\mathbb{R}} \exp((iv + \alpha)k)(S_T - \exp(k))^+ \, dk \right] \]

\[ = \mathbb{E}_{x,j} \left[ (B^D_T)^{-1} \exp((1 + \alpha + iv)X_T) \right] / (\alpha^2 + \alpha - v^2 + i(2\alpha + 1)v) \]

\[ = \frac{e^{x(1+\alpha+iv)}}{\alpha^2 + \alpha - v^2 + i(2\alpha + 1)v} [\exp(T(K(1 + \alpha + iv) - \Lambda_D))1](j). \]

Then for \( k = \log(K) \) and \( \alpha > 0 \) we have

\[ C_T(K) = \frac{\exp(-\alpha k)}{2\pi} \int_{-\infty}^{\infty} e^{-isk} c^*_T(s - i\alpha) \, ds \]

\[ = \frac{\exp(-\alpha k)}{\pi} \int_{0}^{\infty} \Re \left[ e^{-isk} D(s - i\alpha, \log S_0, Z_0) \right] \, ds. \]
The implied volatility surface

**IVol surface** is a graph of a function \((K, T) \mapsto \sigma(K, T)\) defined implicitly by the equation

\[
C^{BS}(S_0, K, T, \sigma(K, T)) = C(K, T),
\]

where \(C(K, T)\) are the market/model specified call option prices and \(C^{BS}(S_0, K, T, \cdot)\) is the Black-Scholes formula.

- \(C(K_{ij}, T_i), i = 1, \ldots, n, j = 1, 2, 3,\) are the most liquid derivative instruments in the financial markets.
- Knowing \(\sigma\) is equivalent to knowing the one-dimensional marginals in a risk-neutral measure of the underlying process.
- To calibrate to the observed IVol surface the model needs to have stochastic volatility AND jumps.
- If \(n = 2\) (i.e. two maturities) typically time-dependence of parameters is needed for calibration.
Simple Markov additive model – Calibration

- \( N_0 = 2 \) (two states only!)
- \( \Lambda(u) \) a \( 2 \times 2 \) diagonal matrix with the \( i \)-th diagonal element

\[
\psi_i(u) := u\mu_i + \sigma_i^2 u^2 / 2 + \lambda_i p_i \left( \frac{\alpha_i^+}{\alpha_i^- - u} - 1 \right) + \lambda_i (1 - p_i) \left( \frac{\alpha_i^-}{\alpha_i^- + u} - 1 \right).
\]

- Recall \( \Lambda_D := \text{diag}(R_D) \), \( \Lambda_F := \text{diag}(R_F) \) and

\[
\mathbb{E}_{0,i} \left[ \frac{\exp(uX_t)}{B_t^D} I\{Z_t=j\} \right] = \left[ \exp(t(Q + \Lambda(u) - \Lambda_D)) \right] (i,j).
\]

- A risk-neutral drift \( \mu : \mathbb{E}^0 \to \mathbb{R} \) is given by the formula

\[
\Lambda(1) = \Lambda_D - \Lambda_F.
\]
Markov additive model – calibration of stochastic rates

- For maturities $T_1 < T_2$ market implies two pairs $P_{0,T_k}^D, P_{0,T_k}^F$, $k = 1, 2$, of domestic and foreign zero coupon bond prices.
- In our model we have

$$P_{0,T_k}^F = \mathbb{E}_{x,i}[(B_{T_k}^D)^{-1}S_{T_k}]/S_0 \quad \text{and} \quad P_{0,T_k}^D = \mathbb{E}_{x,i}[(B_{T_k}^D)^{-1}].$$

- To calibrate $R_D, R_F$ solve the system:

$$P_{0,T_k}^D = e_i' \exp((Q - \Lambda_D)T_k) 1,$$

$$P_{0,T_k}^F = e_i' \exp((Q - \Lambda_F)T_k) 1,$$

where $k = 1, 2$ and $\Lambda_D = \text{diag}(R_D), \Lambda_F = \text{diag}(R_F)$.

- Since $N_0 = 2$, this system determines the risk-neutral drift of $S$, is independent of the calibration to option prices and can be solved accurately very fast.
Market data: $S_0 = 98.05$, domestic rate $r_d = -0.00036$, foreign rate $r_f = 0.0045$, maturity $T = 1/12$.

Model parameters: $N = 2$, $q_1 = 12$, $q_2 = 6$, $B_m(1) = \text{diag}(-45, -300)$, $B_p(1) = -100, b_m(1) = (0.12, 0.88)$, $\lambda_2 = 0$ (chosen), $\sigma = (0.0423, 0.0628)$, $\lambda_1 = 276.5196$, $p_1 = 0.1610$ (calibrated).
Market data: $S_0 = 98.05$, domestic interest rate $r_d = (-0.00036, 0.005)$, foreign interest rate $r_f = (0.0045, 0.0111)$, maturity $T = (1/12, 3/12)$.

Model parameters: $N = 2$, $q_1 = 12$, $q_2 = 6$, $B_m(1) = \text{diag}(-45, -300)$, $b_m(1) = (0.12, 0.88)$, $B_m(2) = -50$, $B_p(1) = -130$, $p_2 = 0$ (chosen), $\sigma = (0.1312, 0)$, $\lambda_1 = 137.4337$, $\lambda_2 = 0.9484$, $p_1 = 0.0386$ (calibrated).
Market data: spot $S_0 = 1.3009$, domestic interest rate $r_d = 0.0045$, foreign interest rate $r_f = 0.0084$, maturity $T = 1/12$.

Model parameters: $N = 2$, $q_1 = 12$, $q_2 = 6$, $B_m(1) = \text{diag}(-45, -300)$, $b_m(1) = (0.1, 0.9)$, $B_p(1) = -130$, $\lambda_2 = 0$ (chosen)

$\sigma = (0.1352, 0.0490)$, $\lambda_1 = 90.6456$, $p_1 = 0.5231$ (calibrated)
Market data: $S_0 = 1.3009$, domestic rate $r_d = (0.0045, 0.0111)$, foreign rate $r_f = (0.0084, 0.0139)$, maturity $T = (1/12, 3/12)$.

Model parameters: $N = 2$, $q_1 = 12$, $q_2 = 6$, $B_m(1) = -70$, $B_p(1) = -70$, $B_m(2) = -30$, $B_p(2) = -30$, $p_2 = 0.5$ (chosen)
$\sigma = (0.1281, 0.0001)$, $\lambda_1 = 10.7141$, $\lambda_2 = 10.2962$, $p_1 = 0.1084$ (calibrated)
Part II

Exotic Derivatives
Implied volatility at extreme strikes

The implied volatility $\sigma_{x,i}(K, T)$ in $(X, Z)$ satisfies

$$C^{\text{BS}}(e^x, K, T, \sigma_{x,i}(K, T)) = \mathbb{E}_{x,i} \left[ (B_T^D)^{-1}(S_T - K)^+ \right].$$

For fixed maturity $T$ define the quantities $F_T := \mathbb{E}_{x,i}[S_T]$ and

$$q_+ := \sup \left\{ u : \mathbb{E}_{x,i} \left[ e^{(1+u)X_T} \right] < \infty \text{ for all } i \in E^0 \right\},$$
$$q_- := \sup \left\{ u : \mathbb{E}_{x,i} \left[ e^{-uX_T} \right] < \infty \text{ for all } i \in E^0 \right\}.$$

Lee formula (under some assumptions):

$$\lim_{K \to \infty} \frac{T \sigma_{x,i}(K, T)^2}{\log(K/F_T)} = 2 - 4 \left( \sqrt{q_+^2 + q_+} - q_+ \right),$$
$$\lim_{K \to 0} \frac{T \sigma_{x,i}(K, T)^2}{|\log(K/F_T)|} = 2 - 4 \left( \sqrt{q_-^2 + q_-} - q_- \right).$$
Ivol at extreme strikes in \((X, Z)\) with phase-type jumps

If \(X\) has double-phase type jumps then, for \(i \in E^0\), \(\psi_i(u)\) is:

\[
i u \mu_i - \sigma_i^2 u^2 / 2 + \lambda_i [p_i (\beta_i^+)' (B_i^+ + iuI)^{-1} B_i^+ 1 + (1-p_i) (\beta_i^-)' (B_i^- - iuI)^{-1} B_i^- 1].
\]

Define \(\alpha_i^\pm := \min \{-\Re(\lambda) : \lambda \text{ eigenvalue of } B_i^\pm\}\) for any state \(i \in E^0\).

- Note \(\psi_i\) has analytic extension to \(\{u \in \mathbb{C} : \Im(u) \in (-\alpha_i^+, \alpha_i^-)\}\).
- If the chain \(Z\) is irreducible, the quantities \(q_{\pm}\) are:

\[
q_+ = \min \{\alpha_i^+ - 1 : i \in E^0, \ p_i \lambda_i > 0\}, \quad q_- = \min \{\alpha_i^- : i \in E^0, \ (1-p_i) \lambda_i > 0\}.
\]

\[
\lim_{K \to \infty} \frac{T \sigma_x,i(K,T)^2}{\log(K/F_T)} = 2 - 4 \left(\sqrt{q_+^2 + q_+ - q_+}\right),
\]

\[
\lim_{K \to 0} \frac{T \sigma_x,i(K,T)^2}{|\log(K/F_T)|} = 2 - 4 \left(\sqrt{q_-^2 + q_- - q_-}\right).
\]
Forward starting options

A payoff of $T_1$-forward starting call option with maturity $T_2 > T_1$ is

$$(S_{T_2} - \kappa S_{T_1})^+, \quad \kappa \in \mathbb{R}_+.$$  

- The Fourier transform in the forward log-strike of $F_{T_1,T_2} (\kappa) = \mathbb{E}_{x,i} \left[ (B_{T_2}^D)^{-1} (S_{T_2} - \kappa S_{T_1})^+ \right]$ is defined by

  $$F_{T_1,T_2}^* (\xi) = \int_{\mathbb{R}} e^{i\xi k} F_{T_1,T_2}(e^k) dk, \quad \text{where} \quad \Im(\xi) < 0.$$  

- For $x = \log S_0$, $Z_0 = j$ and $\xi$ with $\Im(\xi) < 0$ it holds that

  $$F_{T_1,T_2}^* (\xi) = \frac{e^{(1+i\xi)x}}{i\xi - \xi^2} \left[ \exp(T_1(Q - \Lambda_F)) \exp \left\{ (T_2 - T_1)(K(1 + i\xi) - \Lambda_D) \right\} 1 \right](j).$$
Forward starting options

Proof.

\[
F_{T_1,T_2}(\kappa) = \mathbb{E}_{x,i} \left[ (B_{T_2}^D)^{-1} (S_{T_2} - \kappa S_{T_1})^+ \right]
\]

\[
= \mathbb{E}_{x,i} \left[ \frac{S_{T_1}}{B_{T_1}^D} \mathbb{E}_{0,Z_{T_1}} \left[ (B_{T_2-T_1}^D)^{-1} (S_{T_2-T_1} - \kappa)^+ \right] \right]
\]

\[
= \sum_{j \in E^0} \mathbb{E}_{x,i} \left[ \frac{S_{T_1}}{B_{T_1}^D} I\{Z_{T_1} = j\} \right] \mathbb{E}_{0,j} \left[ (B_{T_2-T_1}^D)^{-1} (S_{T_2-T_1} - \kappa)^+ \right]
\]

\[
= S_0 \sum_{j \in E^0} e'_i \exp(T(K(-i) - \Lambda_D)) e_j \mathbb{E}_{0,j} \left[ (B_{T_2-T_1}^D)^{-1} (S_{T_2-T_1} - \kappa)^+ \right]
\]

\[
= S_0 e'_i \exp(T(K(-i) - \Lambda_D)) C_{T_2-T_1}(\kappa; 1),
\]

\(j\)-th entry of vector \(C_{T_2-T_1}(\kappa; 1)\) is \(\mathbb{E}_{0,j} \left[ (B_{T_2-T_1}^D)^{-1} (S_{T_2-T_1} - \kappa)^+ \right].\)
The forward smile

The \textit{forward implied volatility} $\sigma_{x,i}^{fw}(S_T, \kappa, T)$ at a future time $T$: \[
C^{BS}(S_{T_1}, \kappa S_{T_1}, T_2 - T_1, \sigma_{x,i}^{fw}(S_{T_1}, \kappa, T_1)) = \mathbb{E}_{x,i} \left[ \frac{B_{T_1}^D}{B_{T_2}^D} (S_{T_2} - \kappa S_{T_1})^+ \right| S_{T_1} \right],
\]

where $C^{BS}$ the Black-Scholes formula with strike $\kappa S_{T_1}$ and spot $S_{T_1}$. \[
\mathbb{E}_{x,i} \left[ \frac{B_{T_1}^D}{B_{T_2}^D} (S_{T_2} - \kappa S_{T_1})^+ \right| S_{T_1} \right] = S_{T_1} f_{x,i}^{x,i}(X_{T_1}, T_1)' C_{T_2-T_1}(\kappa, 1),
\]

where \[
f_{x,i}^{x,i}(y, T) := \mathbb{P}_{x,i} \left[ Z_T = j \big| X_T = y \right] = \frac{q_{x,i}^{x,i}(y, j)}{q_{x,i}^{x,i}(y)} \quad \text{and ...}
\]
The forward smile

... the joint distribution \( q^{x,i}_{T}(y,j) = \frac{d}{dy} \mathbb{P}_{x,i}[X_T \leq y, Z_T = j] \) at time \( T \) of \((X_T, Z_T)\) is given by

\[
q^{x,i}_{T}(y,j) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi(x-y)} \exp(K(\xi) T)(i,j) \, d\xi, \quad y \in \mathbb{R}, i, j \in E^0. 
\]

\( X_T \) is a continuous random variable with probability density function \( q^{x,i}_{T}(y) = \frac{\mathbb{P}_{x,i}[X_T \in dy]}{dy} \) given by

\[
q^{x,i}_{T}(y) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi(x-y)} \left[\exp(K(\xi) T) \mathbf{1}\right](i) \, d\xi, \quad y \in \mathbb{R}, i \in E^0. 
\]

**Proof.** The characteristic function is in \( L^1(\mathbb{R}) \).
Volatility derivatives

Refining sequence of partitions \((\Pi_n)_{n \in \mathbb{N}}\) of \([0, T]\): \(\Pi_n \subset \Pi_{n+1}\), \(\Pi_n = \{t^n_0 \leq \ldots \leq t^n_n\}\) s.t. \(\lim_{n \to \infty} \max\{|t^n_i - t^n_{i-1}| : 1 \leq i \leq n\} = 0\).

- **Quadratic variation** \(\Sigma_T\) of \(X = \log S\):

\[
\Sigma_T := \lim_{n \to \infty} \sum_{t^n_i \in \Pi_n, i \geq 1} \log \left( \frac{S_{t^n_i}}{S_{t^n_{i-1}}} \right)^2.
\]

- The sequence converges in probability, uniformly on \([0, T]\).

- The limit is given by

\[
\Sigma_T = \int_0^T \sigma(Z_t)^2 \, dt + \sum_{i \in E^0} \sum_{t \leq T} I_{\{Z_t = i\}} (\Delta X^i_t)^2,
\]

where \(\Delta X^i_t := X^i_t - X^i_{t-}\).
Volatility derivatives

\((\Sigma_t)_{t \geq 0}\) is the \textit{quadratic variation (realized variance) process} of \(X\).

- A buyer of a swap on the realized variance pays a premium (the swap rate) to receive at maturity \(T\) a pay-off \(\phi(\Sigma_T)\), where \(\phi : \mathbb{R}_+ \to \mathbb{R}_+\) is a measurable payoff function.

- Most common examples of \(\phi\) are
  (i) variance swap: \(\phi(x) = x/T\).
  (ii) volatility swap: \(\phi(x) = \sqrt{x/T}\).
  (iii) option on variance: \(\phi(x) = (x - \kappa)^+,\) where \(\kappa \in \mathbb{R}_+\).

- The swap rate for the payoff \(\phi\) is \(\mathbb{E}_i \left[ \phi(\Sigma_T)/B^D_T \right] \).
(\(\Sigma_t\))\(\text{t} \geq 0\) is a regime-switching Lévy process with

\[ \Sigma_t = \int_0^t \sigma(Z_s)^2 ds + \sum_{i \in E^0} \int_0^t I\{Z_s = i\} d\tilde{X}^i_s, \]

where \(\tilde{X}^i, i \in E^0\), is a pure-jump subordinator with

\[ \nu^{\Sigma}(dx) = I_{(0,\infty)}(x)[-d\overline{\nu}(\sqrt{x}) + d\nu(-\sqrt{x})] \quad \text{(Lévy measure)} \]

\[ \psi^\Sigma_i(u) = u\sigma^2_i + \int_{\mathbb{R}^+} (1 - e^{-ux})\nu^\Sigma_i(dx) \]

\[ = u\sigma^2_i + \int_{\mathbb{R}} (1 - e^{-uy^2})\nu_i(dy) \quad \text{(characteristic exponent of } \tilde{X}^i). \]

Recall: \(\psi^\Sigma_i(u) = -\log\mathbb{E}[e^{-u\tilde{X}^i_1}], \overline{\nu}(x) = \nu([x, \infty)), \nu(x) = \nu(-\infty, x)]. \)
Volatility derivatives

The Laplace transform of $\Sigma_t$ is given by

$$\mathbb{E}_i [\exp(-u\Sigma_t)] = \exp(tK_\Sigma(u)) \mathbf{1}(i), \quad u > 0,$$

where

- the characteristic matrix $K_\Sigma(u)$ is given by

$$K_\Sigma(u) := Q + \Lambda_\Sigma(u) \quad \text{and}$$

- $\Lambda_\Sigma(u)$ is an $N_0 \times N_0$ diagonal matrix with

$$\Lambda_\Sigma(u)(i, i) = \psi_\Sigma^i(u) = -\log \mathbb{E}[e^{-u\tilde{X}_i}], \quad i \in E^0.$$
Volatility derivatives

$X^i$ jump-diffusion with double phase-type jumps. Then

- $\tilde{X}^i$ is a compound Poisson process with intensity $\lambda_i$
- with positive jump sizes $K_i$ with probability density

$$g_i(x) = \frac{1}{2\sqrt{x}} \left[ p_i \beta_i^+ e^{\sqrt{x}B_i^+} (-B_i^+) 1 + (1-p_i) \beta_i^- e^{\sqrt{x}B_i^-} (-B_i^-) 1 \right] I_{(0,\infty)}(x).$$

- $\Phi(x) := \exp(x^2/2)\mathcal{N}(x)$, $\mathcal{N}$ normal cdf. Then $\mathbb{E} [\exp(-uK_i)]$ is

$$\sqrt{\frac{\pi}{u}} \left[ p_i \beta_i^+ \Phi \left( \frac{1}{\sqrt{2u}} B_i^+ \right) (-B_i^+) + (1-p_i) \beta_i^- \Phi \left( \frac{1}{\sqrt{2u}} B_i^- \right) (-B_i^-) \right] 1$$

- and the characteristic exponent of $\tilde{X}^i$ equals

$$\psi_i^\Sigma(u) := u\sigma_i^2 + \lambda_i (1 - \mathbb{E} [\exp(-uK_i)]).$$
Volatility derivatives - the pricing formulae

Assume $R_D \equiv \text{const}$ (to simplify the formulae) and $Z_0 = i$.

- $\varsigma_{\text{var}}(T, j) = \mathbb{E}_i \left[ \Sigma T / T \right]$ and $\varsigma_{\text{vol}}(T, j) = \mathbb{E}_i \left[ \sqrt{\Sigma T / T} \right]$ are

$$
\varsigma_{\text{var}}(T, j) = \frac{1}{T} \left[ \int_0^T e^{Q t} V dt \right] (j),
$$

$$
\varsigma_{\text{vol}}(T, j) = \frac{1}{2 \sqrt{\pi T}} \int_0^\infty \left\{ \left[ I - \exp(TK_\Sigma(u)) \right] 1 \right\} (j) \frac{du}{u^{3/2}},
$$

where $V \in \mathbb{R}^{N_0}$ with $V(i) = (\psi_i^\Sigma)'(0) = \sigma_i^2 + \int_{\mathbb{R}} y^2 \nu_i(dy)$.

- $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and $\exists a > 0$ s.t the Fourier transform $\phi^*_a$ of $\phi_a(x) = e^{ax} \phi(x)$ is in $L^1(\mathbb{R})$. Then the $\phi$-swap rate is

$$
\varsigma_{\phi}(T, j) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \phi^*_a(\xi) \left[ \exp(T(K_\Sigma(a - i\xi) - \Lambda_D)) 1 \right] (j) d\xi.
$$
Volatility derivatives - remarks

- $X^i$ phase-type Lévy model:

$$V(i) = \sigma^2_i + 2\lambda_i \left(p_i(\beta^+_i)'(B^+_i)^{-2} + (1 - p_i)(\beta^-_i)'(B^-_i)^{-2}\right) \mathbf{1}.$$ 

- Since $Q = U^{-1} \Delta U$ for diagonal $\Delta$ with $\Delta(i, i) = \lambda_i$ and $\lambda_1 = 0$, $\varsigma_{var}(T, j)$ is given by

$$\varsigma_{var}(T, j) = \frac{1}{T} \left[ U^{-1} \begin{pmatrix} T \\ \lambda_2^{-1}(e^{\lambda_2 T} - 1) \\ \vdots \\ \lambda_N^{-1}(e^{\lambda_N T} - 1) \end{pmatrix} UV \right] (j).$$

- The integral in $\varsigma_{vol}(T, j)$ converges at the rate proportional to $1/\sqrt{M}$ for upper bound $M$ (follows from definition of $K_\Sigma(u)$).
Variance swap formula – proof

Condition on the sigma algebra $\mathcal{F}^Z_T = \sigma(\{Z_t : t \in [0, T]\})$:

$$
\mathbb{E}_{x,i} \left[ \int_0^t \sigma(Z_s)^2 ds + \sum_{j \in E^0} \int_0^t I\{Z_s = j\} d\tilde{X}^j_s \right] = \sum_{j \in E^0} \mathbb{E}_{x,i} \left[ \int_0^T I\{Z_s = j\} ds \right] w(j),
$$

where $w(j) := \sigma^2(j) + \mathbb{E}[\tilde{X}_1^j] = (\psi^\Sigma)'(0)$, $j \in E^0$, and note that

$$
\mathbb{E}_{x,i} \left[ \int_0^T I\{Z_s = j\} ds \right] = \left[ \int_0^T \exp(sQ) ds \right] (i, j)
$$

by Fubini’s theorem.
Volatility derivatives - proofs

Elementary integral

\[
\sqrt{x} = \frac{1}{2\sqrt{\pi}} \int_0^{\infty} [1 - \exp(-ux)] \frac{du}{u^{3/2}}, \quad \text{for any} \quad x \geq 0,
\]

and Fubini’s theorem yield

\[
\mathbb{E}_i \left[ \sqrt{\frac{\Sigma T}{T}} \right] = \frac{1}{2\sqrt{\pi T}} \int_0^{\infty} \left[ (\exp(TQ) - \exp(T(K\Sigma(u)))) 1 \right] (i) \frac{du}{u^{3/2}}.
\]

Similarly for \( \phi \)-swap the Fourier inversion formula yields

\[
\phi(S) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \phi_a^*(\xi) e^{-(a + i\xi)S} d\xi \quad \text{and hence}
\]

\[
\mathbb{E}_j [\phi(\Sigma_T)] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \phi_a^*(\xi) [\exp(TK\Sigma(a + i\xi)1)] (j) d\xi, \quad a > 0.
\]
Part III

Fluctuation Theory and Barrier Options
Barrier contracts

A barrier contract with expiry $T > 0$ pays the random cash flow

$$g(S_T)I_{\{\tau_A > T\}} + h(S_{\tau_A})I_{\{\tau_A \leq T\}}, \quad \text{where} \quad \tau_A = \inf\{t \geq 0 : S_t \in A\},$$

- knock-out set $A = (0, \ell] \cup [u, \infty)$, $0 \leq \ell < u \leq \infty$;
- $g, h : (0, \infty) \to \mathbb{R}_+$ payoff and rebate functions respectively.

Examples:
- knock-out double barrier $(0 < \ell, u < \infty, h \equiv 0)$;
- down-and-out $(u = \infty, h \equiv 0)$, up-and-out $(\ell = 0, h \equiv 0)$;
- rebate $(g \equiv 0)$, European $(0 = \ell, u = \infty)$. 

Stochastic Volatility Models with Jumps – p. 50
Double-no-touch options

Double-no-touch (or range bet) pays one unit of domestic currency at $T$ if FX rate $S$ stays in $(\ell, u)$ during $[0, T]$ and zero else.

- DNTs are the most liquid exotic options in financial markets.
- Hence DNTs should be used for calibration of the model $S$.
- The arbitrage-free price in a model $S$ of a double-no-touch:

$$D_{S_0}(T) = \mathbb{E}_S \left[ \frac{I_{\{\tau_{\ell u} > T\}}}{B_T^D} \right], \quad \text{where}$$

$$\tau_{\ell u} := \inf \{ t : S_t \notin (\ell, u) \}.$$  

Warning: price of DNT involves joint law of max and min of $S$. 
Wiener-Hopf factorisation for Brownian motion $X$

Let $e_q$ be exponential rv, $\mathbb{E}[e_q] = 1/q$, independent of $X$. 

$$\frac{q}{q - u^2/2} = \frac{\rho_+(q)}{\rho_+(q) + u} \cdot \frac{\rho_-(q)}{\rho_-(q) - u}, \quad \text{where} \quad \rho_\pm(q) = \pm \sqrt{2q}$$

are the largest and smallest root of the characteristic equation

$$q - \frac{u^2}{2} = 0.$$ 

Define $\overline{X}_t = \max\{X_s : s \in [0, t]\}$, $\underline{X}_t = \min\{X_s : s \in [0, t]\}$.
Moment generating function of $\overline{X}_{eq}$, $\underline{X}_{eq}$ are

$$\mathbb{E} \left[ \exp(-u\overline{X}_{eq}) \right] = \frac{\rho_+(q)}{\rho_+(q) - u}, \quad \mathbb{E} \left[ \exp(u\underline{X}_{eq}) \right] = \frac{\rho_-(q)}{u + \rho_-(q)}, \quad u \geq 0.$$
Wiener-Hopf factorisation for Brownian motion $X$

Therefore $X_{e_q}, -X_{e_q}$ are geometric rvs with params $\rho_+(q), -\rho_-(q)$.

Let $\tau_u := \min\{t \geq 0 : X_t \geq u\}$ and $\tau_\ell := \min\{t \geq 0 : X_t \leq \ell\}$.

$$\{\tau_u < t\} = \{X_t > u\}, \quad \{\tau_\ell < t\} = \{X_t < \ell\} \quad \forall t \in \mathbb{R}_+.$$ 

Hence

$$\mathbb{E}[e^{-q\tau_u}] = \mathbb{E} \left[ \int_{0}^{\infty} I_{\{\tau_u < t\}} q e^{-qt} dt \right] = \mathbb{P}(\tau_u < e_q) = e^{-u\rho_+(q)}$$

$$\mathbb{E}[e^{-q\tau_\ell}] = e^{\ell \rho_-(q)}.$$ 

An application of Doob’s optional stopping theorem yields a closed form for the Laplace transform for the two-sided first passage time

$$\tau_{\ell u} := \inf\{t : X_t \notin (\ell, u)\}.$$
Matrix Wiener-Hopf factorisation

In the general case of the Markov additive process the steps are similar (but the details are very different):

- Fluid-embedding: embed the jumps to get a continuous Markov additive process (phase-type distribution of jumps is used in this step).
- The characteristic equation becomes a quadratic matrix equation.
- The Wiener-Hopf factors can be inverted analytically.
- Closed-form formula for Laplace transform of the one-sided first passage time can be obtained.
- Doob’s optional stopping theorem gives a closed-form formula for the Laplace transform of the two-sided first passage time.
Thank you for your attention!!

Corse notes and problem sheets available at
http://www.warwick.ac.uk/go/amilajatovic