Stochastic Volatility Models with Jumps

Exotic Derivatives in Financial Markets

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Based on joint work with **Martijn Pistorius**

Overview of the Course

Part I: The Models (SVJ)

Part II: Exotic Derivatives (volatility derivatives, forward-starting options, asymptotics of the implied volatility smile)

Part III: Fluctuation Theory and Barrier Options

Part I

The Models

The Models

What are models used for?

Understanding the risk of portfolios of derivative securities:

- Pricing
- Hedging
- Risk Management

Features they must possess:

- Jumps (Gamma Regime)
- Stochasticity of Volatility (Vega Regime, Volatility Clustering)
- Analytical Tractability (Calibration, Hedging and Risk Managemement)

Regime switching Lévy processes: the volatility chain

- State-space $E^0 := \{1, \dots, N_0\}$, $N_0 \in \mathbb{N}$, of a continuous-time Markov chain $Z = (Z_t)_{t \geq 0}$.
- Generator of Z is $Q \in \mathbb{R}^{N_0 \times N_0}$.
- Notation: $M \in \mathbb{C}^{N_0 \times N_0}$, $m \in \mathbb{C}^{N_0}$ are identified with functions

$$M : E^0 \times E^0 \to \mathbb{C}, \qquad M(i,j) = M_{ij} = e_i' M e_j,$$
$$m : E^0 \to \mathbb{C}, \qquad m(j) = m_j = m' e_j,$$

where $i, j = 1, \dots, N_0$, and e_i are the standard basis of $\mathbb{C}^{N_0}$.

Regime switching Lévy processes: the volatility chain

- Let $B : E^0 \to \mathbb{C}$.
- Let Λ_B be a diagonal matrix such that $\Lambda_B(i,i) = B(i)$, $i = 1, \ldots, N_0$. Then it holds that

$$\begin{aligned} \mathbb{P}_i\left[Z_t = j\right] &= \exp\left(tQ\right)(i,j) \\ \mathbb{E}_i\left[\exp\left(\int_0^t B(Z_s)ds\right) I_{\{Z_t = j\}}\right] &= \exp\left(t(Q + \Lambda_B)\right)(i,j) \end{aligned}$$

  for any $i, j \in E^0,\ t \geq 0,$.
- We denote $\mathbb{E}_i[\cdot] = \mathbb{E}[\cdot|Z_0 = i]$, $\mathbb{P}_i[\cdot] = \mathbb{P}[\cdot|Z_0 = i]$, and $I_{\{\cdot\}}$ is the indicator of the set $\{\cdot\}$.
- Note that the former expression is a special case of the latter.

Regime switching Lévy processes

- Let $i \in E^0$ and $X^i = (X_t^i)_{t \geq 0}$ Lévy process with characteristic exponent $\psi_i : \mathbb{R} \to \mathbb{R}$,

$$\mathbb{E}\left[\mathrm{e}^{\mathtt{i}uX_t^i}\right] = \mathrm{e}^{t\psi_i(u)},$$

with the Lévy-Khintchine representation

$$\psi_i(u) = \mathtt{i}\mu_i u - \frac{\sigma_i^2}{2}u^2 + \int_{-\infty}^{\infty} [\mathrm{e}^{\mathtt{i}ux} - 1 - \mathtt{i}uxI_{\{|x| \leq 1\}}]\nu_i(\mathsf{d}x),$$

where $\sigma_i, \mu_i \in \mathbb{R}$ are constants and ν_i is the Lévy measure.

- Hence ν_i satisfies the integrability condition

$$\int_{\mathbb{R}} (1 \wedge x^2)\nu_i(\mathsf{d}x) < \infty.$$

- $(\mu_i, \sigma_i^2, \nu_i)$ is the characteristic triplet of X^i.

Regime switching Lévy processes

- Vanilla option prices must be finite!
- Hence exponential moments must be finite: assume $\exists p_i > 1$

$$\text{such that} \qquad \int_1^\infty \mathrm{e}^{p_i x} \nu_i(\mathrm{d}x) < \infty.$$

- This is equivalent to

$$\mathbb{E}\left[\mathrm{e}^{p_i X_t^i}\right] < \infty \quad \text{for all} \quad t \geq 0.$$

- Then identity $\mathbb{E}[\mathrm{e}^{\mathtt{i} u X_t^i}] = \mathrm{e}^{t\psi_i(u)}$ remains valid for all u in strip

$$\{u \in \mathbb{C} \,:\, \Im(u) \in (-p_i, 0]\} \subset \mathbb{C}$$

where the function ψ_i is analytically extended to this strip.

Regime switching Lévy model

- Model for the foreign exchange rate $S = (S_t)_{t\geq 0}$ is given by

$$S_t := S_0 \exp(X_t) \qquad \text{where} \quad S_0 \in (0,\infty) \quad \text{and}$$
$$X_t := \sum_{i\in E^0} \int_0^t I_{\{Z_s=i\}} \mathrm{d}X_s^i.$$

- Domestic and foreign money market accounts (MMA) $B^D = (B_t^D)_{t\geq 0}$ and $B^F = (B_t^F)_{t\geq 0}$:

$$B_t^D := \exp\left(\int_0^t R_D(Z_s)\mathrm{d}s\right), \qquad B_t^F := \exp\left(\int_0^t R_F(Z_s)\mathrm{d}s\right).$$

- Functions $R_D, R_F, \mu, \sigma : E^0 \to \mathbb{R}$ and Lévy measures ν_i, $i \in E^0$, are given and $R_D, R_F \geq 0$ and $\sigma > 0$.
- X^i are independent Lévy processes with triplets $(\mu(i), \sigma(i)^2, \nu_i)$ for $i \in E^0$.

Regime switching Lévy model: basic observations

- The process X is not Markovian!
- The pair (X, Z), is Markov and task is to understand its law!
- Let J^i, $i \in E^0$, be independent pure jump Lévy processes (i.e. with characteristic triplets $(0, 0, \nu_i)$ and $W = (W)_{t \geq 0}$ standard Brownian motion. Then the process $\widetilde{X}$, defined by

$$\widetilde{X}_t \quad := \quad \int_0^t \mu(Z_s) \mathrm{d}s + \int_0^t \sigma(Z_s) \mathrm{d}W_s + \sum_{i \in E^0} \int_0^t I_{\{Z_s = i\}} \mathrm{d}J_s^i,$$

has the same law as X.

The characteristic matrix exponent

The *characteristic matrix exponent* $K : \mathbb{R} \to \mathbb{C}^{N_0 \times N_0}$ of (X, Z) is

$$K(u) := Q + \Lambda(u), \quad \text{where } \Lambda(u)(i,i) = \psi_i(u),\ i \in E^0,$$

$\Lambda(u)$ is a diagonal matrix and Q the generator of Z.
Define diagonal matrices Λ_D and Λ_F by

$$\Lambda_D(i,i) := R_D(i), \qquad \Lambda_F(i,i) := R_F(i).$$

Theorem 1 *The discounted characteristic function of Markov process (X, Z):*

$$\mathbb{E}_{x,i}\left[\frac{\exp(\mathtt{i}uX_t)}{B_t^D} I_{\{Z_t=j\}}\right] = \exp(\mathtt{i}ux)\cdot\exp(t(K(u)-\Lambda_D))(i,j), \quad u \in \mathbb{R},$$

where $\mathbb{E}_{x,i}[\cdot]$ denotes the conditional expectation $\mathbb{E}[\cdot|X_0 = x, Z_0 = i]$.

The characteristic matrix exponent

Proof. Define $\Psi(i,u) := \psi_i(u),\ i \in E^0$, and condition on $\mathcal{F}_t^Z := \sigma(Z_s : s \in [0,t])$:

$$\mathbb{E}_{x,i}\left[\frac{\exp(\mathtt{i}uX_t)}{B_t^D} I_{\{Z_t=j\}} \middle| \mathcal{F}_t^Z\right] = \exp\left(\mathtt{i}ux + \int_0^t \left(\Psi(Z_s,u) - R_D(Z_s)\right) \mathsf{d}s\right)(i,j)$$

Recall that

$$\mathbb{E}_i\left[\exp\left(\int_0^t B(Z_s)ds\right) I_{\{Z_t=j\}}\right] = \exp\left(t(Q+\Lambda_B)\right)(i,j)$$

for any function $B : E^0 \to \mathbb{C}$, with Λ_B diagonal, $\Lambda_B(i,i) = B(i)$. $\square$

Regime switching Lévy model

- A risk-neutral measure for S makes $(S_t B_t^F / B_t^D)_{t \geq 0}$ into a positive martingale.
- Pricing measure is non-unique (the market is incomplete).
- Natural choice is given by

$$\begin{aligned} \Lambda(-\mathtt{i}) &= \Lambda_D - \Lambda_F \qquad \Longrightarrow \\ \mathbb{E}_{i,x}[S_t B_t^F / B_t^D] &= \mathrm{e}^x \left[\exp\left(tQ\right)\mathbf{1}\right](i) = S_0 B_0^F / B_0^D \end{aligned}$$

  for all $S_0 = \mathrm{e}^x \in (0, \infty)$. This, together with Markov property of (X, Z), implies that $(S_t B_t^F / B_t^D)_{t \geq 0}$ is a martingale.
- Here we are implicitly using the assumption $p_i > 1$ for $i \in E^0$.

Regime switching Lévy model

- The price at time s of a zero coupon bond maturing at $t \geq s$

$$\mathbb{E}_i\left[\frac{1}{B_t^D}\middle|\mathcal{F}_s^{(X,Z)}\right] = \frac{1}{B_s^D} \cdot \left(\exp((t-s)(Q-\Lambda_D))\mathbf{1}\right)(Z_s),$$

where $\mathcal{F}_s^{(X,Z)} = \sigma\left((X_u, Z_u)\ :\ u \in [0,s]\right)$.

- Infinitesimal generator $\mathcal{L}$ of Markov process (X,Z) is for sufficiently smooth functions $f : \mathbb{R} \times E^0 \to \mathbb{R}$ as

$$\begin{aligned}\mathcal{L}f(x,i) \quad &= \quad \frac{\sigma^2(i)}{2} f''(x,i) + \mu(i) f'(x,i) \\ &\quad + \int_{\mathbb{R}} \left[f(x+z,i) - f(x,i) - f'(x,i) z I_{\{|z| \leq 1\}}\right] \nu_i(\mathrm{d}z), \\ &\quad + \sum_{j \in E^0} Q(i,j)[f(x,j) - f(x,i)].\end{aligned}$$

Markov additive process (X, Z)

An important subclass of regime switching Lévy processes:

$$X_t \quad := \quad x + \int_0^t \mu(Z_s)\mathrm{d}s + \int_0^t \sigma(Z_s)\mathrm{d}W_s + \sum_{i \in E^0} \int_0^t I_{\{Z_s = i\}}\mathrm{d}J_s^i.$$

- $J^i = (J_t^i)_{t \geq 0}$ are independent compound Poisson processes with Lévy exponents

$$\psi_i(u) = \lambda_i \left(\Phi_i(u) - 1\right), \qquad u \in \mathbb{R},\ i \in E^0,$$

  where jump intensity $\lambda_i \geq 0$ and $\Phi_i(u)$ is the characteristic function of the jump distribution in regime i with:

$$\Phi_i(-\mathtt{i}) < \infty \iff p_i \geq 1 \iff \quad \text{jump distrib has exp moment.}$$

- $Z = (Z_t)_{t \geq 0}$ a continuous-time MC on $E^0 = \{1, \ldots, N\}$.

Phase-type distributions

Definition A cdf $F : \mathbb{R}_+ \to [0,1]$ is *phase-type* if it is a cdf of the absorption time of a continuous-time MC on $m+1 \in \mathbb{N}$ states, with one state absorbing and the remaining states transient.

- $F \sim PH(\alpha, A)$: vector $\alpha \in [0,1]^m$ satisfies $0 \le \alpha' \mathbf{1} \le 1$ and $A \in \mathbb{R}^{m \times m}$ is a sub-generator matrix, i.e. a generator of the chain restricted to the transient states.
- F is uniquely determined by vector α and matrix $A \in \mathbb{R}^{m \times m}$.
- The initial distribution and generator of the original chain are

$$\begin{pmatrix} \alpha \\ 1 - \alpha' \mathbf{1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A & (-A)\,\mathbf{1} \\ \mathbf{0} & 0 \end{pmatrix}.$$

Phase-type distributions: properties and examples

If $F \sim PH(\alpha, A)$ then

- cdf and pdf take the following form

$$F(t) = 1 - \alpha' e^{tA}\mathbf{1} \quad \text{and} \quad f(t) = -\alpha' e^{tA} A\mathbf{1} \quad \text{for any} \quad t \in \mathbb{R}_+.$$

- Characterisitc function

$$\Phi(-\mathtt{i}u) = \mathbb{E}[\exp(uX)] = \alpha'(A + uI)^{-1}A\mathbf{1} + (1 - \alpha'\mathbf{1}),$$

  exists and is finite if and only if $\Re(u) < -\Re(\lambda_0)$, where λ_0 is the eigenvalue of A with the largest real part.

Examples: Hyper-exponential, Erlang

Double phase-type distributions

$F \sim DPH(p, \beta^+, B^+, \beta^-, B^-)$ is *double phase-type* if its pdf is

$$f(x) \quad = \quad pf^+(x)I_{(0,\infty)}(x) + (1-p)f^-(-x)I_{(-\infty,0)}(x) \quad \text{such that}$$
$$p \in [0,1], \quad f^\pm(x) = -(\beta^\pm)' e^{xB^\pm} B^\pm \mathbf{1} \quad \text{and} \quad \mathbf{1}'\beta^\pm = 1.$$

- Condition $\mathbf{1}'\beta^\pm = 1$ ensures that the distribution of jump sizes has no atom at zero.
- The DPH contains double exponential,

$$f(x) \quad := \quad p\alpha^+ e^{-x\alpha^+} I_{(0,\infty)}(x) + (1-p)\alpha^- e^{x\alpha^-} I_{(-\infty,0)}(x),$$

where $\alpha^\pm > 0$ and $p \in [0,1]$, and double Erlang distributions.

Markov additive process (X, Z)

Proposition 1 *Let F be a probability distribution function on $\mathbb{R}$. Then there exists a sequence $(F_n)_{n\in\mathbb{N}}$ of double-phase-type distributions F_n such that $F_n \Rightarrow F$ as $n \to \infty$.*

- Class of Markov additive process (X, Z) where

$$X_t = x + \int_0^t \mu(Z_s)\mathrm{d}s + \int_0^t \sigma(Z_s)\mathrm{d}W_s + \sum_{i\in E^0} \int_0^t I_{\{Z_s=i\}}\mathrm{d}J_s^i,$$

  and jumps of J^i are DPH, is dense in regime-switching Lévy.
- When generalised appropriately, the lack-of-memory property holds for phase-type distributions.
- Wiener-Hopf theory can be developed for (X, Z).

Regime switching Lévy models

How are regime switching Lévy models used in practice?

- As approximations to general stochastic volatility models with jumps (the chain Z has many states).
- As parsimonious descriptions of risk-neutral probability laws implied by the markets (the chain Z has two or three states).

Stochastic volatility models with jumps

- $v = \{v_t\}_{t\geq 0}$ a Markov process in $\mathbb{R}_+$ (stochastic variance).
- X be a Lévy process (possibly Brownian motion) with characteristic exponent $\psi(u)$, independent of v.

A class of stochastic volatility models in a time interval $[0, T]$

$$S_t := S_0 \exp\left((r-d)t + \int_0^t \sqrt{v_u}\mathtt{d}X_u - \int_0^t \psi(-\mathtt{i}\sqrt{v_s})\mathtt{d}s\right), \qquad \text{where}$$

$$\int_0^T |\psi(-\mathtt{i}\sqrt{v_s})|\mathtt{d}s < \infty \quad \text{a.s.}$$

- If X is BM and v indep. square-root process, then S Heston.
- v scales the jump-size distribution of S and does NOT affect the jump-intensity!

Stochastic volatility models with jumps

$$S_t := S_0 \exp\left((r-d)t + X_{V_t} - \psi(-\mathtt{i})V_t\right), \qquad \text{where}$$

$$V_t := \int_0^t v_u \mathsf{d}u < \infty \quad \text{a.s.}$$

- Stochasticity of volatility is achieved by randomly changing the time-scale.
- If X Brownian motion with drift: the scaling propery of BM implies both SV models are the same.
- v modulates jump-intensity not jump-size.

HOMEWORK: Prove that in both cases $(\mathrm{e}^{-(r-d)t} S_t)_{t\in[0,T]}$ is a martingale.

Two step approximation of SVJ

(i) Approximate variance process v by a finite-state continuous-time Markov chain.

(ii) Approximation of the Lévy process X by a Lévy process with double-phase-type jumps.

Basic idea: approximate the respective generators of v and X and define a Markov additive process that approximates S.

- In (i) fix a state-space and approximate the generator of v locally at every state by a generator matirx.
- In (ii) approximate the Lévy triplet.

European options in regime switching Lévy model

A call option struck at K with expiry T is defined as

$$C_T(K) := C(S_0, i, K, T) := \mathbb{E}_{x,i}\left[(B_T^D)^{-1}(S_T - K)^+\right].$$

- Fourier transform c_T^* in log-strike $k = \log K$ of $C_T(K)$ is

$$c_T^*(\xi) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\xi k} C_T(\mathrm{e}^k)\mathsf{d}k \quad \text{where} \quad \Im(\xi) < 0.$$

- Let $\xi \in \mathbb{C}\backslash\{0, \mathtt{i}\}$, $x \in \mathbb{R}$, $j \in E^0$. Define

$$D(\xi, x, j) := \frac{\mathrm{e}^{(1+\mathtt{i}\xi)x}}{\mathtt{i}\xi - \xi^2} \cdot \left[\exp\left\{T(K(1+\mathtt{i}\xi) - \Lambda_D)\right\}\mathbf{1}\right](j).$$

- If $\Im(\xi) < 0$, then for $x = \log S_0$ and $Z_0 = j$, it holds

$$c_T^*(\xi) = D(\xi, x, j) \qquad \text{since ...}$$

European options in regime switching Lévy model

$$
\begin{aligned}
c_T^*(\xi) &= \int_{\mathbb{R}} \exp((\mathtt{i}v+\alpha)k)\mathbb{E}_{x,j}\left[(B_T^D)^{-1}(S_T-\exp(k))^+\right]\mathsf{d}k \\
&= \mathbb{E}_{x,j}\left[(B_T^D)^{-1}\int_{\mathbb{R}} \exp((\mathtt{i}v+\alpha)k)(S_T-\exp(k))^+\mathsf{d}k\right] \\
&= \mathbb{E}_{x,j}\left[(B_T^D)^{-1}\exp((1+\alpha+\mathtt{i}v)X_T)\right]/(\alpha^2+\alpha-v^2+\mathtt{i}(2\alpha+1)v) \\
&= \frac{\mathrm{e}^{x(1+\alpha+\mathtt{i}v)}}{\alpha^2+\alpha-v^2+\mathtt{i}(2\alpha+1)v}\left[\exp(T(K(1+\alpha+\mathtt{i}v)-\Lambda_D))\mathbf{1}\right](j).
\end{aligned}
$$

Then for $k=\log(K)$ and $\alpha>0$ we have

$$
\begin{aligned}
C_T(K) &= \frac{\exp(-\alpha k)}{2\pi}\int_{-\infty}^{\infty} e^{-\mathtt{i}sk}c_T^*(s-\mathtt{i}\alpha)\mathsf{d}s \\
&= \frac{\exp(-\alpha k)}{\pi}\int_0^{\infty} \Re\left[e^{-\mathtt{i}sk}D(s-\mathtt{i}\alpha,\log S_0,Z_0)\right]\mathsf{d}s.
\end{aligned}
$$

The implied volatility surface

IVol surface is a graph of a function $(K,T) \mapsto \sigma(K,T)$ defined implicitly by the equation

$$C^{\text{BS}}(S_0, K, T, \sigma(K,T)) \quad = \quad C(K,T),$$

where $C(K,T)$ are the market/model specified call option prices and $C^{\text{BS}}(S_0, K, T, \cdot)$ is the Black-Scholes formula.

- $C(K_{ij}, T_i)$, $i = 1, \ldots, n$, $j = 1, 2, 3$, are the most liquid derivative instruments in the financial markets.
- Knowing σ is equivalent to knowing the one-dimensional marginals in a risk-neutral measure of the underlying process.
- To calibrate to the observed IVol surface the model needs to have stochastic volatility AND jumps.
- If $n = 2$ (i.e. two maturities) typically time-dependence of parameters is needed for calibration.

Simple Markov additive model – Calibration

- $N_0 = 2$ (two states only!)
- $\Lambda(u)$ a 2×2 diagonal matrix with the i-th diagonal element

$$\psi_i(u) := u\mu_i + \sigma_i^2 u^2/2 + \lambda_i p_i \left(\frac{\alpha_i^+}{\alpha_i^+ - u} - 1 \right) + \lambda_i (1 - p_i) \left(\frac{\alpha_i^-}{\alpha_i^- + u} - 1 \right).$$

- Recall $\Lambda_D := \mathrm{diag}(R_D)$, $\Lambda_F := \mathrm{diag}(R_F)$ and

$$\mathbb{E}_{0,i}\left[\frac{\exp(uX_t)}{B_t^D} I_{\{Z_t = j\}} \right] = [\exp(t(Q + \Lambda(u) - \Lambda_D))]\,(i,j).$$

- A risk-neutral drift $\mu : E^0 \to \mathbb{R}$ is given by the formula

$$\Lambda(1) = \Lambda_D - \Lambda_F.$$

Markov additive model – calibration of stochastic rates

- For maturities $T_1 < T_2$ market implies two pairs P^D_{0,T_k}, P^F_{0,T_k}, $k = 1, 2$, of domestic and foreign zero coupon bond prices.
- In our model we have

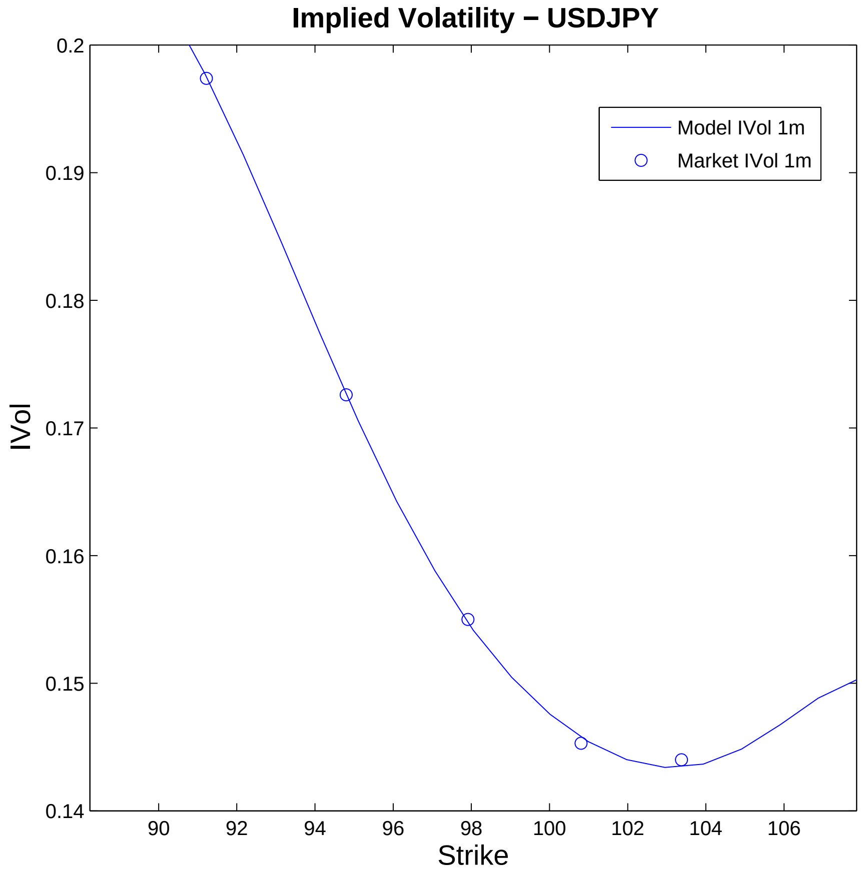
$$P^F_{0,T_k} = \mathbb{E}_{x,i}[(B^D_{T_k})^{-1} S_{T_k}]/S_0 \quad \text{and} \quad P^D_{0,T_k} = \mathbb{E}_{x,i}[(B^D_{T_k})^{-1}].$$

- To calibrate R_D, R_F solve the system:

$$\begin{aligned} P^D_{0,T_k} &= e_i' \exp\left((Q - \Lambda_D)T_k\right) \mathbf{1}, \\ P^F_{0,T_k} &= e_i' \exp\left((Q - \Lambda_F)T_k\right) \mathbf{1}, \end{aligned}$$

  where $k = 1, 2$ and $\Lambda_D = \operatorname{diag}(R_D), \Lambda_F = \operatorname{diag}(R_F)$.
- Since $N_0 = 2$, this system determines the risk-neutral drift of S, is independent of the calibration to option prices and can be solved accurately very fast.

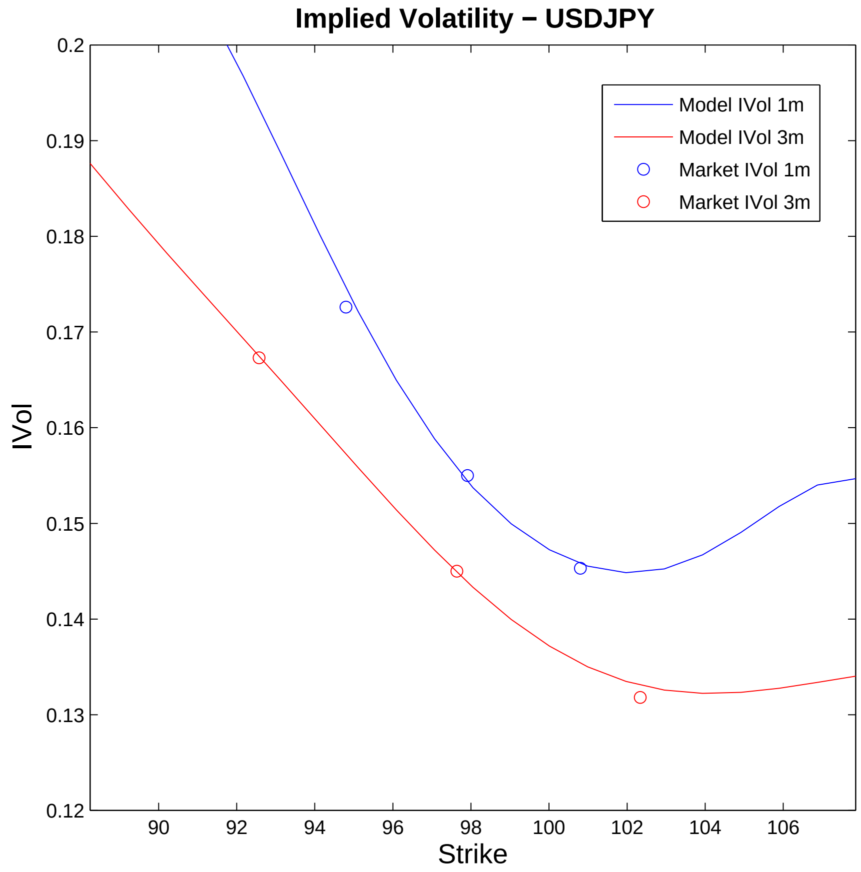
USDJPY – one maturity



Market data: $S_0 = 98.05$, domestic rate $r_d = -0.00036$, foreign rate $r_f = 0.0045$, maturity $T = 1/12$.

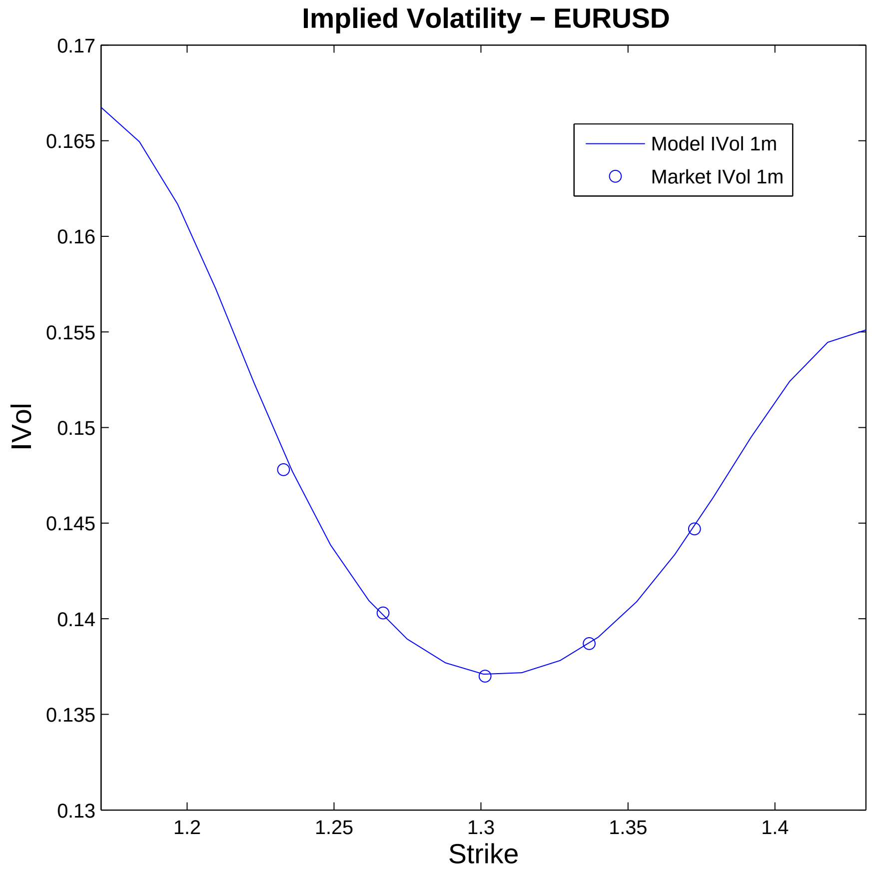
Model parameters: $N = 2$, $q_1 = 12, q_2 = 6$, $B_m(1) = \text{diag}(-45, -300)$, $B_p(1) = -100$,$b_m(1) = (0.12, 0.88)$, $\lambda_2 = 0$ (chosen), $\sigma = (0.0423, 0.0628)$, $\lambda_1 = 276.5196$, $p_1 = 0.1610$ (calibrated).

USDJPY – two maturities



Market data: $S_0 = 98.05$, domestic interest rate $r_d = (-0.00036, 0.005)$, foreign interest rate $r_f = (0.0045, 0.0111)$, maturity $T = (1/12, 3/12)$.
Model parameters: $N = 2$, $q_1 = 12, q_2 = 6$, $B_m(1) = \mathrm{diag}(-45, -300)$, $b_m(1) = (0.12, 0.88)$, $B_m(2) = -50$, $B_p(1) = -130$, $p_2 = 0$ (chosen), $\sigma = (0.1312, 0)$, $\lambda_1 = 137.4337$, $\lambda_2 = 0.9484$, $p_1 = 0.0386$ (calibrated)

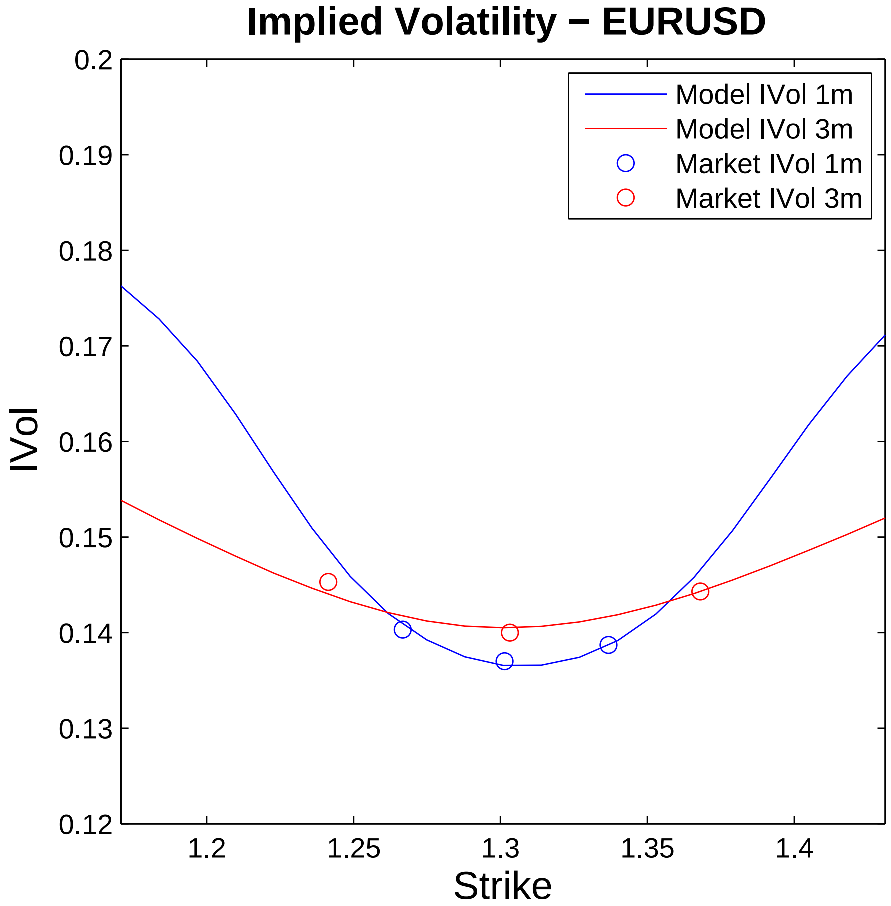
EURUSD – one maturity



Market data: spot $S_0 = 1.3009$, domestic interest rate $r_d = 0.0045$, foreign interest rate $r_f = 0.0084$, maturity $T = 1/12$.

Model parameters: $N = 2$, $q_1 = 12, q_2 = 6$, $B_m(1) = \mathrm{diag}(-45, -300)$, $b_m(1) = (0.1, 0.9)$, $B_p(1) = -130$, $\lambda_2 = 0$ (chosen)
$\sigma = (0.1352, 0.0490)$, $\lambda_1 = 90.6456$, $p_1 = 0.5231$ (calibrated)

EURUSD – two maturities



Market data: $S_0 = 1.3009$, domestic rate $r_d = (0.0045, 0.0111)$, foreign rate $r_f = (0.0084, 0.0139)$, maturity $T = (1/12, 3/12)$.

Model parameters: $N = 2$, $q_1 = 12, q_2 = 6$, $B_m(1) = -70$, $B_p(1) = -70$, $B_m(2) = -30$, $B_p(2) = -30$, $p_2 = 0.5$ (chosen)

$\sigma = (0.1281, 0.0001)$, $\lambda_1 = 10.7141$, $\lambda_2 = 10.2962$, $p_1 = 0.1084$ (calibrated)

Part II

Exotic Derivatives

Implied volatility at extreme strikes

The *implied volatility* $\sigma_{x,i}(K,T)$ in (X,Z) satisfies

$$C^{\mathsf{BS}}(\mathrm{e}^x, K, T, \sigma_{x,i}(K,T)) = \mathbb{E}_{x,i}\left[(B_T^D)^{-1}(S_T-K)^+\right].$$

For fixed maturity T define the quantities $F_T := \mathbb{E}_{x,i}[S_T]$ and

$$\begin{aligned} q_+ &:= \sup\left\{u : \mathbb{E}_{x,i}\left[\mathrm{e}^{(1+u)X_T}\right] < \infty \quad \text{for all} \quad i \in E^0\right\}, \\ q_- &:= \sup\left\{u : \mathbb{E}_{x,i}\left[\mathrm{e}^{-uX_T}\right] < \infty \quad \text{for all} \quad i \in E^0\right\}. \end{aligned}$$

Lee formula (under some assumptions):

$$\begin{aligned} \lim_{K\to\infty} \frac{T\sigma_{x,i}(K,T)^2}{\log(K/F_T)} &= 2 - 4\left(\sqrt{q_+^2 + q_+} - q_+\right), \\ \lim_{K\to 0} \frac{T\sigma_{x,i}(K,T)^2}{|\log(K/F_T)|} &= 2 - 4\left(\sqrt{q_-^2 + q_-} - q_-\right). \end{aligned}$$

Ivol at extreme strikes in (X, Z) with phase-type jumps

If X has double-phase type jumps then, for $i \in E^0$, $\psi_i(u)$ is:

$$\mathtt{i}u\mu_i - \sigma_i^2 u^2/2 + \lambda_i \left[p_i(\beta_i^+)'(B_i^+ + \mathtt{i}uI)^{-1}B_i^+ \mathbf{1} + (1-p_i)(\beta_i^-)'(B_i^- - \mathtt{i}uI)^{-1}B_i^- \mathbf{1} \right].$$

Define $\quad \alpha_i^{\pm} := \min\{-\Re(\lambda) \,:\, \lambda$ eigenvalue of $B_i^{\pm}\} \quad$ for any state $\quad i \in E^0$.

- Note ψ_i has analytic extension to $\{u \in \mathbb{C} \,:\, \Im(u) \in (-\alpha_i^+, \alpha_i^-)\}$.
- If the chain Z is irreducible, the quantities $q_{\pm}$ are:

$$q_+ = \min\{\alpha_i^+ - 1 : i \in E^0,\ p_i\lambda_i > 0\}, \quad q_- = \min\{\alpha_i^- : i \in E^0,\ (1-p_i)\lambda_i > 0\}.$$

$$\lim_{K\to\infty} \frac{T\sigma_{x,i}(K,T)^2}{\log(K/F_T)} = 2 - 4\left(\sqrt{q_+^2 + q_+} - q_+\right),$$

$$\lim_{K\to 0} \frac{T\sigma_{x,i}(K,T)^2}{|\log(K/F_T)|} = 2 - 4\left(\sqrt{q_-^2 + q_-} - q_-\right).$$

Forward starting options

A payoff of T_1-*forward starting call option* with maturity $T_2 > T_1$ is

$$(S_{T_2} - \kappa S_{T_1})^+, \qquad \kappa \in \mathbb{R}_+.$$

- The Fourier transform in the forward log-strike of $F_{T_1,T_2}(\kappa) = \mathbb{E}_{x,i}\left[(B^D_{T_2})^{-1}(S_{T_2} - \kappa S_{T_1})^+\right]$ is defined by

$$F^*_{T_1,T_2}(\xi) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\xi k} F_{T_1,T_2}(\mathrm{e}^k)\mathsf{d}k, \quad \text{where} \quad \Im(\xi) < 0.$$

- For $x = \log S_0$, $Z_0 = j$ and ξ with $\Im(\xi) < 0$ it holds that

$$F^*_{T_1,T_2}(\xi) = \frac{\mathrm{e}^{(1+\mathtt{i}\xi)x}}{\mathtt{i}\xi - \xi^2} \cdot \left[\exp(T_1(Q - \Lambda_F)) \exp\left\{(T_2 - T_1)(K(1 + \mathtt{i}\xi) - \Lambda_D)\right\}\mathbf{1}\right](j).$$

Forward starting options

Proof.

$$
\begin{aligned}
F_{T_1,T_2}(\kappa) &= \mathbb{E}_{x,i}\left[(B^D_{T_2})^{-1}(S_{T_2}-\kappa S_{T_1})^+\right] \\
&= \mathbb{E}_{x,i}\left[\frac{S_{T_1}}{B^D_{T_1}}\mathbb{E}_{0,Z_{T_1}}\left[(B^D_{T_2-T_1})^{-1}(S_{T_2-T_1}-\kappa)^+\right]\right] \\
&= \sum_{j\in E^0}\mathbb{E}_{x,i}\left[\frac{S_{T_1}}{B^D_{T_1}}I_{\{Z_{T_1}=j\}}\right]\mathbb{E}_{0,j}[(B^D_{T_2-T_1})^{-1}(S_{T_2-T_1}-\kappa)^+] \\
&= S_0\sum_{j\in E^0}e_i'\exp(T(K(-\mathtt{i})-\Lambda_D))e_j\mathbb{E}_{0,j}[(B^D_{T_2-T_1})^{-1}(S_{T_2-T_1}-\kappa)^+] \\
&= S_0 e_i'\exp(T(K(-\mathtt{i})-\Lambda_D))C_{T_2-T_1}(\kappa;1),
\end{aligned}
$$

j-th entry of vector $C_{T_2-T_1}(\kappa;1)$ is $\mathbb{E}_{0,j}[(B^D_{T_2-T_1})^{-1}(S_{T_2-T_1}-\kappa)^+]$.

The forward smile

The *forward implied volatility* $\sigma^{fw}_{x,i}(S_T, \kappa, T)$ at a future time T:

$$C^{\mathsf{BS}}(S_{T_1}, \kappa S_{T_1}, T_2 - T_1, \sigma^{fw}_{x,i}(S_{T_1}, \kappa, T_1)) = \mathbb{E}_{x,i}\left[\frac{B^D_{T_1}}{B^D_{T_2}}(S_{T_2} - \kappa S_{T_1})^+ \middle| S_{T_1}\right],$$

where C^{BS} the Black-Scholes formula with strike κS_{T_1} and spot S_{T_1}.

$$\mathbb{E}_{x,i}\left[\frac{B^D_{T_1}}{B^D_{T_2}}(S_{T_2} - \kappa S_{T_1})^+ \middle| S_{T_1}\right] \quad = \quad S_{T_1} f^{x,i}(X_{T_1}, T_1)' C_{T_2 - T_1}(\kappa, 1),$$

where

$$f^{x,i}_j(y, T) := \mathbb{P}_{x,i}\left[Z_T = j \middle| X_T = y\right] = \frac{q^{x,i}_T(y, j)}{q^{x,i}_T(y)}$$

and ...

The forward smile

... the joint distribution $q_T^{x,i}(y,j) = \frac{\mathsf{d}}{\mathsf{d}y}\mathbb{P}_{x,i}[X_T \le y, Z_T = j]$ at time T of (X_T, Z_T) is given by

$$q_T^{x,i}(y,j) \quad = \quad \frac{1}{2\pi}\int_{\mathbb{R}} e^{\mathtt{i}\xi(x-y)} \exp\left(K\left(\xi\right)T\right)(i,j)\,\mathsf{d}\xi, \qquad y \in \mathbb{R}, i,j \in E^0.$$

X_T is a continuous random variable with probability density function $q_T^{x,i}(y) = \frac{\mathbb{P}_{x,i}[X_T \in \mathsf{d}y]}{\mathsf{d}y}$ given by

$$q_T^{x,i}(y) \quad = \quad \frac{1}{2\pi}\int_{\mathbb{R}} e^{\mathtt{i}\xi(x-y)} \left[\exp\left(K\left(\xi\right)T\right)\mathbf{1}\right](i)\,\mathsf{d}\xi, \qquad y \in \mathbb{R}, i \in E^0.$$

Proof. The characteristic function is in $L^1(\mathbb{R})$.

Volatility derivatives

Refining sequence of partitions $(\Pi_n)_{n\in\mathbb{N}}$ of $[0,T]$: $\Pi_n \subset \Pi_{n+1}$, $\Pi_n = \{t_0^n \leq \ldots \leq t_n^n\}$ s.t. $\lim_{n\to\infty} \max\{|t_i^n - t_{i-1}^n| : 1 \leq i \leq n\} = 0$.

- *Quadratic variation* Σ_T of $X = \log S$:

$$\Sigma_T \quad := \quad \lim_{n\to\infty} \sum_{t_i^n \in \Pi_n, i\geq 1} \log\left(\frac{S_{t_i^n}}{S_{t_{i-1}^n}}\right)^2 .$$

- The sequence converges in probability, uniformly on $[0,T]$.
- The limit is given by

$$\Sigma_T = \int_0^T \sigma(Z_t)^2 \mathrm{d}t + \sum_{i\in E^0} \sum_{t\leq T} I_{\{Z_t=i\}} (\Delta X_t^i)^2,$$

where $\Delta X_t^i := X_t^i - X_{t-}^i$.

Volatility derivatives

$(\Sigma_t)_{t\geq 0}$ is the *quadratic variation* (*realized variance*) *process* of X.

- A buyer of a swap on the realized variance pays a premium (the swap rate) to receive at maturity T a pay-off $\phi(\Sigma_T)$, where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a measurable payoff function.
- Most common examples of ϕ are
  - (i) variance swap: $\phi(x) = x/T$.
  - (ii) volatility swap: $\phi(x) = \sqrt{x/T}$.
  - (iii) option on variance: $\phi(x) = (x-\kappa)^+$, where $\kappa \in \mathbb{R}_+$.
- The swap rate for the payoff ϕ is $\mathbb{E}_i\left[\phi(\Sigma_T)/B_T^D\right]$.

Volatility derivatives

$(\Sigma_t)_{t\geq 0}$ is a regime-switching Lévy process with

$$\Sigma_t = \int_0^t \sigma(Z_s)^2 \mathrm{d}s + \sum_{i\in E^0} \int_0^t I_{\{Z_s=i\}} d\widetilde{X}^i_s,$$

where $\widetilde{X}^i$, $i \in E^0$, is a pure-jump subordinator with

$$\begin{aligned} \nu^\Sigma(\mathrm{d}x) &= I_{(0,\infty)}(x)[-\mathrm{d}\overline{\nu}(\sqrt{x}) + \mathrm{d}\underline{\nu}(-\sqrt{x})] \qquad \text{(Lévy measure)} \\ \psi_i^\Sigma(u) &= u\sigma_i^2 + \int_{\mathbb{R}_+} (1-\mathrm{e}^{-ux})\nu_i^\Sigma(\mathrm{d}x) \\ &= u\sigma_i^2 + \int_{\mathbb{R}} (1-\mathrm{e}^{-uy^2})\nu_i(\mathrm{d}y) \qquad \text{(characteristic exponent of } \widetilde{X}^i\text{).} \end{aligned}$$

Recall: $\psi_i^\Sigma(u) = -\log \mathbb{E}[\mathrm{e}^{-u\widetilde{X}^i_1}]$, $\overline{\nu}(x) = \nu([x,\infty)), \underline{\nu}(x) = \nu(-\infty, x])$.

Volatility derivatives

The Laplace transform of Σ_t is given by

$$\mathbb{E}_i\left[\exp(-u\Sigma_t)\right] = \left[\exp(tK_\Sigma(u))\mathbf{1}\right](i), \qquad u > 0,$$

where

- the characteristic matrix $K_\Sigma(u)$ is given by

$$K_\Sigma(u) := Q + \Lambda_\Sigma(u) \qquad \text{and}$$

- $\Lambda_\Sigma(u)$ is an $N_0 \times N_0$ diagonal matrix with

$$\Lambda_\Sigma(u)(i,i) = \psi_i^\Sigma(u) = -\log \mathbb{E}[\mathrm{e}^{-u\widetilde{X}_1^i}], \quad i \in E^0.$$

Volatility derivatives

X^i jump-diffusion with double phase-type jumps. Then

- $\widetilde{X}^i$ is a compound Poisson process with intensity λ_i
- with positive jump sizes K_i with probability density

$$g_i(x) = \frac{1}{2\sqrt{x}}\left[p_i\beta_i^+ \mathrm{e}^{\sqrt{x}B_i^+}(-B_i^+)\mathbf{1} + (1-p_i)\beta_i^- \mathrm{e}^{\sqrt{x}B_i^-}(-B_i^-)\mathbf{1}\right] I_{(0,\infty)}(x).$$

- $\Phi(x) := \exp(x^2/2)\mathcal{N}(x)$, $\mathcal{N}$ normal cdf. Then $\mathbb{E}\left[\exp(-uK_i)\right]$ is

$$\sqrt{\frac{\pi}{u}}\left[p_i\beta_i^+\Phi\left(\frac{1}{\sqrt{2u}}B_i^+\right)(-B_i^+) + (1-p_i)\beta_i^-\Phi\left(\frac{1}{\sqrt{2u}}B_i^-\right)(-B_i^-)\right]\mathbf{1}$$

- and the characteristic exponent of $\widetilde{X}^i$ equals

$$\psi_i^\Sigma(u) := u\sigma_i^2 + \lambda_i\left(1 - \mathbb{E}\left[\exp(-uK_i)\right]\right).$$

Volatility derivatives - the pricing formulae

Assume $R_D \equiv$ const (to simplify the formulae) and $Z_0 = i$.

- $\varsigma_{var}(T,j) = \mathbb{E}_i\left[\Sigma_T/T\right]$ and $\varsigma_{vol}(T,j) = \mathbb{E}_i\left[\sqrt{\Sigma_T/T}\right]$ are

$$\begin{aligned}
\varsigma_{var}(T,j) &= \frac{1}{T}\left[\int_0^T \mathrm{e}^{Qt} V \mathsf{d}t\right](j), \\
\varsigma_{vol}(T,j) &= \frac{1}{2\sqrt{\pi T}}\int_0^\infty \left\{\left[I - \exp(TK_\Sigma(u))\right]\mathbf{1}\right\}(j)\frac{\mathsf{d}u}{u^{3/2}},
\end{aligned}$$

  where $V \in \mathbb{R}^{N_0}$ with $V(i) = (\psi_i^\Sigma)'(0) = \sigma_i^2 + \int_{\mathbb{R}} y^2 \nu_i(\mathsf{d}y)$.

- $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ and $\exists a > 0$ s.t the Fourier transform ϕ_a^* of $\phi_a(x) = \mathrm{e}^{ax}\phi(x)$ is in $L^1(\mathbb{R})$. Then the ϕ-swap rate is

$$\varsigma_\phi(T,j) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \phi_a^*(\xi)\left[\exp(T(K_\Sigma(a - \mathtt{i}\xi) - \Lambda_D))\mathbf{1}\right](j)\mathsf{d}\xi.$$

Volatility derivatives - remarks

- X^i phase-type Lévy model:

$$V(i) = \sigma_i^2 + 2\lambda_i \left(p_i(\beta_i^+)'(B_i^+)^{-2} + (1-p_i)(\beta_i^-)'(B_i^-)^{-2} \right) \mathbf{1}.$$

- Since $Q = U^{-1}\Delta U$ for diagonal Δ with $\Delta(i,i) = \lambda_i$ and $\lambda_1 = 0$, $\varsigma_{var}(T,j)$ is given by

$$\varsigma_{var}(T,j) = \frac{1}{T}\left[U^{-1} \begin{pmatrix} T & & & \\ & \lambda_2^{-1}(\mathrm{e}^{\lambda_2 T}-1) & & \\ & & \dots & \\ & & & \lambda_N^{-1}(\mathrm{e}^{\lambda_N T}-1) \end{pmatrix} UV \right](j).$$

- The integral in $\varsigma_{vol}(T,j)$ converges at the rate proportional to $1/\sqrt{M}$ for upper bound M (follows from definition of $K_\Sigma(u)$).

Variance swap formula – proof

Condition on the sigma algebra $\mathcal{F}_T^Z = \sigma(Z_t : t \in [0,T])$:

$$\mathbb{E}_{x,i}\left[\int_0^t \sigma(Z_s)^2 \mathrm{d}s + \sum_{j\in E^0}\int_0^t I_{\{Z_s=j\}}\mathrm{d}\widetilde{X}_s^j\right] = \sum_{j\in E^0}\mathbb{E}_{x,i}\left[\int_0^T I_{\{Z_s=j\}}\mathrm{d}s\right] w(j),$$

where $w(j) := \sigma^2(j) + \mathbb{E}[\widetilde{X}_1^j] = (\psi^\Sigma)_j'(0)$, $j \in E^0$, and note that

$$\mathbb{E}_{x,i}\left[\int_0^T I_{\{Z_s=j\}}\mathrm{d}s\right] = \left[\int_0^T \exp(sQ)\mathrm{d}s\right](i,j)$$

by Fubini's theorem.

Volatility derivatives - proofs

Elementary integral

$$\sqrt{x} = \frac{1}{2\sqrt{\pi}} \int_0^\infty [1 - \exp(-ux)] \frac{du}{u^{3/2}}, \quad \text{for any} \quad x \geq 0,$$

and Fubini's theorem yield

$$\mathbb{E}_i \left[\sqrt{\frac{\Sigma_T}{T}} \right] = \frac{1}{2\sqrt{\pi T}} \int_0^\infty [(\exp(TQ) - \exp(T(K_\Sigma(u)))\,\mathbf{1}]\,(i) \frac{\mathrm{d}u}{u^{3/2}}.$$

Similarely for ϕ-swap the Fourier inversion formula yields

$$\phi(S) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \phi_a^*(\xi) \mathrm{e}^{-(a+\mathtt{i}\xi)S} \mathrm{d}\xi \quad \text{and hence}$$

$$\mathbb{E}_j [\phi(\Sigma_T)] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \phi_a^*(\xi) [\exp(TK_\Sigma(a + \mathtt{i}\xi)\mathbf{1}]\,(j) \mathrm{d}\xi, \quad a > 0.$$

Part III

Fluctuation Theory and Barrier Options

Barrier contracts

A *barrier contract* with expiry $T > 0$ pays the random cash flow

$$g(S_T)\mathbf{I}_{\{\tau_A > T\}} + h(S_{\tau_A})\mathbf{I}_{\{\tau_A \leq T\}}, \quad \text{where} \quad \tau_A = \inf\{t \geq 0 : S_t \in A\},$$

- knock-out set $A = (0, \ell] \cup [u, \infty), \quad 0 \leq \ell < u \leq \infty$;
- $g, h : (0, \infty) \to \mathbb{R}_+$ payoff and rebate functions respectively.

Examples:

- knock-out double barrier ($0 < \ell, u < \infty, h \equiv 0$);
- down-and-out ($u = \infty$, $h \equiv 0$), up-and-out ($\ell = 0$, $h \equiv 0$);
- rebate ($g \equiv 0$), European ($0 = \ell, u = \infty$).

Double-no-touch options

Double-no-touch (or **range bet**) pays one unit of domestic currency at T if FX rate S stays in (ℓ, u) during $[0, T]$ and zero else.

- DNTs are the most liquid exotic options in financial markets.
- Hence DNTs should be used for calibration of the model S.
- The arbitrage-free price in a model S of a double-no-touch:

$$\begin{aligned} D_{S_0}(T) &= \mathbb{E}_{S_0}\left[\frac{I_{\{\tau_{\ell u} > T\}}}{B_T^D}\right], \quad \text{where} \\ \tau_{\ell u} &:= \inf\{t \, : \, S_t \notin (\ell, u)\}. \end{aligned}$$

Warning: price of DNT involves joint law of max and min of S.

Wiener-Hopf factorisation for Brownian motion X

Let e_q be exponential rv, $\mathbb{E}[e_q] = 1/q$, independent of X.

$$\frac{q}{q - u^2/2} \quad = \quad \frac{\rho_+(q)}{\rho_+(q) + u} \cdot \frac{\rho_-(q)}{\rho_-(q) - u}, \quad \text{where} \quad \rho_\pm(q) = \pm\sqrt{2q}$$

are the largest and smallest root of the characteristic equation

$$q - \frac{u^2}{2} = 0.$$

Define $\overline{X}_t = \max\{X_s \;:\; s \in [0,t]\}, \quad \underline{X}_t = \min\{X_s \;:\; s \in [0,t]\}$.
Moment generating function of $\overline{X}_{e_q}$, $\underline{X}_{e_q}$ are

$$\mathbb{E}\left[\exp(-u\overline{X}_{e_q})\right] = \frac{\rho_+(q)}{\rho_+(q) - u}, \quad \mathbb{E}\left[\exp(u\underline{X}_{e_q})\right] = \frac{\rho_-(q)}{u + \rho_-(q)}, \quad u \geq 0.$$

Wiener-Hopf factorisation for Brownian motion X

Therefore $\overline{X}_{e_q}$, $-\underline{X}_{e_q}$ are geometric rvs with params $\rho_+(q)$, $-\rho_-(q)$.
Let $\tau_u := \min\{t \geq 0 \,:\, X_t \geq u\}$ and $\tau_\ell := \min\{t \geq 0 \,:\, X_t \leq \ell\}$.

$$\{\tau_u < t\} = \{\overline{X}_t > u\}, \quad \{\tau_\ell < t\} = \{\underline{X}_t < \ell\} \quad \forall t \in \mathbb{R}_+.$$

Hence

$$\begin{aligned}
\mathbb{E}[e^{-q\tau_u}] &= \mathbb{E}\left[\int_0^\infty I_{\{\tau_u < t\}} q e^{-qt} dt\right] = \mathbb{P}(\tau_u < e_q) = e^{-u\rho_+(q)} \\
\mathbb{E}[e^{-q\tau_\ell}] &= e^{\ell\rho_-(q)}.
\end{aligned}$$

An application of Doob's optional stopping theorem yields a closed form for the Laplace transform for the two-sided first passage time

$$\tau_{\ell u} := \inf\{t \,:\, X_t \notin (\ell, u)\}.$$

Matrix Wiener-Hopf factorisation

In the general case of the Markov additive process the steps are similar (but the details are very different):

- Fluid-embedding: embed the jumps to get a continuous Markov additive process (phase-type distribution of jumps is used in this step).
- The characteristic equation becomes a quadratic matrix equation.
- The Wiener-Hopf factors can be inverted analytically.
- Closed-form formula for Laplace transform of the one-sided first passage time can be obtained.
- Doob’s optional stopping theorem gives a closed-form formula for the Laplace transform of the two-sided first passage time.

Thank you for your attention!!

Corse notes and problem sheets available at
`http://www.warwick.ac.uk/go/amijatovic`