Innovative retirement products

An Chen, University of Ulm

joint with:

Peter Hieber (Technical University of Munich)

Jakob Klein (Allianz Life)

Manuel Rach (University of Ulm)

Thorsten Sehner (University of Ulm)

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Content

- This talk “Innovative retirement products” is based on two working papers:
  - An Chen, Peter Hieber and Jakob Klein (2019): “Tonuity: A Novel Individual-Oriented Retirement Plan”. *Astin Bulletin*, Forthcoming.
  - An Chen, Manuel Rach and Thorsten Sehner (2019): “On the optimal combination of annuities and tontines”, *Preprint*

We start with the first paper

Chen & Hieber & Klein (2019)

Motivation I

Ageing society:
How to ensure
pension security?

- Desirable products (from policyholders' perspective)
  - not too costly
  - providing good protection against longevity risk
  - secure cash flows in advanced ages

Motivation II: retirement products

- Annuity
  - longevity protection ($\surd$)
  - **Solvency II**: Annuity products get more expensive (more risk capital needed).
- Tontine
  - Popular 17th century (FR, GB), today **"Le Conservateur"**, Sabin (2010), Milevsky and Salisbury (2015, 2016)
  - not good longevity protection
  - low risk capital required ($\surd$)

$\Rightarrow$ Chen & Hieber & Klein (2019): Ton~~tine/ann~~uity = Tonuity
Chen & Rach & Sehner (2019): Other combinations of tontines and annuities

Annuity and Tontine: Payoff

Single premium P_0 at time $t = 0$.

Annuity: payoff $c(t)$ ($t \geq 0$) until death (residual life time $\zeta > 0$):

$$b_A(t) := \mathbf{1}_{\{\zeta > t\}}\, c(t)\,.$$

Tontine: **homogeneous cohort** of size n receives payoff $nd(t)$ ($t \geq 0$). Each **tontine** holder receives:

$$b_{OT}(t) := \begin{cases} \mathbf{1}_{\{\zeta > t\}} \frac{nd(t)}{N_t} & \text{if } N_t > 0, \\ 0, & \text{else} \end{cases} .$$

where N_t is the number of surviving policyholders at time t.

Tontine: example

1st year

$d(1) = 800$, $N_1 = 8$

$nd(1)/N_1 = 800$

2nd year

$d(2) = 800$, $N_2 = 7$

$nd(2)/N_2 \approx 914$

3rd year

$d(3) = 720$, $N_3 = 7$

$nd(3)/N_3 \approx 823$

Actuarially fair pricing I: a simple mortality model

...is used by the insurer to price the retirement products (c.f. Lin and Cox (2005)):

(1) Get survival probabilities ${}_tp_x = P(\zeta > t)$, $t \geq 0$ from past data (best-estimate survival probability)

(2) Draw a mortality shock ϵ, true survival probabilities are $({}_tp_x)^{1-\epsilon}$. (**systematic mortality risk**)

- ϵ is a r.v. with density $f_\epsilon(\varphi)$ and support on $(-\infty, 1)$

(3) Conditional on $\epsilon = \varphi$, the number of survivors is binomially distributed i.e. $N_\varphi(t) \sim \text{Bin}(n, {}_tp_x^{1-\varphi})$, (**unsystematic mortality risk**)

Actuarially fair pricing II

- Premium of the annuity:

$$P_0 = P_0^A = \mathbb{E}\left[\int_0^\infty e^{-rt}\mathbf{1}_{\{\zeta>t\}}c(t)dt\right]$$
$$= \int\limits_0^\infty e^{-rt}c(t)\int\limits_{-\infty}^1 {}_tp_x^{1-\varphi} f_\epsilon(\varphi)\,\mathrm{d}\varphi\,\mathrm{d}t$$

- Premium of the tontine:

$$P_0 = P_0^{OT} = \int_0^\infty e^{-rt}\int_{-\infty}^1 \left(1-\left(1-{}_tp_x^{1-\varphi}\right)^n\right) f_\epsilon(\varphi)\,d\varphi\,d(t)\,dt$$

Policyholder's utility

- Policyholder follows constant relative risk aversion (**CRRA**) utility

$$u(x) = \frac{x^{1-\gamma}}{1-\gamma},$$

with risk aversion $\gamma \in [0, \infty) \setminus \{1\}$.

- Assumption: the policyholder **without bequest motives** would choose $c(t)$ **or** $d(t)$ to maximize

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t}\, \mathbf{1}_{\{\zeta > t\}}\, u\big(\chi(t)\big)\, \mathrm{d}t\right],$$

with $\chi(t) = c(t)$ (annuity) or $\chi(t) = nd(t)/N(t)$ (tontine), subjective discount factor ρ, given an **actuarially fair** premium.

Theorem (Optimal payout function: Annuity and Tontine)

(a) *For an annuity product, we obtain*

$$c^*(t) = e^{\frac{1}{\gamma}(r-\rho)t} \cdot P_0 \cdot \left(\int_0^\infty e^{(\frac{r-\rho}{\gamma} - r)t} {}_t\bar{p}_x \, \mathrm{d}t \right)^{-1},$$

where ${}_t\bar{p}_x := \mathbb{E}[{}_tp_x{}^{1-\epsilon}]$*. (e.g. Yaari (1965))*

(b) *For a tontine product, we obtain*

$$d^*(t) = \frac{e^{\frac{1}{\gamma}(r-\rho)t} \cdot P_0}{\left(\lambda^*\right)^{\frac{1}{\gamma}}} \cdot \frac{\kappa_{n,\gamma,\epsilon}({}_tp_x)}{\mathbb{E}\left[1 - (1 - {}_tp_x{}^{1-\epsilon})^n\right]^{\frac{1}{\gamma}}},$$

with suitable $\kappa_{n,\gamma,\epsilon}$ *and* λ^**. (e.g. Chen, Hieber and Klein (2019))*

Sketch of a proof, (a), well-known (e.g. Yaari (1965)):

- Budget constraint:

$$P_0 \overset{!}{=} \int_0^\infty e^{-rt} c(t) \int_{-\infty}^{1} {}_tp_x^{1-\varphi} f_\epsilon(\varphi)\, \mathrm{d}\varphi\, \mathrm{d}t = \int_0^\infty e^{-rt}\, {}_tp_x\, m_\epsilon(-\log {}_tp_x)\, c(t)\, dt.$$

- Write down the Lagrangian function for $\lambda > 0$:

$$L(c,\lambda) := \int_0^\infty e^{-\rho t} \int_{-\infty}^{1} {}_tp_x^{1-\varphi} f_\epsilon(\varphi)\, \mathrm{d}\varphi \cdot u(c(t))\, \mathrm{d}t + \lambda \left(P_0 - \int_0^\infty e^{-rt} c(t) \int_{-\infty}^{1} {}_tp_x^{1-\varphi} f_\epsilon(\varphi)\, \mathrm{d}\varphi\, \mathrm{d}t \right)$$

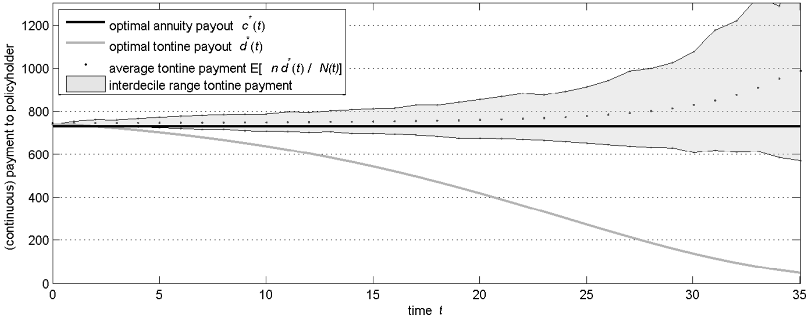
- First-order condition: $c^*(t) = \left(\lambda \cdot e^{(\rho - r)t}\right)^{-\frac{1}{\gamma}}$.
- From budget constraint:
$\lambda^* = P_0^{-\gamma} \big(\int_0^\infty e^{(\frac{r-\rho}{\gamma} - r)t}\, {}_tp_x \cdot m_\epsilon(-\log {}_tp_x)\, \mathrm{d}t \big)^\gamma$.

Numerical example: parameter choices

net premium $P_0 = 10\,000$	pool size $n = 100$	risk aversion $\gamma = 10$
risk-free rate $r = 4\%$	subjective discount rate $\rho = 4\%$	cost of capital rate $CoC = 6\%$
initial age $x = 65$	Gompertz-law $m = 88.721,\ b = 10$	mortality shock $\epsilon \sim \mathcal{N}_{(-\infty,1)}(\mu, \sigma^2)$ $\mu = -0.0035,$ $\sigma = 0.0814$

$$_tp_x = e^{e^{\frac{x-m}{b}}\left(1-e^{\frac{t}{b}}\right)}$$

Numerical example



Optimal payouts $c^*(t)$ and $d^*(t)$. Distribution $n \cdot d^*(t)/N(t)$.

Risk capital charge: Risk margin according to Solvency II

product		risk capital charge F_0
tontine	$n = 10$	101.32
	$n = 100$	10.89
	$n = 1\,000$	1.33
annuity		483.51

Risk capital charges

$$F_0 = \text{CoC} \cdot \sum_{t=0}^{\infty} e^{-r(t+1)} \cdot SCR(t)$$

for different pool sizes n.

Drawbacks Tontine/Annuity

Both products have advantages / disadvantages, mainly:

- For an annuity, the insurance company takes the **aggregate mortality risk**. This increases the cost of risk capital provision (a tontine does not).
- A tontine leads to a **volatile payoff at old ages** (an annuity does not).

Chen, Hieber and Klein (2019) suggest one way of combining both products (**Ton~~tine/Ann~~uity = Tonuity**)?

Tonuity: Payoff

Idea: Switch between tontine and annuity payoff:

$$b_{[\tau]}(t) := \mathbb{1}_{\{0 \leq t < \min\{\tau,\zeta\}\}} \frac{n d_{[\tau]}(t)}{N(t)} + \mathbb{1}_{\{\tau \leq t < \zeta\}} c_{[\tau]}(t),$$

with **switching time** τ:

- A tonuity with switching time $\tau = 0$ is an **annuity**
- A tonuity with switching time $\tau \to \infty$ is a **tontine**
- **Volatile tontine payoff at old ages** is **replaced by** a secure **annuity** payoff

Conclusion of Chen, Hieber and Klein (2019)

- **Tonuities combine beneficial features** of annuities, tontines:
  - **Reduced solvency capital** provision (tontine).
  - **Secure income at old ages** (annuity).
- Each individual can choose an optimal tonuity product (with a corresponding switching time τ), depending on **longevity risk aversion**, **pool size**, **cost-of-capital rate**.

Moving to the second paper

Chen & Rach & Sehner (2019): In addition to tonuities, further innovative products are introduced/analyzed:

- Antine
- Portfolio of Annuities and Tontines

Antine: Payoff

Alternative Idea: Switch between annuity and tontine payoff:

$$b_{[\sigma]}(t) = \mathbb{1}\{0 \leq t < \min\{\sigma, \zeta\}\} c_{[\sigma]}(t) + \mathbb{1}\{\sigma \leq t < \zeta\} \frac{n}{N(t)} d_{[\sigma]}(t)$$

with **switching time** σ:

- An antine with switching time $\sigma = 0$ is a **tontine**
- An antine with switching time $\sigma \to \infty$ is an **annuity**

Portfolio

- The policyholder can now combine annuities and tontines by simultaneously investing in both products to a certain extent.
- The resulting payoff of this portfolio is given by

$$b_{AT}(t) = b_A(t) + b_{OT}(t).$$

Expected discounted lifetime utility

- A policyholder with an initial wealth v follows constant relative risk aversion (**CRRA**) utility $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$ with risk aversion $\gamma \in [0, \infty) \setminus \{1\}$.
- Assumption: the policyholder **without bequest motives** would choose $b(t)$ to maximize

$$U\big(\{b(t)\}_{t\geq 0}\big) := \mathbb{E}\left[\int_0^\infty e^{-\rho t} u\left(b(t)\right) \mathbb{1}_{\{\zeta_\epsilon > t\}} \mathrm{d}t\right],$$

under a budget constraint, where $b(t)$ is the contract payoff from the various retirement products.

Budget constraint: Expected value principle

- Premium of the annuity ($m_\epsilon(s) = \mathbb{E}\left[e^{s\epsilon}\right]$)

$$P_0^A = \mathbb{E}\left[\int_0^\infty e^{-rt} b_A(t) dt\right] = \int_0^\infty e^{-rt}\,{}_tp_x\, m_\epsilon(-\log {}_tp_x)\, c(t)\, dt$$

$$\widetilde{P}_0^A = (1 + C_A) P_0^A$$

- Premium of the tontine:

$$P_0^{OT} = \int_0^\infty e^{-rt} \int_{-\infty}^1 \left(1 - \left(1 - {}_tp_x^{1-\varphi}\right)^n\right) f_\epsilon(\varphi)\, d\varphi\, d(t)\, dt$$

$$\widetilde{P}_0^{OT} = (1 + C_{OT}) P_0^{OT}$$

- Note: $C_A > C_{OT} \geq 0$

Budget constraint: Expected value principle

- Premium of the tonuity:

$$
\begin{aligned}
P_0^{[\tau]} &= \mathbb{E}\left[\int_0^\infty e^{-rt} b_{[\tau]}(t)dt\right] \\
&= \int_0^\tau e^{-rt} \int_{-\infty}^1 \left(1-\left(1-{}_tp_x^{1-\varphi}\right)^n\right) f_\epsilon(\varphi)d\varphi\, d_{[\tau]}(t)dt \\
&\quad + \int_\tau^\infty e^{-rt}{}_tp_x m_\epsilon(-\log {}_tp_x) c_{[\tau]}(t)dt \\
&=: P_0^{OT,\tau} + P_0^{A,\tau} \\
\widetilde{P}_0^{[\tau]} &= (1+C_{OT})P_0^{OT,\tau} + (1+C_A)P_0^{A,\tau}
\end{aligned}
$$

Budget constraint: Expected value principle

- Premium of the antine:

$$P_0^{[\sigma]} = \mathbb{E}\left[\int_0^{\infty} e^{-rt} b_{[\sigma]}(t)dt\right]$$

$$= \int_0^{\sigma} e^{-rt} {}_tp_x m_\epsilon(-\log {}_tp_x) c_{[\sigma]}(t)dt$$

$$+ \int_{\sigma}^{\infty} e^{-rt} \int_{-\infty}^{1} \left(1 - \left(1 - {}_tp_x^{1-\varphi}\right)^n\right) f_\epsilon(\varphi)d\varphi d_{[\sigma]}(t)dt$$

$$=: P_0^{A,\sigma} + P_0^{OT,\sigma}$$

$$\widetilde{P}_0^{[\sigma]} = (1 + C_A)P_0^{A,\sigma} + (1 + C_{OT})P_0^{OT,\sigma}$$

Optimization Problems

- Tonuity:

$$\max_{c_{[\tau]}(t), d_{[\tau]}(t)} \mathbb{E}\left[\int_0^\infty e^{-\rho t} \left(\mathbb{1}_{\{0 \le t < \min\{\tau, \zeta_\epsilon\}\}} u\left(\frac{n}{N_\epsilon(t)} d_{[\tau]}(t) \right) + \mathbb{1}_{\{\tau \le t < \zeta_\epsilon\}} u(c_{[\tau]}(t)) \right) dt \right]$$

$$\text{subject to} \quad v = \widetilde{P}_0^{[\tau]} = (1 + C_A)\, P_0^{A,\tau} + (1 + C_{OT})\, P_0^{OT,\tau}.$$

- Closed-form solution to $c^*_{[\tau]}(t)$, $d^*_{[\tau]}(t)$ available

Proof sketch

- Write down the Lagrangian function for the optimization problem.

$$\mathcal{L} = \int_0^{\tau} e^{-\rho t} u(d_{[\tau]}(t))\mathbb{E}\left[\mathbb{1}\{\zeta_\epsilon > t\}\left(\frac{n}{N_\epsilon(t)}\right)^{1-\gamma}\right] dt + \int_{\tau}^{\infty} e^{-\rho t}\mathbb{E}\left[\mathbb{1}\{\zeta_\epsilon > t\}\right] u(c_{[\tau]}(t))dt$$
$$+ \lambda_{[\tau]}\left(v - (1 + C_{OT})\int_0^{\tau} e^{-rt}\int_{-\infty}^{1}\left(1 - \left(1 - {}_tp_x^{1-\varphi}\right)^n\right) f_\epsilon(\varphi)d\varphi d_{[\tau]}(t)dt\right.$$
$$\left. - (1 + C_A)\int_{\tau}^{\infty} e^{-rt}{}_tp_x m_\epsilon(-\log {}_tp_x)c_{[\tau]}(t)dt\right)$$

- Taking partial derivatives with respect to $d = d_{[\tau]}(t)$ and $c = c_{[\tau]}(t)$, we obtain:

$$\frac{\partial \mathcal{L}}{\partial d} = e^{-\rho t}\kappa_{n,\gamma,\epsilon}({}_tp_x)d^{-\gamma} - \lambda_{[\tau]}\left(1 + C_{OT}\right)e^{-rt}\int_{-\infty}^{1}\left(1 - \left(1 - {}_tp_x^{1-\varphi}\right)^n\right) f_\epsilon(\varphi)d\varphi \stackrel{!}{=} 0$$
$$\frac{\partial \mathcal{L}}{\partial c} = e^{-\rho t}{}_tp_x m_\epsilon(-\log {}_tp_x)c^{-\gamma} - \lambda_{[\tau]}\left(1 + C_A\right)e^{-rt}{}_tp_x m_\epsilon(-\log {}_tp_x) \stackrel{!}{=} 0$$

- Solving for the optimal $d_{[\tau]}(t)$ and $c = c_{[\tau]}(t)$ and substituting them back to the budget constraint, we obtain $\lambda^*_{[\tau]}$.

Optimization Problems

- Antine:

$$\max_{c_{[\sigma]}(t), d_{[\sigma]}(t)} \mathbb{E}\Bigg[\int_0^\infty e^{-\rho t}\Bigg(\mathbb{1}_{\{\sigma \le t < \zeta_\epsilon\}} u(c_{[\sigma]}(t)) + \mathbb{1}_{\{0 \le t < \min\{\sigma, \zeta_\epsilon\}\}} u\left(\frac{n}{N_\epsilon(t)} d_{[\sigma]}(t)\right)\Bigg) dt\Bigg]$$

$$\text{subject to} \quad {\color{red}v} = \widetilde{P}_0^{[\sigma]} = (1 + C_A)\, P_0^{A,\sigma} + (1 + C_{OT})\, P_0^{OT,\sigma}.$$

- Closed-form solution to $c^*_{[\sigma]}(t), d^*_{[\sigma]}(t)$ available

Optimization Problems

- Portfolio:

$$\max_{c(t),d(t)} \mathbb{E}\left[\int_0^\infty e^{-\rho t}\mathbb{1}_{\{\zeta_\epsilon>t\}} u\left(\frac{n}{N_\epsilon(t)}d(t)+c(t)\right)dt\right]$$
$$\text{subject to} \quad v=\widetilde{P}_0^{AT}:=\widetilde{P}_0^{A}+\widetilde{P}_0^{OT}$$

- v is the initial wealth
- The optimal fraction of wealth invested in the annuity and tontine are determined by the choice of $c(t)$ and $d(t)$
- No closed-form solution available

Main result

Proposition 1

We denote by U_{AT}, $U_{[\tau]}$ and $U_{[\sigma]}$ the optimal levels of expected utility resulting from the optimal portfolio, tonuity and antine, respectively. Then it holds

$$U_{AT} \geq U_{[\tau]}, \quad U_{AT} \geq U_{[\sigma]}$$

for all switching times τ, σ and risk loadings C_A and C_{OT}.

Proof: taking tonuity as an example

- Consider a tonuity with a switching time $\tau \in [0, \infty]$ and payoffs $d_{[\tau]}(t)$ for $0 \le t \le \tau$ and $c_{[\tau]}(t)$ for $t > \tau$ which satisfy the budget constraint $v = (1 + C_A)P_0^{A,\tau} + (1 + C_{OT})P_0^{OT,\tau}$ for fixed v, C_A and C_{OT}
- We can define

$$c(t) := \begin{cases} 0 & \text{for } t \in [0, \tau] \\ c_{[\tau]}(t) & \text{for } t > \tau \end{cases}, \quad d(t) := \begin{cases} d_{[\tau]}(t) & \text{for } t \in [0, \tau] \\ 0 & \text{for } t > \tau \end{cases}$$

as the payoffs of the portfolio.

$\rightarrow$ Tonuity describes one possible choice for the portfolio.

Base case parameters

Initial wealth $v = 10000$	Pool size $n = 50$	Risk aversion $\gamma = 6$
Risk-free rate $r = 0.02$	Subjective discount rate $\rho = 0.02$	Risk loadings $C_A = 4\%$, $C_{OT} = 1\%$
Initial age $x = 65$	Gompertz-law $m = 80.5$, $\beta = 10$	Longevity shock $\epsilon \sim \mathcal{N}_{(-\infty,1]}(-0.0035, 0.0814)$

Tabelle: Parameters of the standard example

Certainty Equivalents

- CE is determined by

$$U\big(\{\mathrm{CE}\}_{t\geq 0}\big) = U\big(\{b(t)\}_{t\geq 0}\big)$$

  or equivalently,

$$\mathrm{CE} = \left((1-\gamma)\left(\int_0^{\infty} e^{-\rho t}{}_t p_x m_{\epsilon}(-\ln {}_t p_x)\mathrm{d}t\right)^{-1} \cdot U\big(\{b(t)\}_{t\geq 0}\big)\right)^{\frac{1}{1-\gamma}}$$

- $U\left(\{b(t)\}_{t\geq 0}\right)$ is the expected discounted lifetime utility of the individual

Comparison of Annuity and Tontine

Pool size, C_{OT}	Tonuity	Antine	Portfolio
n=30, $C_{OT} = 0.015$	796.23, $\tau = 7$	790.37, $\sigma = 39$	797.35, $\widetilde{P}_0^A / v = 0.46$
n=50, $C_{OT} = 0.01$	800.63, $\tau = 10$	790.37, $\sigma = 39$	801.75, $\widetilde{P}_0^A / v = 0.32$
n=100, $C_{OT} = 0.005$	806.85, $\tau = 14$	792.90, $\sigma = 0$	807.73, $\widetilde{P}_0^A / v = 0.19$
n=500, $C_{OT} = 0.001$	814.59, $\tau = 19$	809.34, $\sigma = 0$	815.04, $\widetilde{P}_0^A / v = 0.08$

Tabelle: Certainty equivalents of the tonuity, antine and portfolio of annuities and tontines along with the optimal stopping times and the fraction of wealth invested in the annuity, respectively, for different pool sizes n and different tontine loadings C_{OT}.

Comparison of Annuity and Tontine

	Tonuity	Antine	Portfolio
$\gamma = 0.8$	809.56, $\tau = 20$	808.04, $\sigma = 0$	809.72, $\widetilde{P}_0^A/v = 0.06$
$\gamma = 2$	805.88, $\tau = 16$	799.60, $\sigma = 0$	806.42, $\widetilde{P}_0^A/v = 0.14$
$\gamma = 4$	802.61, $\tau = 12$	790.37, $\sigma = 37$	803.53, $\widetilde{P}_0^A/v = 0.24$
$\gamma = 6$	800.63, $\tau = 10$	790.37, $\sigma = 39$	801.75, $\widetilde{P}_0^A/v = 0.32$
$\gamma = 8$	799.27, $\tau = 9$	790.37, $\sigma = 40$	800.49, $\widetilde{P}_0^A/v = 0.37$
$\gamma = 10$	798.27, $\tau = 8$	790.37, $\sigma = 41$	799.57, $\widetilde{P}_0^A/v = 0.42$

Tabelle: Certainty equivalents of the tonuity, antine and portfolio of annuities and tontines along with the optimal stopping times and the fraction of wealth invested in the annuity, respectively, for different risk aversions γ.

Main results

- Among the three products: portfolio of annuities and tontines, tonuity, and antine, the antine performs worst.
- The portfolio of annuities and tontines outperforms the tonuity and antine, as the payoff of any of these two can be replicated by a portfolio of the classical products.
- The optimal portfolio never consists solely of annuities or tontines.

  A combination of both products appears to be the future.

Literature

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Thank you very much for your attention!