ulm university universität ulm

Lévy driven linear SPDEs and CARMA generalized processes

David Berger | 10th October 2019 | Institute of Mathematical Finance

CARMA processes

A stationary CARMA(p,q) process is a stationary stochastic process Y of the form

$$Y_t = b' X_t, \text{ where}$$
$$X_t = \int_{-\infty}^{t} e^{A(t-s)} dL(s),$$

where A is a matrix, b is a vector and L is the driving Lévy process. The process Y is the solution of a (formal) SDE

$$p(D)Y = q(D)L,$$

where p and q are real polynomials and L is the driving Lévy process. See Brockwell for the definition and many applications.

CARMA random fields

There exists at least two definitions of a CARMA random field. The first definition was given by Brockwell and Matsuda. The CARMA random field is given by

$$X(t) := \int_{\mathbb{R}^d} \sum_{r=1}^{p} \frac{b(\lambda_r)}{a'(\lambda_r)} e^{\lambda_r \|t-u\|} dL(u),$$

where dL denotes the integration over a Lévy bases, a and b are polynomials and $(\lambda_i)_{1 \leq 1 \leq p} \in \mathbb{C}^p$.

CARMA random fields

Pham follows another way and defines a CARMA random field Y as a mild solution of the system of SPDEs given by

$$Y(t) = b'X(t),\ t \in \mathbb{R}^d,$$
$$(I_p\partial_d - A_d)\cdots(I_p\partial_1 - A_1)X(t) = c\dot{L}(t),\ t \in \mathbb{R}^d,$$

where $\dot{L}$ is a Lévy basis, $A_1, \ldots, A_d \in \mathbb{R}^{p\times p}$ are matrices and I_p is the identity matrix.

CARMA random fields

Pham speaks of causal CARMA random fields, as the solution of the system depends only on the past in the sense that the solution at a point x depends solely on the behavior of $\dot{L}$ on $(-\infty, x_1] \times \cdots \times (-\infty, x_d]$. So we can see directly that there is a big difference between these two definitions.

Goal

Our Goal of this talk is to give a suitable random field solution (and sufficient conditions for the existence) of the stochastic partial differential equation

$$p(D)s = q(D)\dot{L}, \tag{2.1}$$

where $p(D)$ and $q(D)$ are linear partial differential operators and $\dot{L}$ is a so called Lévy white noise.

Ansatz

Let φ be a sufficiently nice function and assume that s is a random field regular enough. Then we multiply $p(D)s$ with φ and by partial integration we obtain

$$\int_{\mathbb{R}^d} p(D)s(x)\varphi(x)\,\lambda^d(dx) = \int_{\mathbb{R}^d} s(x)p(-D)\varphi(x)\,\lambda^d(dx) =: s(p(-D)\varphi).$$

At the end we obtain

$$\langle s, p(-D)\varphi\rangle := s(p(-D)\varphi) = \dot{L}(q(-D)\varphi) =: \langle \dot{L}, q(-D)\varphi\rangle.$$

Definition

We denote by $\mathcal{D}(\mathbb{R}^d)$ the space of infinitely differentiable functions with compact support and equip it with its usal topology, i.e. a sequence $(\varphi_n)_{n\in\mathbb{N}} \subset \mathcal{D}(\mathbb{R}^d)$ converges to $\varphi \in \mathcal{D}(\mathbb{R}^d)$ such that there exists a compact set K with $\mathrm{supp}\ \varphi_n \subset K$ and

$$\sup_{x\in\mathbb{R}^d} \|D^\alpha(\varphi_n(x) - \varphi(x))\| \to 0$$

for $n \to \infty$ for every $\alpha \in \mathbb{N}_0^d$.

Definition (see Fageot, Definition 2.1)

A generalized random process is a linear and continuous function $s : \mathcal{D}(\mathbb{R}^d) \to L^0(\Omega)$. The linearity means that, for every $\varphi_1, \varphi_2 \in \mathcal{D}(\mathbb{R}^d)$ and $\gamma \in \mathbb{R}$,

$$s(\varphi_1 + \gamma\varphi_2) = s(\varphi_1) + \gamma s(\varphi_2) \text{ almost surely.}$$

The continuity means that if $\varphi_n \to \varphi$ in $\mathcal{D}(\mathbb{R}^d)$, then $s(\varphi_n) \to s(\varphi)$ in $L^0(\Omega)$.

Definition

A Lévy white noise $\dot{L}$ is a generalized random process, such that

$$\mathbb{E}e^{iz\dot{L}(\varphi)} = \exp\left(\int_{\mathbb{R}^d} \psi(z\varphi(x))\lambda^d(dx)\right), \quad z \in \mathbb{R},$$

for every $\varphi \in \mathcal{D}(\mathbb{R}^d)$, where $\psi : \mathbb{R} \to \mathbb{C}$ is given by

$$\psi(z) = i\gamma z - \frac{1}{2}az^2 + \int_{\mathbb{R}} (e^{ixz} - 1 - ixz\mathbf{1}_{|x|\leq 1})\nu(dx)$$

with $a \in \mathbb{R}^+$, $\gamma \in \mathbb{R}$ and ν is a Lévy-measure.

Definition

Let $\dot{L}$ be a Lévy white noise, $n, m \in \mathbb{N}_0$ and $p, q : \mathbb{R}^d \to \mathbb{R}$ be polynomials of the form

$$p(x) = \sum_{|\alpha| \leq n} p_\alpha x^\alpha \text{ and } q(x) = \sum_{|\alpha| \leq m} q_\alpha x^\alpha.$$

A generalized process $s : \mathcal{D}(\mathbb{R}^d) \to L^0(\Omega)$ is called a CARMA(p, q) generalized process if s solves (2.1) which means that

$$\langle s, p(-D)\varphi \rangle = \langle \dot{L}, q(-D)\varphi \rangle \text{ a.s. for every } \varphi \in \mathcal{D}(\mathbb{R}^d).$$

1-dimensional case

In the 1-dimensional case there exists a similar model from Brockwell and Hannig, which was defined for Gaussian white noise. We will see later that our model contains their results.

Assumption

The rational function $q(i\cdot)/p(i\cdot)$ has a holomorphic extension in a strip $\{z \in \mathbb{C}^d : \|\Im z\| < \varepsilon\}$ for some $\varepsilon > 0$.

Main theorem

Theorem (B., 2019)

Let p, q be real multivariate polynomials such that the assumption above holds true. Furthermore, let $\dot{L}$ be a Lévy white noise with characteristic triplet (a, γ, ν) with

$$\int_{\mathbb{R}} \mathbf{1}_{|r|>1} \log(|r|)^d \nu(dr) < \infty.$$

Then there exists a stationary CARMA(p, q) generalized process.

Corollary (B., 2019)

Let $d = 1$ and p and q be two real polynomials, such that p/q has no roots on the imaginary axis.
Then there exists a stationary generalized solution s of the equation

$$p(\frac{d}{dx})s = q(\frac{d}{dx})\dot{L}$$

for every Lévy white noise $\dot{L}$ with characteristic triplet (a, γ, ν) such that $\int_{|r|>1} \log(|r|)\nu(dr) < \infty$.
Therefore the results of Brockwell and Hannig are included.

Definition (see Walsh)

Let $p(D)$ and $q(D)$ be partial differential operators and let $G : \mathbb{R}^d \to \mathbb{R}$ be a locally integrable fundamental solution of the equation $p(D)u = q(D)\delta_0$.

We say that $(X_t)_{t\in\mathbb{R}^d}$ defined by

$$X_t = \int_{\mathbb{R}^d} G(t-s)\,dL(s),$$

where dL denotes a Lévy basis, is the mild solution of the equation $p(D)X = q(D)dL$, provided that the integral exists.

Theorem (B., 2019)

Let dL be a Lévy basis in $\mathbb{R}^d$ *with characteristic triplet* (a, γ, ν) *such that* $\int_{\mathbb{R}} \mathbf{1}_{|r|>1} \log(|r|)^d \nu(dr) < \infty$*. Assume furthermore that there exists* $\varepsilon > 0$ *such that*

$$\sup_{\eta \in B_\varepsilon(0)} \left\| \frac{q(i \cdot + \eta)}{p(i \cdot + \eta)} \right\|_{L^2} < \infty. \tag{5.1}$$

Then there exists a mild solution of the equation

$$p(D)X = q(D)\, dL. \tag{5.2}$$

Proposition (B., 2019)

Let dL be a Lévy basis with existing first moment and p and q be as above.
Then the mild solution X of (5.2) gives rise to a generalized solution X of the SPDE $p(D)X = q(D)\dot{L}$ via

$$\langle X, \varphi \rangle := \int_{\mathbb{R}^d} X_s \varphi(s) \lambda^d(ds),\ \varphi \in \mathcal{D}(\mathbb{R}^d).$$

CARMA random fields in the sense of Brockwell and Matsuda

Let $0 \leq q < p$ and assume that $\lambda_i \neq \kappa_j$ are distinct zeroes of two real polynomials a^* and b^* for all $1 \leq i \leq p$ and $1 \leq j \leq q$. Define the functions

$$a(z) = \prod_{i=1}^{p}(z^2 - \lambda_i^2) \text{ and } b(z) = \prod_{i=1}^{q}(z^2 - \kappa_i^2).$$

Let L be a Lévy basis in $\mathbb{R}^d$ with finite second moment. Then the isotropic CARMA(p,q) field driven by L is given by

$$X_t = \int_{\mathbb{R}^d} \sum_{i=1}^{p} \frac{b(\lambda_i)}{a'(\lambda_i)} e^{\lambda_i ||t-u||} \, dL(u) \tag{5.3}$$

for every $t \in \mathbb{R}^d$. Here, a' denotes the derivative of the polynomial a

Proposition

Let $X = (X_t)_{t \in \mathbb{R}^d}$ be defined by (5.3). Then X is the mild solution of the (fractional) SPDE

$$\prod_{i=1}^{p} a'(\lambda_i)(-\Delta + \lambda_i^2)^{\frac{d+1}{2}} X = c_d \sum_{i=1}^{p} 2\lambda_i b(\lambda_i) \prod_{j=1, j\neq i}^{p} a'(\lambda_j)(-\Delta + \lambda_j^2)^{\frac{d+1}{2}} \dot{L}$$

for some constant c_d depending on the dimension d.

Thank you very much for your attention!

References

- D. Berger, *Lévy driven CARMA generalized processes and stochastic partial differential equations*, arXiv 2019
- D. Berger, *Lévy driven linear and semilinear stochastic partial differential equations*, arXiv 2019
- P.J. Brockwell,*Recent results in the theory and applications of CARMA processes*, Ann Inst Stat Math (2014) 66, 647-685
- P.J. Brockwell and Y. Matsuda, *Continuous auto-regressive moving average random fields on* $\mathbb{R}^n$,J. R. Stat. Soc. B (2016), 79, 833-857.
- J. Fageot and T. Humeau, *Unified view on Lévy white noise: general integrability conditions and applications to linear SPDE*, (2018) arXiv:1708.02500

V.S. Pham, *Lévy driven causcal CARMA random fields*, (2018) arXiv:1805.08807

J.B. Walsh, *An introduction to stochastic partial differential equations*, École d'Été de Probabilités de Saint Flour (1986) XIV-1984, pages 265-439. Springer.