On quasi-infinitely div. distributions

→ joint work with K. Sato and L. Pan
→ and of D. Berger

Def A distr. μ on $\mathbb{R}$ is infinitely divisible (i.d.) :⟺ $\forall n \in \mathbb{N}$ $\exists$ distr. μ_n on $\mathbb{R}$: $\mu = \mu_n^{*n}$

Lévy-Khintchine formula

μ i.d. ⟺ $\exists\, a \geq 0$, $\gamma \in \mathbb{R}$, Lévy measure ν s.t.

$$\hat{\mu}(z) = \int e^{ixz}\mu(dx) = \exp\left\{ i\gamma z - a z^2/2 + \int_{\mathbb{R}} (e^{ixz} - 1 - ixz\, 1_{|x|\leq 1})\, \nu(dx) \right\}$$

Def... A distr. μ on $\mathbb{R}$ is quasi-infinitely divisible (q.i.d.) :⟺

$\exists\, a \in \mathbb{R}$, $\gamma \in \mathbb{R}$ two Lévy measures ν_1, ν_2:

$$\hat{\mu}(z) = \exp\Big\{ \underbrace{i\gamma z - a z^2/2 + \int_{\mathbb{R}} (e^{ixz} - 1 - ixz\, 1_{|x|\leq 1})\, (\nu_1(dx) - \nu_2(dx))}_{\psi(z)} \Big\}$$

$\nu := \nu_1 - \nu_2$ quasi-Lévy measure ("signed Lévy measure")

One can show: (a, γ, ν) is unique.

Observe a) μ q.i.d. ⟺ $\hat{\mu}(z) = \dfrac{\hat{\mu}_1(z)}{\hat{\mu}_2(z)}$ for some i.d. distr. μ_1, μ_2

⟺ $\exists\, \mu_1, \mu_2$ i.d.: $\mu_2 * \mu = \mu_1$

b) $\hat{\mu}(z) \neq 0 \quad \forall z$

c) Since $\dfrac{\psi(z)}{z^2} \to -a/2$ necessarily $a \geq 0$.

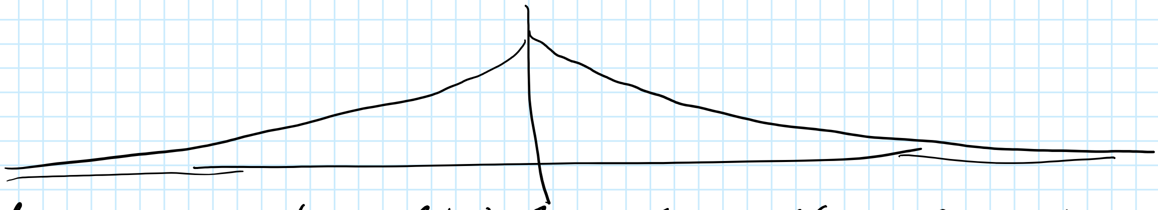
d) Not all triplets (a, ν, γ) give rise to q.i.d. distributions μ (even if $a \geq 0$)
For otherwise $\frac{1}{n}(a, \nu, \gamma)$ triplet of some q.i.d. distr. μ_n. Check $\mu_n^{*n} = \mu \Rightarrow \mu$ i.d.

$\Rightarrow \quad \nu \geq 0$

Thm (R. Cuppens, 1969)
If μ has an atom of mass $> 1/2$, then μ is q.i.d. with $a=0$ and finite quasi Lévy measure.

Cor · Bernoulli-distr. $b(1,p)$ q.i.d. $\Leftrightarrow p \neq 1/2$
· $b(n,p)$ q.i.d $\Leftrightarrow p \neq 1/2 \Leftrightarrow \widehat{b(n,p)}(z) \neq 0$ $\forall z$

Ex Let
$$\varphi(z) = \begin{cases} \frac{1}{7}\exp(1-z^4), & |z| \geq 1 \\ \frac{2}{7}z^2 - \frac{8}{7}|z| + 1, & |z| < 1 \end{cases}$$



φ even, cont., $\varphi(0)=1$, $\lim_{z\to\infty}\varphi(z)=0$, φ is C^2 on $(0,\infty)$ and strictly convex there

$\overset{\text{Polya}}{\Longrightarrow}$ φ is the c.f. of some distr. μ
(with $\varphi(z) = \hat{\mu}(z) \neq 0$ $\forall z$)

But: μ is not q.i.d., since $\lim z^{-2} \underbrace{\log\varphi(z)}_{\psi(z)} = -\infty$

Thm Let μ be a distr. on $\mathbb{Z}$. Then
μ q.i.d. $\Longleftrightarrow$ $\hat{\mu}(z) \neq 0$ $\forall z \in \mathbb{R}$
In that case, $a=0$, ν is finite

Cor The set of all q.i.d. distributions is dense in the set of all distr. with respect to weak convergence.

Thm Convex combinations of normal distributions with mean 0 are q.i.d. I.e.
$\sum_{j=1}^{n} p_j N(0, a_j)$ $\quad (0 < p_j \leq 1, \ \Sigma p_j = 1)$ is
$a_1 < a_2 < \dots < a_n$

q.i.d. (but not i.d. if $n \geq 2$)

$$\sum_{i=1}^{n} p_i \, X_{a_i} = X_{a_1} * \left(\bar{p}_1 \delta_0 + \sum_{i=2}^{n} p_i \, X_{a_i - a_1} \right)$$