Generalized Dynamic Deviation Measures in Risk Analysis

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Motivation

- One traditional way of thinking about risk is in terms of the extent that random realisations deviate from the mean.
- In this context an axiomatic framework for static (generalized) *deviation measures* was introduced and developed in Rockafellar, Uryasev, Zabarankin (2006a).
- They were inspired by the axiomatic approach by Artzner et al. (1999) for coherent risk measures which can be seen as (generalized) *expectations*, see also Föllmer and Schied (2002).
- Various aspects of portfolio optimisation and financial decision making under general deviation measures have been explored in the literature, in particular regarding CAPM, asset betas, one- and two-fund theorems and equilibrium theory;

Motivation

- We present an axiomatic approach to deviation measures in *dynamic* continuous-time settings.
- We give characterisations of dynamic deviation measures in terms of additively m-stable sets and certain (backward) SDEs.
- We give applications for portfolio optimisation in continuous-time.

Conditional Deviation Measures (Rockafellar et al. (2006a)).

For any given $t \in [0, T]$, $D_t : L^2(\mathcal{F}_T) \to L^2_+(\mathcal{F}_t)$ is called an *$\mathcal{F}_t$-conditional deviation measure* if it is *normalised* ($D_t(0) = 0$) and the following properties are satisfied:

(D1) *Translation Invariance*: $D_t(X + m) = D_t(X)$ for any $m \in L^\infty(\mathcal{F}_t)$;

(D2) *Positive Homogeneity*: $D_t(\lambda X) = \lambda D_t(X)$ for any $X \in L^2(\mathcal{F}_T)$ and $\lambda \in L^\infty_+(\mathcal{F}_t)$;

(D3) *Subadditivity*: $D_t(X + Y) \le D_t(X) + D_t(Y)$ for any $X, Y \in L^2(\mathcal{F}_T)$;

(D4) *Positivity*: $D_t(X) \ge 0$ for any $X \in L^2(\mathcal{F}_T)$, and $D_t(X) = 0$ if and only if X is $\mathcal{F}_t$-measurable.

(D5) Lower Semi-Continuity: *If X^n converges to X in $L^2(\mathcal{F}_T)$ then $D_t(X) \le \liminf_n D_t(X^n)$.*

Conditional Risk Measures (Artzner et al. (1999), Föllmer and Schied (2002)).

Rockafellar et. al (2006a) were inspired by the axiomatic approach of Artzner et al. (1999) and Föllmer and Schied (2002) for coherent and convex risk measures (generalized expectations). ρ_t is a coherent risk measure if the following properties are satisfied:

(R1) *Translation Invariance*: $\rho_t(X+m) = \rho_t(X) + m$ for any $m \in L^\infty(\mathcal{F}_t)$;

(R2) *Positive Homogeneity*: $\rho_t(\lambda X) = \lambda \rho_t(X)$ for any $X \in L^2(\mathcal{F}_T)$ and $\lambda \in L^\infty_+(\mathcal{F}_t)$;

(R3) *Subadditivity*: $\rho_t(X+Y) \leq \rho_t(X) + \rho_t(Y)$ for any $X, Y \in L^2(\mathcal{F}_T)$;

(R4) Monotonicity: If $X \leq Y$ then $\rho(X) \geq \rho(Y)$.

Dynamic Deviation Measures

In a dynamic setting we need the following two axioms:

(D6) Consistency: *For all* $s, t \in [0, T]$ *with* $s \leq t$ *and* $X \in L^2(\mathcal{F}_T)$

$$D_s(X) = D_s(\mathbb{E}[X|\mathcal{F}_t]) + \mathbb{E}[D_t(X)|\mathcal{F}_s].$$

Dynamic Deviation Measures

Definition

A family $(D_t)_{t\in[0,T]}$ is called a *dynamic deviation measure* if D_t, $t \in [0, T]$, are $\mathcal{F}_t$-conditional deviation measures satisfying (D6).

Assume from now on that we are in a continuous-time setting with two independent stochastic processes:

(i) A standard d-dimensional Brownian motion $W = (W^1, \dots, W^d)^\intercal$.

(ii) A real-valued Poisson random measure p on $[0, T] \times \mathbb{R}^k \setminus \{0\}$. We denote by $N(ds, dx)$ the associated random (counting) measure with Lévy measure $\nu(dx)$ and set $\tilde{N}(ds, dx) := N(ds, dx) - \nu(dx)ds$.

By the martingale representation theorem for any $X \in L^2(\mathcal{F}_T)$ there exist unique predictable square integrable H^X and $\tilde{H}^X$ satisfying

$$X = \mathbb{E}[X] + \int_0^T H_s^X dW_s + \int_0^T \int_{\mathbb{R}^k \setminus \{0\}} \tilde{H}_s^X(x) \tilde{N}_p(ds, dx).$$

Henceforth, we will refer to $(H^X, \tilde{H}^X)$ as the representing pair.

g-Deviation Measures

Definition

We call a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{U}$-measurable function

$$\begin{array}{lllllllll} g: & [0,T] & \times & \Omega & \times & \mathbb{R}^d & \times & L^2(\nu(\mathrm{d}x)) & \to & \mathbb{R}_+ \\ & (t, & & \omega, & & h, & & \tilde{h}) & \longmapsto & g(t,\omega,h,\tilde{h}) \end{array}$$

a *driver function* if for $\mathrm{d}\mathbb{P} \times \mathrm{d}t$ a.e. $(\omega, t) \in \Omega \times [0,T]$:

(i) *(Positivity)* For any $(h, \tilde{h}) \in \mathbb{R}^d \times L^2(\nu(\mathrm{d}x))$ $g(t,h,\tilde{h}) \geq 0$ with equality if and only if $(h, \tilde{h}) = 0$.

(ii) *(Lower semi-continuity)* If $h^n \to h$, $\tilde{h}^n \to \tilde{h}$ $L^2(\nu(\mathrm{d}x))$-a.e. then $g(t,h,\tilde{h}) \leq \liminf_n g(t,h^n,\tilde{h}^n)$.

Definition

We call a driver function g *convex* if $g(t, h, \tilde{h})$ is convex in $(h, \tilde{h})$, $\mathrm{d}\mathbb{P} \times \mathrm{d}t$ a.e.; *positively homogeneous* if $g(t, h, \tilde{h})$ is positively homogeneous in $(h, \tilde{h})$, i.e., for $\lambda > 0$, $g(t, \lambda h, \lambda \tilde{h}) = \lambda g(t, h, \tilde{h})$, $\mathrm{d}\mathbb{P} \times \mathrm{d}t$ a.e. and of *linear growth* if for some $K > 0$ we have $\mathrm{d}\mathbb{P} \times \mathrm{d}t$ a.e.

$$|g(t, h, \tilde{h})|^2 \leq K^2 + K^2|h|^2 + K^2 \int_{\mathbb{R}^k \setminus \{0\}} \tilde{h}(x)^2 \nu(\mathrm{d}x).$$

g-Deviation Measures

Suppose that g is a convex and positively homogeneous driver function of linear growth. Let $(Y, Z, \tilde{Z})$ be the unique solution of the SDE with terminal condition 0 given in terms of the representing pair $(H^X, \tilde{H}^X)$ of X by

$$\mathrm{d}Y_t = -g(t, H_t^X, \tilde{H}_t^X)\mathrm{d}t + Z_t \mathrm{d}W_t + \int_{\mathbb{R}^k \setminus \{0\}} \tilde{Z}_t(x)\tilde{N}(\mathrm{d}t \times \mathrm{d}x),\ t \in [0, T),$$
$$Y_T = 0.$$

The *g-deviation measure* $D^g = (D_t^g)_{t\in[0,T]}$ is equal to the collection $D_t : L^2(\mathcal{F}_T) \to L_+^2(\mathcal{F}_t)$, $t \in [0, T]$, given by

$$D_t^g(X) = Y_t, \qquad X \in L^2(\mathcal{F}_T).$$

Remark: g-expectations

We remark that there is very rich literature on g-expectations. g-expecations can be seen as generalized expectations and as special examples of coherent risk measures satisfying the tower property. A g-expectation $\rho_t(X) := \mathcal{E}_t^g(X) := Y_t$ for a terminal payoff X is defined as a the first component Y of a unique triple $(Y, Z, \tilde{Z})$ satisfying

$$\begin{aligned} \mathrm{d}Y_t &= -g(t, Z_t, \tilde{Z}_t)\mathrm{d}t + Z_t \mathrm{d}W_t + \int_{\mathbb{R}^k \setminus \{0\}} \tilde{Z}_t(x)\tilde{N}(\mathrm{d}t \times \mathrm{d}x),\ t \in [0, T), \\ Y_T &= X. \end{aligned}$$

A Large Class of Dynamic Deviation Measures

Proposition

Let g be a convex and positively homogeneous driver function of linear growth.

(i) *D^g is a dynamic deviation measure. In particular, D^g satisfies (D6).*

(ii) *For given $X \in L^2(\mathcal{F}_T)$, we have*

$$D_t^g(X) = \mathbb{E}\left[\int_t^T g(s, H_s^X, \tilde{H}_s^X)\mathrm{d}s \middle| \mathcal{F}_t\right], \qquad t \in [0, T].$$

Properties of g-Deviation Measures

Proposition

Let g and $\tilde{g}$ be driver functions of linear growth.

(i) *D^g is convex if and only if g is convex.*

(ii) *D^g is positively homogeneous if and only if g is positively homogeneous.*

(iii) *$D^g \geq D^{\tilde{g}}$ if and only if $g \geq \tilde{g}$ $\mathrm{d}\mathbb{P} \times \mathrm{d}t$ a.e.*

Dynamic Deviation Measures and g Deviation Measures

Theorem

Let $D = (D_t)_{t\in[0,T]}$ be a collection of maps $D_t : L^2(\mathcal{F}_T) \to L^0(\mathcal{F}_t)$, $t \in [0, T]$. Then D is a dynamic deviation measure if and only if there exists a convex positively homogeneous driver function g such that $D = D^g$.

Additively m-stable Sets

Define $\mathcal{Q}_{\mathcal{F}_t} = \{\xi \in L^2(\mathcal{F}_T) | \mathbb{E}[\xi|\mathcal{F}_t] = 0\}$ and
$\mathcal{Q} = \mathcal{Q}_{\mathcal{F}_0} = \{\xi \in L^2(\mathcal{F}_T) | \mathbb{E}[\xi] = 0\}$.

Definition

We call a set $\mathcal{S} \subset \mathcal{Q}$ addititively m-stable if for $\xi^1, \xi^2 \in \mathcal{S}$ with associate martingales $\xi^i_t = \mathbb{E}\left[\xi^i|\mathcal{F}_t\right]$ for $i = 1, 2$ and for each time t taking values in $[0, T]$, the element L defined as $L_s = \xi^1_s$ for $s \leq t$ and $L_s = \xi^1_t + \xi^2_s - \xi^2_t$ for $s > t$ is a martingale that defines an element L_T in $\mathcal{S}$.

A Duality Result

Theorem

Let $D = (D_t)_{t\in[0,T]}$ be a collection of maps $D_t : L^2(\mathcal{F}_T) \to L^0(\mathcal{F}_t)$, $t \in [0, T]$, satisfying (D4). Then D is a dynamic deviation measure if and only if for some convex, bounded, closed set $\mathcal{S}^D$ that contains zero and is additively m-stable we have

$$D_t(X) \quad = \quad \operatorname{ess\,sup}_{\xi\in\mathcal{S}^D\cap\mathcal{Q}_{\mathcal{F}_t}} \mathbb{E}\left[\xi X|\mathcal{F}_t\right], \quad t \in [0, T].$$

Note that as dynamic coherent risk measures correspond to multiplicative m-stable sets (see for instance Riedel (2001), Chen and Epstein (2002) or Delbaen (2006)) dynamic deviation measure correspond to additively m-stable sets.

Characterizations of Additively m-Stable Sets

Theorem

A convex closed set $\mathcal{S}^D \subset \mathcal{Q}$ is additively m-stable if and only if there exists a set valued mapping $C_t^D(\omega)$ with convex and closed sets (with $C = (C_t)_t$ being $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{U}$-measurable), such that

$$\mathcal{S}^D = \left\{ \xi \in \mathcal{Q} \middle| (H_t^\xi, \tilde{H}_t^\xi) \in C_t^D \text{ for all } t \in [0,T] \right\}.$$

An analogous result has been shown by Delbaen (2006) for the representing pairs of stochastic logarithms of multiplicative m-stable sets.

The Financial Market

Suppose that we have a financial market that consists of a bank-account that pays interest at a fixed rate $r \geq 0$ and n risky stocks (with $1 \leq n \leq \min\{d, k\}$) with price processes $S^i = (S^i_t)_{t\in[0,T]}$, $i = 1, \dots, n$, satisfying the SDEs given by

$$\frac{\mathrm{d}S^i_t}{S^i_{t-}} = \mu_i \,\mathrm{d}t + \sum_{j=1}^{d} \sigma_{ij} \mathrm{d}W^j_t + \sum_{j=1}^{k} \rho_{ij} \,\mathrm{d}L^j_t, \; t \in (0, T].,$$

with $L^j_t = \int_{[0,t]\times\mathbb{R}^k\setminus\{0\}} x_j \tilde{N}(\mathrm{d}s \times \mathrm{d}x)$, $j = 1, \dots, k$, where x_j is the jth coordinate of $x \in \mathbb{R}^k$, is a vector of pure-jump $(\mathcal{F}_t)$-martingales.

The Performance Criterion

Denote by π_t^i the fraction of the wealth which is invested in asset i. We assume that short-selling and borrowing is not permitted. The wealth process corresponding to a trading strategy $\pi = (\pi_t^1, \ldots, \pi_t^n)_t \in \mathcal{B}$ is given by

$$\frac{\mathrm{d}X_t^\pi}{X_{t-}^\pi} = [r + (\mu - r\mathbf{1})^\intercal \pi_t]\,\mathrm{d}t + \pi_t^\intercal \Sigma\,\mathrm{d}W_t + \pi_t^\intercal R\,\mathrm{d}L_t, t \in (0, T],$$

with $\mathcal{B} = \left\{x \in \mathbb{R}^{1\times n} : \min_{i=1,\ldots,n} x_i \geq 0, \sum_{i=1}^n x_i \leq 1\right\}$, $\Sigma = (\sigma_{ij})_{ij}$, and $R = (\rho_{ij})_{ij}$.

To a given admissible allocation strategy $\pi \in \Pi$ we associate the dynamic performance criterion:

$$J_t^\pi := E[X_T^\pi | \mathcal{F}_t] - \gamma D_t(X_T^\pi), \quad t \in [0, T]. \tag{4.1}$$

Equilibrium Policies

Definition

(i) An allocation strategy $\pi^* \in \Pi$ is an *equilibrium policy* for the dynamic mean-deviation problem with objective J if

$$\liminf_{h \searrow 0} \frac{J_t^{\pi^*} - J_t^{\pi(h)}}{h} \geq 0$$

for any $t \in [0, T)$ and any policy $\pi(h) \in \Pi$ satisfying, for some $\pi \in \Pi$,

$$\pi(h)_s = \pi_s I_{[t,t+h)}(s) + \pi_s^* I_{[t+h,T]}(s), \quad s \in [t, T].$$

See Ekeland and Pirvu (2008) and Björk and Murgoci (2010).

Equilibrium Policies

Definition

(ii) An equilibrium policy π^* is of *feedback type* if, for some *feedback function* $\pi_* : [0, T] \times \mathbb{R}_+ \to \mathcal{B}$ such that the SDE for X with π_t replaced by $\pi_*(t, X_{t-})$ has a unique solution $X^* = (X_t^*)_{t\in[0,T]}$, we have

$$\pi_t^* = \pi_*(t, X_{t-}^*), \quad t \in [0, T],$$

with $X_{0-}^* = X_0^*$.

To any vector $\pi \in \mathcal{B}$ we associate the operators $\mathcal{L}^\pi : f \mapsto \mathcal{L}^\pi f$ and $\mathcal{G}^\pi : f \mapsto \mathcal{G}^\pi f$ that map $C^{0,2}([0,T] \times \mathbb{R}_+, \mathbb{R})$ to $C^{0,0}(\mathbb{R}_+, \mathbb{R})$ and are given by

$$\mathcal{L}^\pi f(t,x) = \mu_\pi x f'(t,x) + \frac{\sigma_\pi^2}{2} x^2 f''(t,x) + \int_{\mathbb{R}^k \setminus \{0\}} [f(t, x + x\pi^\intercal R y) - f(t,x) - x\pi^\intercal R y f'(t,x)] \nu(\mathrm{d}y),$$

$$\mathcal{G}^\pi f(t,x) = g(x f'(t,x)\, \pi^\intercal \Sigma, \delta_{x\pi^\intercal R I} f(t,x)),$$

for $(t,x) \in [0,T] \times \mathbb{R}_+$, where $\delta_y f : \mathbb{R}_+ \to \mathbb{R}$ and $I : \mathbb{R}^{k\times 1} \to \mathbb{R}^{k\times 1}$ are given by

$$\delta_y f(x) = f(t, y+x) - f(t,x), \quad I(z) = z, \quad z \in \mathbb{R}^{k\times 1}, x \in \mathbb{R}_+, y \in \mathbb{R},$$

and where

$$\mu_\pi = r + (\mu - r\mathbf{1})^\intercal \pi, \qquad \sigma_\pi^2 = \pi^\intercal \Sigma \Sigma^\intercal \pi, \qquad \pi \in \mathcal{B}.$$

The Extended HJB Equation

Consider the *extended Hamilton-Jacobi-Bellman equation* for a triplet (π_*, V, h) of a feedback function π_*, the corresponding value function V and auxiliary function h given by (denoting $\dot{V} = \frac{\partial V}{\partial t}$):

$$
\begin{aligned}
&\dot{V}(t,x) + \sup_{\pi \in \mathcal{B}} \{ \mathcal{L}^{\pi} V(t,x) - \gamma \mathcal{G}^{\pi} h(t,x) \} = 0, && \\
&\dot{h}(t,x) + \mathcal{L}^{\pi_*(t,x)} h(t,x) = 0, && \\
&V(T,x) = h(T,x) = x, && x \in \mathbb{R}_+, \\
&V(t,0) = h(t,0) = 0, && t \in [0,T].
\end{aligned}
$$

Theorem

Let (π_*, h, V) be a triplet satisfying the extended HJB equation, let X^* be the wealth process corresponding to $\pi^*_t = \pi_*(t, X_{t-})$. Assume $V \in C^{1,2}([0,T] \times \mathbb{R}_+, \mathbb{R})$ with h', V' bounded and that $\pi^* = (\pi^*_t)_{t\in[0,T]} \in \Pi$. Then π^* is an equilibrium policy of feedback type and h and V are given by $V(t,x) = \mathbb{E}_{t,x}[X_T^{\pi^*}] - \gamma \tilde{D}_{t,x}(X_T^{\pi^*})$ and $h(t,x) = \mathbb{E}_{t,x}[X_T^{\pi^*}]$ for $(t,x) \in [0,T] \times \mathbb{R}_+$.

Thank you for your attention!

Assumption

For some countable set A and any $a \in [0, \gamma^{-1}] \backslash A$, the function $T_a : \mathcal{B} \to \mathbb{R}$ given by

$$T_a(c) := a\,(\mu - r\mathbf{1})^\intercal c - g(c^\intercal \Sigma, c^\intercal RI), \quad c \in \mathcal{B},$$

achieves its maximum over ∂B at a unique $c^* \in \partial\mathcal{B}$.

Lemma

For any $f : [0,T] \to \mathcal{B}$ *denote by* $A_f, d_f, b_f, F_f : [0,T] \to \mathbb{R}$ *the functions given by*

$$
\begin{aligned}
b_f(t) &:= \exp\left(\int_t^T \{r + (\mu - r\mathbf{1})^\intercal f(s)\}\mathrm{d}s\right), \\
d_f(t) &:= b_f(t)\int_t^T \hat{g}(f(s)^\intercal \Sigma, f(s)^\intercal RI)\mathrm{d}s, \\
A_f(t) &:= \gamma^{-1} - (b_f(t))^{-1} d_f(t), \\
F_f(t) &:= A_{C_f}(t), \quad \textit{with} \\
C_f(t) &:= \begin{cases} \arg\sup_{c\in\partial\mathcal{B}}\left\{T_{f(t)}(c)\right\}, & \textit{if } f(t) \notin A, \\ \mathrm{Centroid}(\arg\sup_{c\in\partial\mathcal{B}}\left\{T_{f(t)}(c)\right\}), & \textit{if } f(t) \in A, \end{cases}
\end{aligned}
$$

where for any Borel set $A' \subset \mathbb{R}^d$, $\mathrm{Centroid}(A')$ *is equal to the mean of* $U \sim \mathrm{Unif}(A')$*. Then there exists a continuous non-decreasing function* $a^* : [0,T] \to \mathbb{R}_+$ *such that* $a^* = F_{a^*}$*.*

Theorem

With $T_a(c)$ *and* a^* *given in* (1) *and in Lemma 13, we let* $s(a) := \sup_{c\in\partial\mathcal{B}} T_a(c)$, $a_- := \sup\{a \in [0,\gamma^{-1}] : s(a) \leq 0\}$, *and* $t^* := \sup\{t \in [0,T] : a^*(t) \leq a_-\}$ *(where* $\sup\emptyset := -\infty$*).*

(i) *If* $s(1/\gamma) \leq 0$ *then* $\pi^* \equiv 0$ *with value-function given by* $V(t,x) = x\exp(r(T-t))$ *for* $(t,x) \in [0,T]\times\mathbb{R}_+$.

(ii) *If* $s(1/\gamma) > 0$ *define the function* $C^* : [0,T] \to \mathcal{B}$ *by*

$$C^*(t) = \begin{cases} C_{a^*(t)}, & \textit{if } t \in [t^* \vee 0, 1], \\ 0, & \textit{otherwise}, \end{cases}$$

where $C_{a^*(t)}$ *is given in* (4.2) *with* $f = a^*$. *Then* $\pi^* = C^*$ *is an equilibrium policy with value function given by* $V(t,x) = x(b_{C^*}(t) - \gamma d_{C^*}(t))$ *for* $(t,x) \in [0,T]\times\mathbb{R}_+$, *where* b_{C^*} *and* d_{C^*} *are given in* (4.2) *and* (4.2) *with* $f = C^*$.

Distribution Invariance

D is distribution invariant whenever $D_0(X) = D(Y)$ if X has the same distribution as Y.

Theorem

If Positive Homogeneity and Subadditivity is replaced by convexity, then the only convex deviation measure being distribution invariant is a constant multiple of the variance, i.e, there exists $\alpha > 0$ such that

$$D_0(X) = \alpha Var(X).$$