Weak dependence of mixed moving average processes and applications

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Continuous time stochastic volatility models

Consider the log price process of a financial asset

$$p_t = a_t + M_t,$$

where M is a local martingale and a is a cádlág adapted process of locally bounded variation.

We take M to have a stochastic volatility:

$$M_t = \int_0^t \sqrt{X_s} dW_s$$

where the non-negative spot volatility X is assumed to have cádlág sample paths (which implies it can posses jumps!)

Continuous time stochastic volatility models

- X_s independent of W
- X_s can be Itô diffusion as in the Heston model (1993)

$$dX_s = \alpha(\beta - X_s)\,ds + \nu\sqrt{X_s}\,dZ_s$$

  where Z is a Brownian motion correlated with W, and $2\alpha\beta > \nu^2$.
- X_s can be a Lévy driven Ornstein Uhlenbeck process

$$dX_s = -a\,X_s\,ds + dL_s, \quad a > 0$$

  where L is a subordinator independent of W, (Lévy process with positive increments and no drift), Barndorff-Nielsen and Shepard (2001).
  For $\mathbb{E}[\log(|L_1| \vee 1)] < \infty$ and $a > 0$, a unique stationary solution exists:

$$X_t = \int_{-\infty}^{t} \mathrm{e}^{-a(t-s)} dL_s$$

Lévy driven Ornstein Uhlenbeck Process

For L a Lévy process with characteristic triplet (γ, Σ, ν),

$$X_t = \int_{-\infty}^{t} \mathrm{e}^{-a(t-s)} dL_s$$

- The parameter a is called mean reversion parameter
- $Corr(X_0, X_r) = \mathrm{e}^{-ar}$ with $r > 0$.

Superposition of Ornstein-Uhlenbeck process

- Typically, the autocovariance function of the squared returns of financial prices decays much faster at the beginning than at higher lags.

Hence:

- Add up countably many independent OU-type processes

$$X_t = \sum_{k=1}^{\infty} w_i \int_{-\infty}^{t} \mathrm{e}^{-a_i(t-s)} dL_{i,s}$$

with independent identically distributed Lévy processes $(L_i)_{i \in \mathbb{N}}$ and appropriate $a_i > 0$, $w_i > 0$ with $\sum_{i=1}^{\infty} w_i = 1$.

Superposition of Ornstein-Uhlenbeck process

More generally we can "integrate" over all possible mean reversion parameters.

$$X_t = \int_{\mathbb{R}^-} \int_{-\infty}^{t} \mathrm{e}^{A(t-s)} \Lambda(dA, ds)$$

where Λ is called a Lévy basis and the mean reversion parameter A becomes a random variable.

The supOU process was first introduced by Barndorff-Nielsen (2001) and further investigated in Barndorff-Nielsen and St.(2011) and Fuchs and St. (2013).

Lévy basis I

Let $\mathcal{B}(S)$ the Borel σ-field on S and π some probability measure on $(S, \mathcal{B}(S))$.

Definition

A family $\Lambda = \{\Lambda(B) : B \in \mathcal{B}_b(S \times \mathbb{R})\}$ of real-valued random variables is called a d-dimensional Lévy basis on $S \times \mathbb{R}$ if:

- the distribution of $\Lambda(B)$ is infinitely divisible for all $B \in \mathcal{B}_b(S \times \mathbb{R})$,
- for arbitrary $n \in \mathbb{N}$ and pairwise disjoint sets $B_1, \ldots, B_n \in \mathcal{B}_b(S \times \mathbb{R})$ the random variables $\Lambda(B_1), \ldots, \Lambda(B_n)$ are independent and
- for any pairwise disjoint sets $B_1, B_2, \ldots \in \mathcal{B}_b(S \times \mathbb{R})$ with $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}_b(S \times \mathbb{R})$ we have, almost surely, $\Lambda(\bigcup_{n\in\mathbb{N}} B_n) = \sum_{n\in\mathbb{N}} \Lambda(B_n)$.

Lévy basis II

We restrict ourselves to time-homogeneous and factorisable Lévy bases, i.e. Lévy bases with characteristic function

$$\mathbb{E}[e^{i\langle u,\Lambda(B)\rangle}] = e^{\Phi(u)\Pi(B)} \tag{1}$$

for all $u \in \mathbb{R}^d$ and $B \in \mathcal{B}_b(S \times \mathbb{R})$, where $\Pi = \pi \times \lambda$ is the product of the probability measure π on S and the Lebesgue measure λ on $\mathbb{R}$ and

$$\Phi(u) = i\langle \gamma, u\rangle - \frac{1}{2}\langle u, \Sigma u\rangle + \int_{\mathbb{R}^d} e^{i\langle u,x\rangle} - 1 - i\langle u, x\rangle 1_{[0,1]}(\|x\|)\, \nu(dx),$$

where $\gamma \in \mathbb{R}^d$, $\Sigma \in \mathbb{S}_d^+$ - i.e. the space of the positive semi-definite matrix- and ν is a Lévy measure. By L we denote the underlying Lévy process with characteristic triplet (γ, Σ, ν). The quadruple $(\gamma, \Sigma, \nu, \pi)$ determines the distribution of the Lévy bases completely and therefore it is called the generating quadruple.

Mixed Moving Average Processes

The process

$$X_t = \int_S \int_{\mathbb{R}} f(A, t-s)\, \Lambda(dA, ds),$$

is infinitely divisible and strictly stationary and called a MMA process. f is a deterministic kernel function and integrable in the sense of Rajput and Rosiński (1989).

- The class of mixed moving average processes allows to obtain models with flexible autocorrelation structure and that at the same time can generate many kinds of marginal distribution by choosing an appropriate Lévy basis.
- In Fuchs and St. (2013), it is shown that a MMA process is mixing and consequently ergodic.

Example

Let us assume that π is a probability distribution with support in $\mathbb{R}^-$ defined as $B\xi$ where $B \in \mathbb{R}^-$ and ξ is $\Gamma(\alpha, 1)$ with $\alpha > 1$ (distribution function of the random mean reverting parameter A). The autocovariance of the supOU process

$$X_t = \int_{\mathbb{R}^-} \int_{-\infty}^{t} \mathrm{e}^{A(t-s)} \Lambda(dA, ds)$$

is

$$Cov(X_0, X_k) = -\frac{\sigma^2 (1 - Bk)^{1-\alpha}}{2B(\alpha - 1)},$$

where $\sigma^2 = Var[L_1]$.

Weak dependence

Let

$$\mathcal{F} = \bigcup_{u \in \mathbb{N}^*} \mathcal{F}_u \quad \text{and} \quad \mathcal{G} = \bigcup_{v \in \mathbb{N}^*} \mathcal{G}_v$$

where $\mathcal{F}_u$ and $\mathcal{G}_v$ are respectively two classes of measurable functions from $(\mathbb{R}^d)^u$ to $\mathbb{R}$ and $(\mathbb{R}^d)^v$ to $\mathbb{R}$.

Ψ-weak dependence

A process $X = (X_t)_{t\in\mathbb{R}}$ with values in $\mathbb{R}^d$ is called a Ψ-weakly dependent process if there exists a sequence $(\epsilon(r))_{r\in\mathbb{R}^+}$ converging to 0, satisfying

$$|Cov(F(X_{i_1},\ldots,X_{i_u}),G(X_{j_1},\ldots,X_{j_v}))| \leq c\,\Psi(F,G,u,v)\,\epsilon(r)$$

for all

$$\begin{cases} (u,v)\in\mathbb{N}^*\times\mathbb{N}^*; \\ r\in\mathbb{R}^+; \\ (i_1,\ldots,i_u)\in\mathbb{R}^u \text{ and } (j_1,\ldots,j_v)\in\mathbb{R}^v, \\ \text{with } i_1\leq\ldots\leq i_u\leq i_u+r\leq j_1\leq\ldots\leq j_v; \\ \text{functions } F\colon(\mathbb{R}^d)^u\to\mathbb{R} \text{ and } G\colon(\mathbb{R}^d)^v\to\mathbb{R} \\ \text{ belonging respectively to } \mathcal{F} \text{ and } \mathcal{G} \end{cases}$$

and where c is a constant independent of r.
The sequence $(\epsilon(r))_{r\in\mathbb{R}^+}$ corresponds to different sequences of weak dependence coefficients

η-weak dependence

Let $\mathcal{F} = \mathcal{G}$ and $\mathcal{F}_u$ be the class of bounded Lipschitz functions. We consider $\mathbb{R}^d$ equipped with the Euclidean norm and $Lip(F) = \sup_{x \neq y} \frac{|F(x)-F(y)|}{\|x_1-y_1\|+\|x_2-y_2\|+\ldots+\|x_d-y_d\|}$.

The η-**coefficients** correspond to

$$\Psi(F, G, u, v) = u\|G\|_\infty Lip(F) + v\|F\|_\infty Lip(G)$$

and have been introduced in Doukhan and Louhichi (1999).

θ-weak dependence

Let $\mathcal{F}_u$ the class of bounded measurable functions, $\mathcal{G}_v$ be the Lipschitz functions.

The θ-**coefficients** correspond to

$$\Psi(F, G, u, v) = v\|F\|_\infty Lip(G).$$

and have been introduced in Dedecker and Doukhan (2003).

Remarks

- Let $(\mathcal{A}_t)_{t\in\mathbb{R}}$ be the filtration generated by Λ defined as the σ-algebras $\mathcal{A}_t$ generated by the set of random variables $\{\Lambda(B) : B \in \mathcal{B}(S \times (-\infty, t])\}$ for $t \in \mathbb{R}$. If an MMA process is adapted to $(\mathcal{A}_t)_{t\in\mathbb{R}}$, we call it **causal**. Otherwise it is referred to as being **non-causal**.
- An MMA process is (under moment assumptions) always η-weakly dependent and in the causal case also θ-weakly dependent.
- Different versions and proofs of the above statement can be found in Curato and St. (2019) in function of different moment conditions on the underlying Lévy process.

MMA: θ-weak dependence conditions (Curato and St., 2019)

Let Λ be an $\mathbb{R}^d$-valued Lévy basis with characteristic quadruple $(\gamma, \Sigma, \nu, \pi)$ such that $\mathbb{E}[L_1] = 0$ and $\int_{\|x\|>1} \|x\|^2 \nu(dx) < \infty$, $f : S \times \mathbb{R}^+ \to M_{n\times d}(\mathbb{R})$ a $\mathcal{B}(S \times \mathbb{R}^+)$-measurable function and $f \in L^2(S \times \mathbb{R}^+, \pi \otimes \lambda)$. Then, the resulting causal MMA process X is a θ-weakly dependent process with coefficients

$$\theta_X(r) = \Big(\int_S \int_{-\infty}^{-r} tr(f(A,-s)\Sigma_L f(A,-s)')\, ds\, \pi(dA) \Big)^{\frac{1}{2}}$$

for all $r \geq 0$, where $\mathbb{E}[L_1 L_1'] = \Sigma_L = \Sigma + \int_{\mathbb{R}^d} xx'\nu(dx)$.

Sample mean: asymptotics

Let Λ be an $\mathbb{R}^d$-valued Lévy basis with characteristic quadruple $(\gamma, \Sigma, \nu, \pi)$ such that $\mathbb{E}[L_1] = 0$ and $\int_{\|x\|>1} \|x\|^{2+\delta}\nu(dx) < \infty$, for some $\delta > 0$, $f : S \times \mathbb{R}^+ \to M_{1\times d}(\mathbb{R})$ a $\mathcal{B}(S \times \mathbb{R}^+)$-measurable function and $f \in L^{2+\delta}(S \times \mathbb{R}^+, \pi \otimes \lambda) \cap L^2(S \times \mathbb{R}^+, \pi \otimes \lambda)$. If $(X_i)_{i\in\mathbb{Z}}$ is a θ-weakly dependent process with coefficients $\theta_X(r) = O(r^{-\alpha})$ and $\alpha > 1 + \frac{1}{\delta}$, then

$$\sigma_\theta^2 = \sum_{k\in\mathbb{Z}} Cov(X_0, X_k)$$

is finite, non-negative and as $N \to \infty$

$$\frac{1}{\sqrt{N}} \sum_{i=1}^{N} X_i \xrightarrow{d} \mathcal{N}(0, \sigma_\theta^2).$$

Proof: Apply Dedecker and Rio (2000) and Dedecker and Douckhan (2003).

Proposition: hereditary properties (Curato and St., 2019)

Let $(X_t)_{t\in\mathbb{R}}$ be an $\mathbb{R}^n$-valued stationary process and assume there exists some constant $C>0$ such that $\mathbb{E}[|X_0|^p]^{\frac{1}{p}} \leq C$, with $p>1$, $h\colon \mathbb{R}^n \to \mathbb{R}^m$ be a function such that $h(0)=0$, $h(x)=(h_1(x),\ldots,h_m(x))$ and

$$\|h(x)-h(y)\| \leq c\|x-y\|(1+\|x\|^{a-1}+\|y\|^{a-1}),$$

for $x,y\in\mathbb{R}^n$, $c>0$ and $1\leq a<p$. Define $(Y_t)_{t\in\mathbb{R}}$ by $Y_t=h(X_t)$. If $(X_t)_{t\in\mathbb{R}}$ is an η or θ-weakly dependent process, then $(Y_t)_{t\in\mathbb{R}}$ is a η or θ-weakly dependent process such that

$$\forall\, r\geq 0,\;\; \eta_Y(r)=\mathcal{C}\,\eta_X(r)^{\frac{p-a}{p-1}},$$

or

$$\forall\, r\geq 0,\;\; \theta_Y(r)=\mathcal{C}\,\theta_X(r)^{\frac{p-a}{p-1}},$$

with the constant $\mathcal{C}$ independent of r.

Sample autocovariance function at lag k

$$\frac{1}{N}\sum_{j=1}^{N}(X_{j\Delta}-\mathbb{E}[X_0])(X_{(j+k)\Delta}-\mathbb{E}[X_0]).$$

W.l.o.g we consider that $\mathbb{E}[X_0]=0$ and $\Delta=1$ and when the asymptotic properties of the autocovariance functions are investigated, we focus on the features of the processes

$$Y_{j,k}=X_jX_{j+k}-\mathbb{E}[X_0X_k].$$

Sample autocovariance: asymptotics

Let Λ be an $\mathbb{R}^d$-valued Lévy basis with characteristic quadruple $(\gamma, \Sigma, \nu, \pi)$ such that $\mathbb{E}[L_1] = 0$, $\int_{\|x\|>1} \|x\|^{4+\delta} \nu(dx) < \infty$, for some $\delta > 0$, $f : S \times \mathbb{R} \to M_{1\times d}(\mathbb{R})$ a $\mathcal{B}(S \times \mathbb{R})$-measurable function and $f \in L^{4+\delta}(S \times \mathbb{R}, \pi \otimes \lambda) \cap L^2(S \times \mathbb{R}, \pi \otimes \lambda)$. Let $\mathcal{Z}_j = (Y_{j,0}, \ldots, Y_{j,k})$ for all $j \in \mathbb{Z}$. If $(X_i)_{i\in\mathbb{Z}}$ is η-weakly dependent with coefficients $\eta_X(r) = O(r^{-\beta})$ such that $\beta > (4 + \frac{2}{\delta})(\frac{3+\delta}{2+\delta})$ or it is θ-weakly dependent with coefficients $\theta_X(r) = O(r^{-\alpha})$ such that $\alpha > (1 + \frac{1}{\delta})(\frac{3+\delta}{2+\delta})$, then respectively for each $p, q \in \{0, \ldots, k\}$ with $k \in \mathbb{N}$,

$$\Xi = \sum_{l\in\mathbb{Z}} Cov(X_0 X_p, X_l X_{l+q}) < \infty$$

and as $N \to \infty$

$$\frac{1}{\sqrt{N}} \sum_{j=1}^{N} \mathcal{Z}_j \xrightarrow{d} \mathcal{N}_{k+1}(0, \Xi).$$

Back to the supOU SV model

Let us choose a Lévy basis having as underlying Lévy process a subordinator and consider a supOU process X. We define the logarithmic asset price

$$J_t = \int_0^t \sqrt{X_s}\, dW_s, \quad J_0 = 0,$$

where $(W_t)_{t\in\mathbb{R}^+}$ is a standard Brownian motion and $(X_t)_{t\in\mathbb{R}^+}$ is an adapted, stationary and square-integrable process with values in $\mathbb{R}^+$ being independent of W.

Return process

In Curato and St. (2019), it is shown that, over equidistant time intervals $[(t-1)\Delta, t\Delta]$ for $t \in \mathbb{R}$,

$$Y_t = J_{t\Delta} - J_{(t-1)\Delta} = \int_{(t-1)\Delta}^{t\Delta} \sqrt{X_s} dW_s$$

is θ-weakly dependent with coefficients

$$\theta_Y(r) = \sqrt{\Delta\, \theta_X((r-1)\Delta)}.$$

Assumption 1

Let us assume that the mean reversion parameter A is Gamma distributed. That is, we assume that π is the distribution of $B\xi$ where $B \in \mathbb{R}^-$ and ξ is $\Gamma(\alpha_\pi, 1)$ distributed with $\alpha_\pi > 2$.

Moment function

We work now with a sample $\{Y_t : t = 1, \ldots, N\}$ and define $Y_t^{(m)} = (Y_{t+1}, Y_{t+2}, \ldots, Y_{t+m+1})$ for $t = 1, \ldots, N - m$.
The moment function is given by the measurable function $\tilde{h} : \mathbb{R}^{m+1} \times \Theta \to \mathbb{R}^{m+2}$ as

$$\tilde{h}(Y_t, \theta) = \begin{pmatrix} \tilde{h}_{Var}(Y_t^{(m)}, \theta) \\ \tilde{h}_0(Y_t^{(m)}, \theta) \\ \tilde{h}_1(Y_t^{(m)}, \theta) \\ \vdots \\ \tilde{h}_m(Y_t^{(m)}, \theta) \end{pmatrix} = \begin{pmatrix} Y_{t+1}^2 + \frac{\mu\Delta}{B(\alpha_\pi - 1)} \\ Y_{t+1}^4 - 3\left(\frac{\Delta\mu}{B(\alpha_\pi - 1)}\right)^2 + 3\sigma^2 \frac{(1-B\Delta)^{3-\alpha_\pi} - 1 - \Delta B(\alpha_\pi - 3)}{B^3(\alpha_\pi - 1)(\alpha_\pi - 2)(\alpha_\pi - 3)} \\ Y_{t+1}^2 Y_{t+2}^2 - \left(\frac{\Delta\mu}{B(\alpha_\pi - 1)}\right)^2 + \sigma^2 \frac{f_2 - 2f_1 + f_0}{2B^3(\alpha_\pi - 1)(\alpha_\pi - 2)(\alpha_\pi - 3)} \\ \vdots \\ Y_{t+1}^2 Y_{t+1+m}^2 - \left(\frac{\Delta\mu}{B(\alpha_\pi - 1)}\right)^2 + \sigma^2 \frac{f_{m+1} - 2f_m + f_{m-1}}{2B^3(\alpha_\pi - 1)(\alpha_\pi - 2)(\alpha_\pi - 3)} \end{pmatrix},$$

where $f_k := (1 - B\Delta k)^{3-\alpha_\pi}$.

Sample moments

In this case, the sample moment function of the return process is

$$g_{N,m}(Y,\theta) = \begin{pmatrix} \frac{1}{N-m}\sum_{t=1}^{N-m}\left(Y_{t+1}^2 + \frac{\mu\Delta}{B(\alpha_\pi - 1)}\right) \\ \frac{1}{N-m}\sum_{t=1}^{N-m}\left(Y_{t+1}^4 - 3\left(\frac{\Delta\mu}{B(\alpha_\pi-1)}\right)^2 + 3\sigma^2\frac{(1-B\Delta)^{3-\alpha_\pi}-1-\Delta B(\alpha_\pi-3)}{B^3(\alpha_\pi-1)(\alpha_\pi-2)(\alpha_\pi-3)}\right) \\ \frac{1}{N-m}\sum_{t=1}^{N-m}\left(Y_{t+1}^2 Y_{t+2}^2 - \left(\frac{\Delta\mu}{B(\alpha_\pi-1)}\right)^2 + \sigma^2\frac{f_2-2f_1+f_0}{2B^3(\alpha_\pi-1)(\alpha_\pi-2)(\alpha_\pi-3)}\right) \\ \vdots \\ \frac{1}{N-m}\sum_{t=1}^{N-m}\left(Y_{t+1}^2 Y_{t+1+m}^2 - \left(\frac{\Delta\mu}{B(\alpha_\pi-1)}\right)^2 + \sigma^2\frac{f_{m+1}-2f_m+f_{m-1}}{2B^3(\alpha_\pi-1)(\alpha_\pi-2)(\alpha_\pi-3)}\right) \end{pmatrix},$$

and θ_0 can be estimated by minimizing the objective function

$$\hat{\theta}_0^{*N,m} = \operatorname{argmin} g_{N,m}(Y,\theta)' A_{N,m} g_{N,m}(Y,\theta)$$

where $A_{N,m}$ is a positive definite matrix to weight the $m+2$ different moments collected in $g_{N,m}(Y,\theta)$.

Asymptotic properties

- The consistency of the GMM estimator is shown in St., Tosstorff, Wittlinger (2015).
- We show the asymptotic normality of the GMM estimator in Curato and St. (2019).

Assumptions

- **Assumption 2:** the parameter space Θ is compact and large enough to include the true parameter vector θ_0.
- **Assumption 3:** the matrix $A_{N,m}$ converges in probability to a positive definite matrix of constants A.
- **Assumption 4:** the parameter vector θ_0 is identifiable, i.e. $\mathbb{E}[\tilde{h}(Y,\theta)] = 0$ for all Y if and only if $\theta = \theta_0$.
- **Assumption 5:** the matrix W_Σ is positive definite.

Note: It is shown in St. et al (2011) that the supOU SV model is asymptotically identifiable!

Theorem: asymptotic normality of the GMM estimator

Let Λ be a real valued Lévy basis with generating quadruple $(\gamma, 0, \nu, \pi)$, Assumptions (H) be satisfied such that $\int_{|x|>1} |x|^{4+\delta} \nu(dx) < \infty$, for some $\delta > 0$, and let Assumption 1 hold with $\alpha_\pi - 1 > (1 + \frac{1}{\delta})(\frac{6+2\delta}{\delta})$. If, moreover, Assumptions 2, 3, 4 and 5 hold, then as N goes to infinity

$$\sqrt{N}(\hat{\theta}_0^{*N,m} - \theta_0) \xrightarrow{d} \mathcal{N}(0, MW_\Sigma M')$$

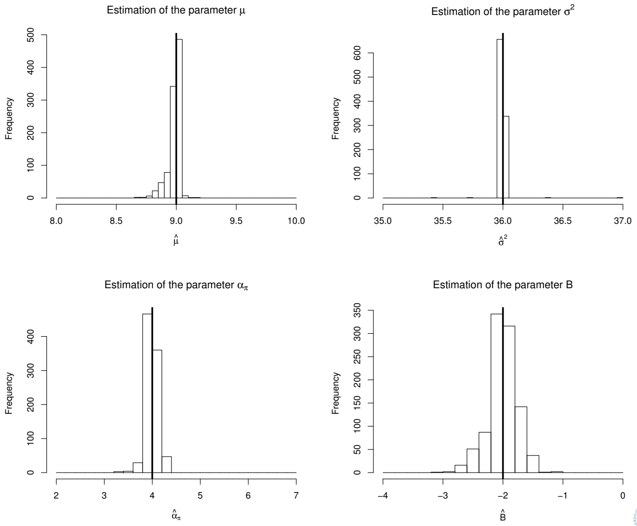
where

$$M = \mathbb{E}[G_0^{*\prime} A G_0^*]^{-1} G_0^{*\prime} A, \;\; G_0^* = \mathbb{E}[\frac{\partial \tilde{h}(Y_t, \theta)}{\partial \theta'}]_{\theta=\theta_0},$$

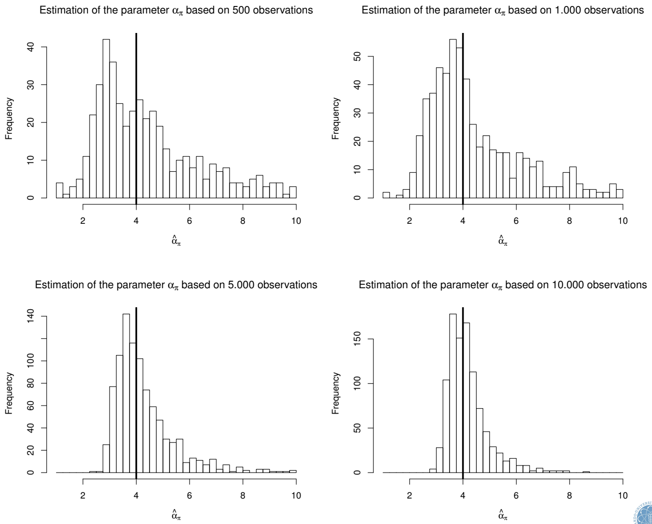
and

$$W_\Sigma = \sum_{l \in \mathbb{Z}} Cov(\tilde{h}(Y_0, \theta_0), \tilde{h}(Y_l, \theta_0)).$$

GMM parameter estimates from 1000 simulated paths with 2000 observations: short memory



Convergence of the estimator for α in a supOU process: short memory



Thank you very much for your attention!

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