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Weak dependence of mixed moving average fields and applications

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Based on joint work with Imma Curato and Robert Stelzer

Motivation

1. Let Λ be a Lévy basis, $(\mathcal{A}_t)_{t\in\mathbb{R}}$ the σ-algebra generated by the set of random variables $\{\Lambda(B), B \in \mathcal{B}(S \times (-\infty, t])\}$.
2. X is called causal if X_t is adapted to $\mathcal{A}_t$.
3. Causal MMA processes are (under moment assumptions) θ-weakly dependent.
4. Weak dependence properties are used to derive central limit theorems.
5. Aim: Generalize the concept of causality and give a suitable definition of weak dependence. Derive distributional limit theorems for such random fields.

Notation

- $\mathcal{F}_u^*$ is the class of bounded functions from $(\mathbb{R}^n)^u$ to $\mathbb{R}$.
- $\mathcal{F}_u$ is the class of bounded, Lipschitz functions from $(\mathbb{R}^n)^u$ to $\mathbb{R}$.
- $\mathcal{F} = \bigcup_{u\in\mathbb{N}^*} \mathcal{F}_u$ and $\mathcal{F}^* = \bigcup_{u\in\mathbb{N}^*} \mathcal{F}_u^*$.
- $Lip(G) = \sup_{x\neq y} \frac{|G(x)-G(y)|}{\|x_1-y_1\|+\ldots+\|x_n-y_n\|}$, where $G : \mathbb{R}^n \to \mathbb{R}$.

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Definition (θ-weakly dependent processes)

*Let $X = (X_t)_{t\in\mathbb{R}}$ be an $\mathbb{R}^n$-valued stochastic process. Then, X is called θ-**weakly dependent** if the θ-coefficients*

$$\theta(h) = \sup_{u,v\in\mathbb{N}^*} \theta_{u,v}(h) \underset{h\to\infty}{\longrightarrow} 0,$$

where

$$\theta_{u,v}(h) = \sup\Big\{ \frac{|Cov(F(X_{i_1},\dots,X_{i_u}), G(X_{j_1},\dots,X_{j_v}))|}{\|F\|_\infty Lip(G)}, F \in \mathcal{F}_u^*, G \in \mathcal{F}_u, \\ (i_1,\dots,i_u) \in \mathbb{R}^u, (j_1,\dots,j_v) \in \mathbb{R}^v, i_1 \leq \dots i_u \leq i_u + h \leq j_1 \leq \dots \leq j_v \Big\}.$$

Under θ-weak dependence central limit theorems can be proven under slower decay of the weak dependence coefficient compared to η-weak dependence.

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Lexicographic order on $\mathbb{R}^m$

Consider $y = (y_1, \ldots, y_m) \in \mathbb{R}^m$ and $z = (z_1, \ldots, z_m) \in \mathbb{R}^m$.
We say $y <_{lex} z$ if and only if $y_1 < z_1$ or $y_p < z_p$ and $y_q = z_q$ for some $p \in \{2, \ldots, m\}$ and $q = 1, \ldots, p-1$.
Define the sets $V_t = \{s \in \mathbb{R}^m : s <_{lex} t\} \cup \{t\}$ and $V_t^h = V_t \cap \{s \in \mathbb{R}^m : \|t-s\|_\infty \geq h\}$ for $h > 0$.

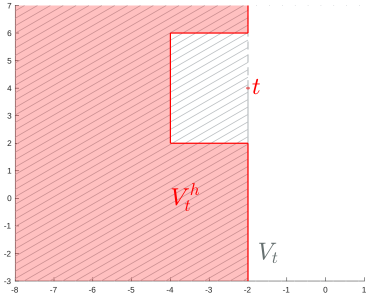

Figure: V_t and V_t^h for $m = 2$, $h = 2$ and $t = (-2, 4)$

Definition (θ-lex-weak dependence (Curato, Stelzer and St.))

*Let $X = (X_t)_{t \in \mathbb{R}^m}$ be an $\mathbb{R}^n$-valued random field. Then, X is called θ**-lex-weakly dependent** if*

$$\theta_X^{lex}(h) = \sup_{u \in \mathbb{N}^*} \theta_u(h) \underset{h \to \infty}{\longrightarrow} 0,$$

where

$$\theta_u(h) = \sup \Bigg\{ \frac{|Cov(F(X_\Gamma), G(X_j))|}{\|F\|_\infty Lip(G)}, \\ F \in \mathcal{F}^*, G \in \mathcal{F}, j \in \mathbb{R}^m, \Gamma \subset V_j^h, |\Gamma| \le u \Bigg\}.$$

Central Limit Theorem

Let $(D_n)_{n\in\mathbb{N}}$ be a sequence of finite subsets of $\mathbb{Z}^m$ with

$$\lim_{n\to\infty} |D_n| = \infty \text{ and } \lim_{n\to\infty} \frac{|D_n|}{|\partial D_n|} = 0.$$

Consider the random quantity

$$\frac{1}{|D_n|^{\frac{1}{2}}} \sum_{j\in D_n} X_j.$$

What can we say about its asymptotic distribution?

Central Limit Theorem (Curato, Stelzer and St.)

Let $X = (X_t)_{t\in\mathbb{Z}^m}$ be a stationary centered real-valued random field such that $E[|X_t|^{2+\delta}] < \infty$ for some $\delta > 0$.
Assume that $\theta_X^{lex}(h) \in \mathcal{O}(h^{-\alpha})$ with $\alpha > m(1+\frac{1}{\delta})$.
Let $\sigma^2 = \sum_{k\in\mathbb{Z}^m} E[X_0X_k|\mathcal{I}]$, where $\mathcal{I}$ is the σ-algebra of shift invariant sets. Then

$$\frac{1}{|\Gamma_n|^{\frac{1}{2}}}\sum_{j\in D_n} X_j \xrightarrow[n\to\infty]{\text{d}} \varepsilon\sigma,$$

with ε standard Gaussian, independent of σ^2.

(A, Λ)-influenced random fields

Let $X = (X_t)_{t \in \mathbb{R}^m}$ be a random field, $A = (A_t)_{t \in \mathbb{R}^m} \subset \mathbb{R}^m$ a family of Borel sets and $M = \{M(B), B \in \mathcal{B}_b(S \times \mathbb{R}^m)\}$ a random measure.
Assume X_t to be measurable w.r.t. $\sigma(M(B), B \in \mathcal{B}_b(S \times A_t))$.
Then, A is the **sphere of influence** and X an (A, M)**-influenced random field**.
If A is translation invariant ($A_t = t + A_0$), the sphere of influence is described by the set A_0. We call A_0 the **initial sphere of influence**.

For $m = 1$ and $A_t = V_t$ the above definition equals the class of causal processes.

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Definition (Mixed moving average field)

Let Λ be an $\mathbb{R}^d$-valued Lévy basis with characteristic quadruplet $(\gamma, \Sigma, \nu, \pi)$ and $f : S \times \mathbb{R}^m \to M_{n\times d}(\mathbb{R})$ be a $\mathcal{B}(S \times \mathbb{R}^m)$-measurable Λ-integrable function. Then,

$$X_t = \int_S \int_{\mathbb{R}^m} f(A, t-s)\Lambda(dA, ds),$$

is called a mixed moving average (MMA) field with kernel f.

(A, Λ)-influenced MMA field

Let $(A_t)_{t\in\mathbb{R}^m}$ be a full dimensional, translation invariant sphere of influence with initial sphere of influence $A_0 \subset V_0$. This also implies $A_t \subset V_t$. If X is adapted to the σ-algebra generated by $\{\Lambda(B), B \in \mathcal{B}(S \times A_t)\}$ it can be written as

$$X_t = \int_S \int_{A_t} f(A, t-s)\Lambda(dA, ds)$$
$$= \int_S \int_{V_t} f(A, t-s)\mathbb{1}_{A_0}(s-t)\Lambda(dA, ds).$$

Theorem (Curato, Stelzer and St.)

Let X be an (A,Λ)-influenced MMA field. Assume that $\int_{\|x\|>1}\|x\|^2\nu(dx)<\infty$, $f\in L^2$ and $A_j\subset K\subset V_j$ for a closed proper cone.

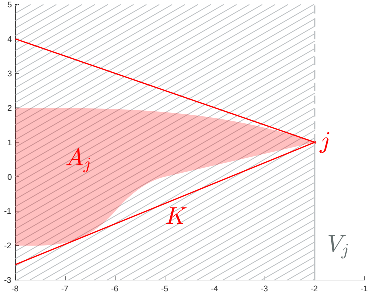

Figure: A_j and K for $m=2$ and $j=(-2,1)$

Theorem (Curato, Stelzer and St.)

Then, X is θ-lex-weakly dependent with coefficients

$$\theta_X^{lex}(h) \leq 2\bigg(\int_S \int_{A_0 \cap V_0^{\psi(h)}} \mathrm{tr}(f(A,-s)\Sigma_\Lambda f(A,-s)')ds\pi(dA) + \Big\| \int_S \int_{A_0 \cap V_0^{\psi(h)}} f(A,-s)\mu_\Lambda ds\pi(dA)\Big\|^2\bigg)^{\frac{1}{2}},$$

for all $h > 0$ and $\Sigma_\Lambda = \Sigma + \int_{\mathbb{R}^d} xx'\nu(dx)$, where $\psi(h) = c_K h$ for a constant $c_K > 0$.

Corollary (Asymptotic normality of the sample mean)

Let $(X_u)_{u\in\mathbb{Z}^m}$ be a zero mean (A,Λ)-influenced MMA field with $f\in L^{2+\delta}\cap L^2$ for $\delta>0$, $\int_{\|x\|>1}\|x\|^{2+\delta}\nu(dx)<\infty$ and $A_0\subset K$ for a closed proper cone $K\subset V_0$.
Additionally assume $\theta_X^{lex}(h)=\mathcal{O}(h^{-\alpha})$, $\alpha>m(1+\frac{1}{\delta})$.

Then $\Sigma=\sum_{k\in\mathbb{Z}^m}E[X_0X_k']$ is finite and

$$\frac{1}{|D_n|^{\frac{m}{2}}}\sum_{j\in D_n}X_j \xrightarrow[n\to\infty]{d} \mathcal{N}(0,\Sigma).$$

Limit distribution of the sample autocovariance

For a zero mean random field define

$$Y_{j,k} = X_j X_{j+k} - E[X_0 X_k], j \in \mathbb{Z}^m, k \in \mathbb{N}^m.$$

Consider the random quantity

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What can we say about its asymptotic distribution?

Proposition (θ-lex-coefficients have hereditary properties)

- *$(X_t)_{t\in\mathbb{R}^m}$ stationary with $X \in L^p$ for $p > 1$.*
- *$h : \mathbb{R}^n \to \mathbb{R}^k$, s.t. $h(0) = 0$ and for $1 \leq a < p$*

$$\|h(x) - h(y)\| \leq c\|x - y\|(1 + \|x\|^{a-1} + \|y\|^{a-1}),$$

for $x, y \in \mathbb{R}^n$, $c > 0$.

- *If X is θ-lex-weakly dependent, then $Y_t = h(X_t)$ is θ-lex-weakly dependent with*

$$\theta_Y^{lex}(h) = \mathcal{C}\theta_X^{lex}(h)^{\frac{p-a}{p-1}},$$

for all $r > 0$ and a constant $\mathcal{C}$.

- *Extend asymptotic results to higher sample moments.*

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Let $(X_u)_{u\in\mathbb{Z}^m}$ be a zero mean (A,Λ)-influenced MMA field with $f\in L^{4+\delta}\cap L^2$ for $\delta>0$, $\int_{\|x\|>1}\|x\|^{4+\delta}\nu(dx)<\infty$ and $A_0\subset K$ for a closed proper cone $K\subset V_0$.
Additionally assume $\theta_X^{lex}(h)=\mathcal{O}(h^{-\alpha})$, $\alpha>m\left(1+\frac{1}{\delta}\right)\left(\frac{3+\delta}{2+\delta}\right)$.

Then, $\Sigma_k=\sum_{j\in\mathbb{Z}^m}Cov(Y_{0,k},Y_{j,k})$ is finite and

$$\frac{1}{|D_n|^{\frac{m}{2}}}\sum_{j\in D_n}Y_{j,k}\xrightarrow[N\to\infty]{d}N\left(0,\Sigma_k\right).$$

Example (MSTOU processes)

- *Introduced in [Nguyen and Veraart, 2018].*
- *Let* $A = (A_t(x))_{(t,x)\in\mathbb{R}\times\mathbb{R}^m}$ *be an ambit set, i.e.* $A_t(x) \subset \mathbb{R}\times\mathbb{R}^{m-1}$

$$\begin{cases} A_t(x) = A_0(0) + (t,x), \; \textit{(Translation invariant)} \\ A_s(x) \subset A_t(x), s < t \\ A_t(x) \cap ((t,\infty)) \times \mathbb{R}^{m-1} = \emptyset. \; \textit{(Non-anticipative)} \end{cases}$$

- *The* (A,Λ)*-influenced MMA field*

$$Y_t(x) = \int_0^\infty \int_{A_t(x)} \exp(-\lambda(t-s))\Lambda(d\lambda, ds, d\xi)$$

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- *In the following we consider a zero mean c-class MSTOU process, i.e.* $A_t(x) = \{(s,\xi) : s \leq t, \|x-\xi\| \leq c|t-s|\}$.
- *Assume* $\int_{|x|>1} |x|^{2+\delta}\nu(dx) < \infty$, *for some* $\delta > 0$ *and* $\pi(d\lambda, ds, d\xi) = ds\, d\xi\, f(\lambda) d\lambda$.

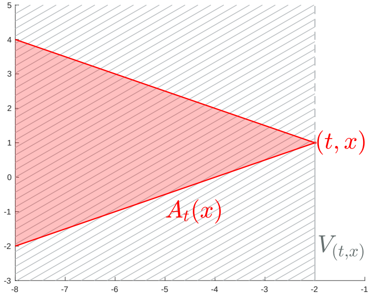

Figure: *c-class MSTOU process for* $c = \frac{1}{2}$ *and* $(t,x) = (-2,1)$

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- *Then, $Y_t(x)$ is θ-lex-weakly dependent with coefficients*

$$\theta_Y^{lex}(h) \leq \Bigg(V_{m-1}(c)\Sigma_\Lambda \int_0^\infty \frac{(m-1)!\sum_{k=0}^{m-1}\frac{1}{k!}(2\frac{\lambda\psi(h)}{c})^k}{(2\lambda)^m} e^{-2\frac{\lambda\psi(h)}{c}} f(\lambda)d\lambda$$

$$-\mathbb{1}_{\{c>1\}}(2\psi(h))^{m-1}\int_0^\infty \frac{e^{-2\frac{\lambda\psi(h)}{c}} - e^{-2\lambda\psi(h)}}{2\lambda} f(\lambda)d\lambda \Bigg)^{\frac{1}{2}},$$

  V_{m-1} is the volume of the $m-1$-dimensional ball with radius c.

- *Consider a Gamma(α,β) distributed mean reversion parameter λ ($\alpha > m$ and $\beta > 0$ ensure existence).*
- *The sample mean of $Y_t(x)$ is asymptotically normal if $\alpha > m\left(3+\frac{2}{\delta}\right)$.*

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Thank you for your attention!