Perturbation of the expected Minkowski functional and its applications

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Contents of talk

I. Smooth isotropic random field and Minkowski functional

II. Expectation of the Minkowski functional under skewness

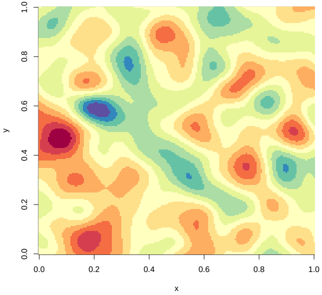
III. Numerical studies

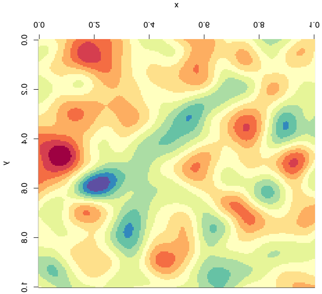
Contents of talk

- Isotropic random field $X(t)$, $t \in E \subset \mathbb{R}^n$:
  for any $P \in O(n)$ and $b \in \mathbb{R}^n$,

  $$\{X(t)\}_{t \in E' \subset \mathbb{R}^n} \overset{d}{=} \{X(Pt + b)\}_{t \in E' \subset \mathbb{R}^n},$$

  where E' is any finite set of E.
- We assume that $t \mapsto X(t)$ is smooth.



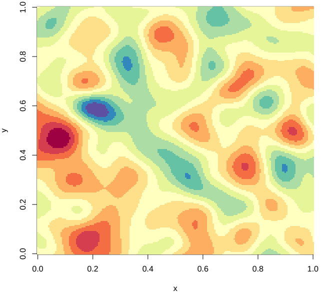


reflected

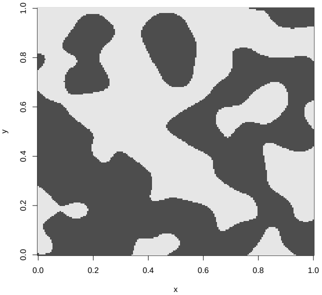
- The sup-level set of a function $X(t)$ on E:

$$E_v = \{t \in E \mid X(t) \geq v\} = X^{-1}([v, \infty))$$

  is referred to as the excursion set.
- By changing the level (threshold) v, we have a filtration.



Original RF

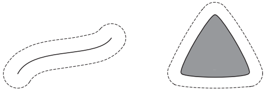


Excursion set

Minkowski functional (MF)

- Let $M \subset \mathbb{R}^n$ be a closed set. Tube about M with radius ρ:

$$\mathrm{Tube}(M,\rho) = \left\{ x \in \mathbb{R}^n \mid \mathrm{dist}(x,M) \le \rho \right\}$$



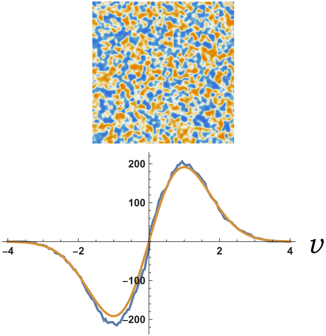
- Steiner's formula (Schneider, 2013): For small $\rho > 0$,

$$\mathrm{Vol}_n(\mathrm{Tube}(M,\rho)) = \sum_{j=0}^{n} \rho^j \binom{n}{j} \mathcal{M}_j(M)$$

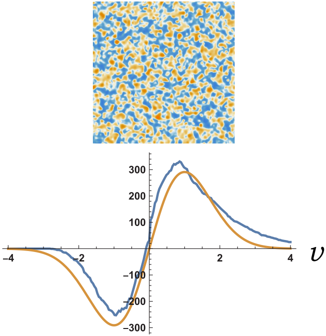
  where $\mathcal{M}_j(M)$ is the j-th Minkowski functional of M
- The Euler characteristic (EC) of M is

$$\chi(M) = \mathcal{M}_n(M)/\omega_n \quad \text{(Gauss-Bonnet theorem)}$$

- From now on, we consider the Minkowski functional $\mathcal{M}_j(E_v)$ of the excursion set E_v.
  $\mathcal{M}_j(E_v)$ can be used as a statistic for testing Gaussianity.



Gaussian



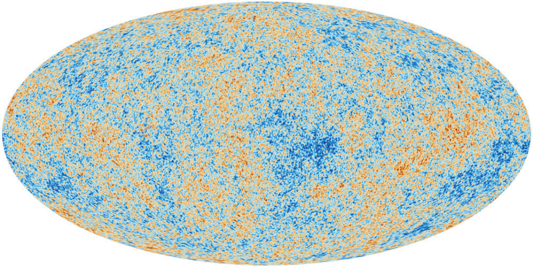
non-Gaussian

— $\chi(E_v)$

— $\mathbb{E}[\chi(E_v)]$ under the assumption of Gaussianity

Applications in cosmology: Cosmic Random field

- Cosmic microwave background (CMB) (mode: 160.2GHz)



http://planck.cf.ac.uk/

- Cosmic inflation theory:

  (normalized) density: $X(t) = \varphi(t) + a_2\varphi^2(t) + a_3\varphi^3(t) + \cdots, \; t \in \mathbb{R}^3$

  $\varphi(t)$: isotropic Gaussian field, $\varphi^2(t) = \int \varphi(s)\varphi(t)K(s-t)\mathrm{d}s$, etc.

Isotropic "Gaussian" random field ?

- For $k \geq 2$,

$$\mathrm{cum}(X(t_1), \ldots, X(t_k)) = O(\sigma^{k-2}) \quad (\sigma \ll 1)$$

  (Decay order is the same as the CLT)

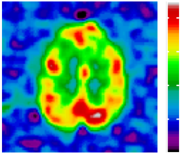
- Many versions of the inflation models exist. Some of them claim Gaussianity (i.e., $a_i \approx 0$), and some of them claim non-Gaussianity.
- In astronomy, $\mathbb{E}[\mathcal{M}_j(E_v)]$ is evaluated under each model, and is compared with the CMB observation.

Expected Euler characteristic method

- The expected EC of the excursion set is used for the approximation of the upper tail probability of the maximum of the random field:

$$\Pr\left(\sup_{t\in E} X(t) \geq v\right) \approx \mathbb{E}\big[\chi(E_v)\big] = \mathbb{E}\big[\mathcal{M}_n(E_v)\big]/\omega_n$$

  (Adler & Taylor, 2007; Takemura & Kuriki, 2002)
- This gives a p-value of the VBM data (installed in SPM):



`http://www.math.mcgill.ca/keith/`

- The purpose of this talk: To provide the formula for $\mathbb{E}[\mathcal{M}_j(E_v)]$ when $X(\cdot)$ is not Gaussian.

2- and 3-point correlation

- The correlation functions of an isotopic random field depend only on the pairwise distances:

$$\mathbb{E}[X(s)] = 0$$

$$\mathbb{E}[X(s)X(t)] = \rho\Big(\frac{1}{2}\|s-t\|^2\Big), \quad \rho(0) = 1$$

$$\mathbb{E}[X(s)X(t)X(u)] = \kappa\Big(\frac{1}{2}\|s-t\|^2, \frac{1}{2}\|s-u\|^2, \frac{1}{2}\|t-u\|^2\Big)$$

  $\kappa(x, y, z)$ is symmetric wrt x, y, z.

- We assume $\kappa \approx 0$ but $\kappa \neq 0$ (skewness$\neq 0$)

Moving average field of a Levy measure

- Suppose that $X(t)$ is generated as the Levy measure as

$$X(t) = \int_{\mathbb{R}^n} g\Big(\frac{1}{2}\|t-s\|^2\Big)Y(ds),$$

where $Y(ds)$ is a Levy measure on $\mathbb{R}^n$ with the moment structures:

$$\begin{aligned}
&\mathbb{E}[Y(ds)] = 0 \\
&\mathrm{cum}(Y(ds), Y(ds')) = \delta(s-s')ds \\
&\mathrm{cum}(Y(ds), Y(ds'), Y(ds'')) = \kappa_0 \cdot \delta(s-s')\delta(s-s'')ds
\end{aligned}$$

- When $\kappa_0 \neq 0$ but $|\kappa_0| \ll 1$, $X(t)$ is a non-Gaussian isometric field with weak skewness.
- $\mathrm{cum}(X(s), X(t), X(u))$ is shown to be a symmetric function in $\|s-t\|$, $\|s-u\|$, $\|t-u\|$ (not trivial).

Expected Minkowski functional under skewness

Theorem

Suppose that $X(t)$ is a zero mean, variance one smooth isotropic random field on $E \subset \mathbb{R}^n$ with covariance function ρ and 3-point correlation function κ. Then

$$\begin{aligned}\mathbb{E}[\mathcal{M}_j(E_v)] =& |E|\,\gamma^{j/2} 2^{-j/2} \frac{\Gamma(\frac{n-j}{2}+1)}{\Gamma(\frac{n}{2}+1)} \times \phi(v) \Big[h_{j-1}(v) \\ &+ 2^{-1}\gamma^{-2}\kappa_{11} j(j-1) h_{j-2}(v) - 2^{-1}\gamma^{-1}\kappa_1 j h_j(v) \\ &+ 6^{-1}\kappa_0 h_{j+2}(v) + o(\kappa) \Big], \quad j = 1, \ldots, n,\end{aligned}$$

where $\phi(x)$: pdf of $N(0,1)$, $h_n(x)$: Hermite poly., $\gamma = -\rho'(0)$, $\kappa_0 = \kappa(0,0,0)$, $\kappa_1 = \frac{\mathrm{d}\kappa(x,0,0)}{\mathrm{d}x}\big|_{x=0}$, $\kappa_{11} = \frac{\mathrm{d}^2\kappa(x,y,0)}{\mathrm{d}x\mathrm{d}y}\big|_{x=y=0}$.

- The Gaussian case ($\kappa \equiv 0$) is well known (Tomita, 1986).
- The case of $n = 2, 3$ was proved by Matsubara (2003).

Derivatives of ρ and κ in the moving average field

- For the moving average field

$$X(t) = \int_{\mathbb{R}^n} g\Big(\frac{1}{2}\|t-s\|^2\Big) Y(ds),$$

the derivatives of 2- and 3-correlation functions appearing in the perturbation formula:

$$\gamma = -\rho'(0) = \frac{\Omega_n}{n}\int_0^\infty g'(r^2/2)^2 r^{n-3} dr$$

$$\kappa_0 = \partial\kappa(x,y,x)\big|_0 = c\,\Omega_n \int_0^\infty g(r^2/2)^3 r^{n-1} dr$$

$$\kappa_1 = \frac{\partial\kappa(x,y,x)}{\partial x}\Big|_0 = -c\frac{\Omega_n}{n}\int_0^\infty g(r^2/2) g'(r^2/2)^2 r^{n-3} dr$$

$$\kappa_{11} = \frac{\partial^2\kappa(x,y,x)}{\partial x \partial y}\Big|_0 = c\frac{\Omega_n}{n(n+2)}\int_0^\infty g'(r^2/2)^2 g''(r^2/2) r^{n-5} dr$$

Outline of the Proof of the Theorem

Step 0. Represent the Minkowski functional $\mathcal{M}_j(E_v)$ in terms of

$$(X(t), \nabla X(t), \nabla^2 X(t)) \in \mathbb{R}^{1+n+n(n+1)/2}$$

Step 1. Obtain the joint cumulant of $(X(t), \nabla X(t), \nabla^2 X(t))$

Step 2. Obtain the moment generating function of $(X(t), \nabla X(t), \nabla^2 X(t))$

Step 3. Obtain the joint pdf of $(X(t), \nabla X(t), \nabla^2 X(t))$

Step 4. Taking expectation of $\mathcal{M}_j(E_v)$

Proof: Step 0. Minkowski Functional

- By taking tube coordinates, the Minkowski Functional is shown to be

$$\mathcal{M}_j(E_v) = \int_E \frac{1}{n} \det(-P^\top R P + \rho'(0) v I_{j-1}) \|V\|^{-j+2} \times p_{X(t)}(v) \, \mathrm{d}t$$

where $p_{X(t)}$ is the pdf of $X(t)$, $V = \nabla X(t)$,

$$R = R(t) = \nabla^2 X(t) - \rho'(0) X(t) I_n$$

and $P = P(t)$ is $n \times (j-1)$ such that $P^\top P = I_{j-1}$ and $P^\top \nabla X(t) = 0$

- That is, $\mathcal{M}_j(E_v)$ is represented in terms of

$$(X(t), \nabla X(t), \nabla^2 X(t)) \in \mathbb{R}^{1+n+n(n+1)/2}$$

Proof: Step 1. Joint cumulant

- Let $X_i = \partial X(t)/\partial t_i$, $X_{ij} = \partial^2 X(t)/\partial t_i \partial t_j$.
- For example,

$$\mathbb{E}[X_i X_j] = \frac{\partial}{\partial s_i}\frac{\partial}{\partial t_j}\mathbb{E}[X(s)X(t)]|_{s=t}$$
$$= \frac{\partial}{\partial s_i}\frac{\partial}{\partial t_j}\rho\Big(\frac{1}{2}\|s-t\|^2\Big)|_{s=t} = -\rho'(0)\delta_{ij}$$

- Similarly,

$$\mathbb{E}[XX] = 1 \qquad \mathbb{E}[X_i X_j] = -\rho'(0)\delta_{ij} \qquad \mathbb{E}[XX_{ij}] = \rho'(0)\delta_{ij}$$
$$\mathbb{E}[X_{ij}X_{kl}] = \rho''(0)(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$
$$\mathbb{E}[XXX] = \kappa_0 \qquad \mathbb{E}[XX_iX_j] = -\kappa_1\delta_{ij} \qquad \mathbb{E}[XXX_{ij}] = 2\kappa_1\delta_{ij}$$
$$\mathbb{E}[XX_{ij}X_{kl}] = (3\kappa_{11} + \kappa_2)\delta_{ij}\delta_{kl} + \kappa_2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$
$$\mathbb{E}[X_iX_jX_{kl}] = -2\kappa_{11}\delta_{ij}\delta_{kl} + \kappa_{11}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$
$$\mathbb{E}[X_{ij}X_{kl}X_{mn}] = (2\kappa_{111} + 6\kappa_{21})\delta_{ij}\delta_{kl}\delta_{mn} + 2\kappa_{21}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\delta_{mn}[3]$$
$$+ (-\kappa_{111})\delta_{il}\delta_{jn}\delta_{km}[8]$$

Proof: Step 2. Moment generating function

- Moment generating function of $X = X(t)$, $V = \nabla X(t)$, $R = \nabla^2 X(t) - \rho'(0)X(t)I_n$:

$$\mathbb{E}\big[\exp\{tX + \langle T, V\rangle + \mathrm{tr}(\Theta R)\}\big]$$
$$= \exp\left\{\frac{t^2}{2} + \frac{-\rho'(0)}{2}\|T\|^2 + \frac{\alpha}{2}\mathrm{tr}(\Theta^2) + \frac{\beta}{2}\mathrm{tr}(\Theta)^2\right\}$$
$$\times \left\{1 + Q(t, T, \Theta) + \cdots\right\}$$

  where $\alpha = 2\rho''(0)$, $\beta = \rho''(0) - \rho'(0)^2$
- $Q(t, T, \Theta)$ is a linear combination of

  t^3, t, $\|T\|^2$, $t^2\mathrm{tr}(\Theta)$, $t\mathrm{tr}(\Theta)^2$, $t\mathrm{tr}(\Theta^2)$, $\|T\|^2\mathrm{tr}(\Theta)$,
  $T^\top\Theta T$, $\mathrm{tr}(\Theta)^3$, $\mathrm{tr}(\Theta)\mathrm{tr}(\Theta^2)$, $\mathrm{tr}(\Theta^3)$

  of the order $O(\max(|\kappa_0|, |\kappa_1|, |\kappa_{11}|))$

Proof: Step 3. Joint pdf

- By inverting the moment generating function, we have the pdf of $X = X(t)$, $V = \nabla X(t)$, $R = \nabla^2 X(t) - \rho'(t)X(t)I_n$:

$$p(X,V,R) = \phi(X)p_{0V}(V)p_{0R}(R)\{1 + q(X,V,R) + \cdots\}$$

  $\phi(X)$: pdf of $N(0,1)$, $p_{0V}(V)$: pdf of $N_n(0, -\rho'(0)I_n)$

$$p_R(R) \propto \exp\Big\{-\frac{1}{2\alpha}\mathrm{tr}(R^2) + \frac{\beta}{2\alpha(\alpha+n\beta)}\mathrm{tr}(R)^2\Big\}$$

  where $\alpha = 2\rho''(0)$, $\beta = \rho''(0) - \rho'(0)^2$

- $q(X,V,R)$ is a linear combination of

$$h_1(X),\ h_3(X),\ \mathrm{tr}(R),\ h_2(X)\mathrm{tr}(R),\ h_1(X)\|V\|^2,$$
$$\|V\|^2\mathrm{tr}(R),\ V^\top RV,\ h_1(X)\mathrm{tr}(R)^2,\ h_1(X)\mathrm{tr}(R^2),$$
$$\mathrm{tr}(R)^3,\ \mathrm{tr}(R)\mathrm{tr}(R^2),\ \mathrm{tr}(R^3)$$

  of the order $O(\max(|\kappa_0|, |\kappa_1|, |\kappa_{11}|))$

Proof: Step 4. Expectation

- We take expectation of

$$\mathcal{M}_j(E_v) = \int_E \frac{1}{n}\det(-P^\top RP+\rho'(0)vI_{j-1})\|V\|^{-j+2}\times p_X(v)\,\mathrm{d}t$$

  with respect to $p(X,V,R)$ in the previous step.
- The most difficult part is to handle the random matrix R. The following formulas are crucial.

Lemma

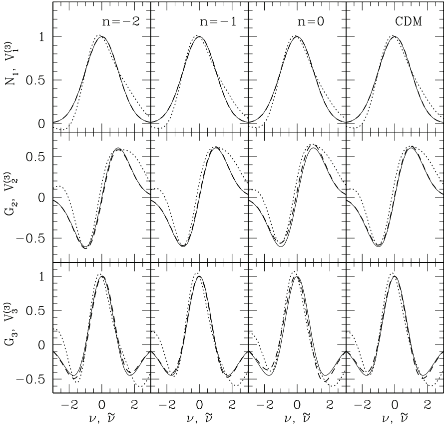
Let $A=(a_{ij})$ be the $n\times n$ GOE random matrix, that is, $a_{ii}\sim N(0,1)$ and $a_{ij}=a_{ji}\sim N(0,1/2)$ $(i<j)$ independently. Let $\mathcal{H}_n$ be physicist's Hermite poly. $\mathcal{H}_n(x)=2^n x^n+\cdots$. Then

$$\begin{aligned}\mathbb{E}[\det(xI_n+A)] &= 2^{-n}\mathcal{H}_n(x)\\ \mathbb{E}[\det(xI_n+A)\mathrm{tr}(A)] &= n2^{-(n-1)}\mathcal{H}_{n-1}(x)\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[\det(xI_n + A)\mathrm{tr}(A)^2] =& n2^{-n}\mathcal{H}_n(x) \\
&+ (n-1)n2^{-(n-2)}\mathcal{H}_{n-2}(x) \\
\mathbb{E}[\det(xI_n + A)\mathrm{tr}(A)^3] =& 3n^2 2^{-(n-1)}\mathcal{H}_{n-1}(x) \\
&+ (n-2)(n-1)n2^{-(n-3)}\mathcal{H}_{n-3}(x) \\
\mathbb{E}[\det(xI_n + A)\mathrm{tr}(A^2)] =& \frac{1}{2}n(n+1)2^{-n}\mathcal{H}_n(x) \\
&- \frac{1}{2}(n-1)n2^{-(n-2)}\mathcal{H}_{n-2}(x) \\
\mathbb{E}[\det(xI_n + A)\mathrm{tr}(A^3)] =& \frac{3}{2}n(n+1)2^{-(n-1)}\mathcal{H}_{n-1}(x) \\
&+ \frac{1}{4}(n-2)(n-1)n2^{-(n-3)}\mathcal{H}_{n-3}(x) \\
\mathbb{E}[\det(xI_n + A)\mathrm{tr}(A^2)\mathrm{tr}(A)] =& \frac{1}{2}(n^2+n+4)n2^{-(n-1)}\mathcal{H}_{n-1}(x) \\
&- \frac{1}{2}(n-2)(n-1)n2^{-(n-3)}\mathcal{H}_{n-3}(x)
\end{aligned}$$

Matsubara's (2003) analysis

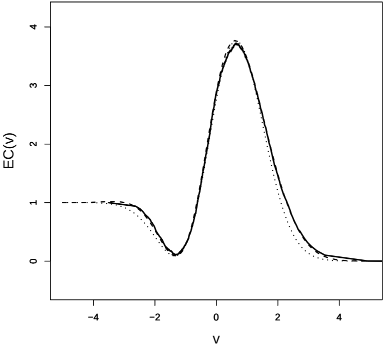
- $\mathbb{E}[\mathcal{M}_j(E_v)]$ $(j = 1, 2, 3)$ of 3D cosmic field under power law model $(n = -2, 1, 0)$ and CDM-like model:



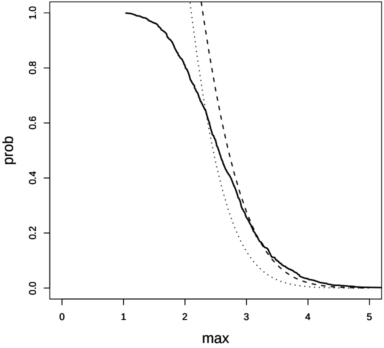
by simulator (solid) and expectation (dashed)

Simulation

- $Z(\cdot)$: 2D Gaussian random field on $E = [0,1]^2$ with covariance function $\mathbb{E}[Z(s)Z(t)] = \exp(-g\|s-t\|^2)$, $g = 50$.
- Let $X(t) = \{Z(t) - \delta(Z(t)^2 - 1)\}/c_\delta$, $\delta = 0.05$



solid: sample EC
dashed: expected EC
dotted: expected EC (Gauss)



solid: upper prob of $\max X(t)$
dashed: expected EC
dotted: expected EC (Gauss)

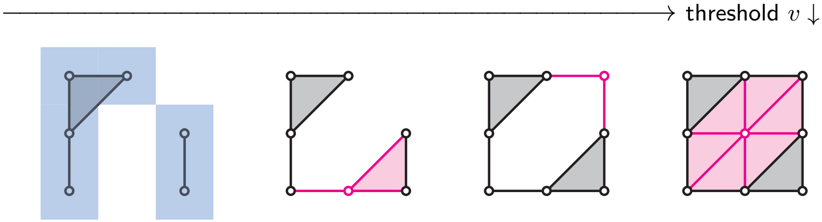
Remark: How to calculate the EC of 2D image

0. The excursion set image is represented as 0/1 at each pixel.
1. We convert the image into a simplicial complex by connecting adjacent vertices and by filling triangles. Then,

$$\chi = \#\text{vertices} - \#\text{edges} + \#\text{triangles}$$

2. By increasing the threshold, one new vertex is generated. Incidentally, new edges and triangles are produced.

$$\Delta\chi = 1 - \#\text{new edges} + \#\text{new triangles}$$



$\chi = 6 - 5 + 1$
$= 2$

$\Delta\chi = 1 - 3 + 1 = -1$
$\chi = 1$

$\Delta\chi = 1 - 2 + 0 = -1$
$\chi = 0$

$\Delta\chi = 1 - 6 + 6 = 1$
$\chi = 1$

Summary

- We introduced "isotropic random field", its "excursion set", and its "Minkowski functional (MF)" including "Euler characteristic (EC)".
- We provided a perturbation formula of the expected MF under skewness.
- We conducted simulation studies. The expected Euler characteristic method to approximate the upper tail probability of the maximum $\max_{t \in E} X(t)$ works well under weak skewness.
- Currently we are trying to derive the next order terms (i.e., under kurtosis).

Discussion: Remaining problems

- As a test statistic, we need to evaluate the variance of $\mathcal{M}_j(E_v)$. The variance formula is not local, i.e., not expressed by the derivatives of correlation functions evaluated at a point only.
- Astronomy people believes that the Minkowski functional is fit to their purpose, i.e., the analysis of CMB and the large-scale structure of the universe. But is it enough?
- Other candidates would be: Tensorial Minkowski functionals? Betti number, and its extension (persistent homology)? The Betti number is not local and more difficult.
- The validity of the EC method (i.e., evaluation of the approximation error) should be examined. (Typically, the approximation error depends on the tail behavior.)

References

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