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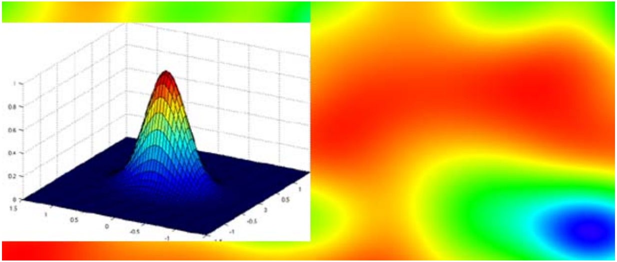

Long range dependence for heavy tailed random functions

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Overview

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Introduction: Random functions with long memory

Random function = Set of random variables indexed by $t \in T$.

Let $X = \{X_t, t \in T\}$ be a wide sense stationary random function defined on an abstract probability space $(\Omega, \mathcal{F}, P)$, e.g., $T \subseteq \mathbb{R}^d$, $d \geq 1$. The property of long range dependence (LRD) can be defined as

$$\int_T |C(t)|\, dt = +\infty$$

where $C(t) = \mathrm{cov}(X_0, X_t)$, $t \in T$ (McLeod, Hipel (1978); Parzen (1981)). Sometimes one requires that $C \in RV(-a)$, i.e., $\exists a \in (0, d)$ such that

$$C(t) = \frac{L(t)}{|t|^a}, \quad |t| \to +\infty,$$

where $L(\cdot)$ is a slowly varying function.

Various approaches to define LRD

- Unbounded spectral density at zero.
- Growth order of sums' variance going to infinity.
- Phase transition in certain parameters of the function (stability index, Hurst index, heaviness of the tails, etc.) regarding the different limiting behaviour of some statistics such as
  - Partial sums
  - Partial maxima.

These approaches are not equivalent, often statistically not tractable and tailored for a particular class of random functions (e.g., time series, square integrable, stable, etc.)

Various approaches to define LRD

LRD for heavy tailed random functions:

- Phase transitions in the limiting behaviour of partial sums and maxima of inf. divisible random processes and their ergodic properties (Samorodnitsky 2004, Samorodnitsky & Roy 2008, Roy 2010).
- α-spectral covariance approach for linear random fields with innovations lying in the domain of attraction of α–stable law (Paulauskas (2016), Damarackas, Paulauskas (2017))

LRD: Infinite variance case

For a stationary random function X with $\mathrm{E}\, X_t^2 = +\infty$ introduce

$$\mathrm{cov}_X(t,u,v) = \mathrm{cov}\left(\mathbb{1}(X_0 > u), \mathbb{1}(X_t > v)\right), \quad t \in T,\ x, v \in \mathbb{R}.$$

It is always defined as the indicators involved are bounded functions.
A random function X is called SRD (LRD, resp.) if

$$\sigma^2_{\mu,X} = \int\limits_T \int\limits_{\mathbb{R}^2} |\mathrm{cov}_X(t,u,v)|\, \mu(du)\, \mu(dv)\, dt < +\infty \qquad (= +\infty)$$

for all finite measures (for a finite measure, resp.) μ on $\mathbb{R}$. For discrete parameter random fields (say, if $T \subseteq \mathbb{Z}^d$), the $\int_T dt$ in the above line should be replaced by a $\sum_{t \in T : t \neq 0}$.

Motivation

Assume that X is wide sense stationary with covariance function $C(t) = \text{cov}(X_0, X_t)$, $t \in T$, and moreover,

$$\text{cov}_X(t, u, v) \geq 0 \text{ or } \leq 0 \text{ for all } t \in T,\ u, v \in \mathbb{R}.$$

Examples of X with this property are all **PA** or **NA**- random functions. W. Hoeffding (1940) proved that

$$C(t) = \int_{\mathbb{R}^2} \text{cov}_X(t, u, v)\, du\, dv. \tag{1}$$

Then, X is long range dependent if

$$\int_T |C(t)|\, dt = \int_T \int_{\mathbb{R}^2} |\text{cov}_X(t, u, v)|\, du\, dv\, dt = +\infty.$$

Motivation: memory and excursions

Level (excursion) sets and their volumes:

Let $a_n(u) = \nu_d\left(A_u\left(X, W_n\right)\right)$ be the volume of the excursion set

$$A_u\left(X, W_n\right) = \{t \in T \cap W_n : X_t > u\}$$

of a random field X at level u in an observation window $W_n = n \cdot W$ where $W \subset \mathbb{R}^d$ is a convex body.

Motivation: excursions and SRD

Multivariate CLT for level sets' volumes (Bulinski, S., Timmermann, Karcher, 2012):
For a stationary centered weakly dependent random function X satisfying some additional conditions (square integrable, α- or max-stable, inf. divisible) we have for any levels $u, v \in \mathbb{R}$ that

$$\frac{(a_n(u), a_n(v))^\top - (\mathrm{P}(X_0 \geq u), \mathrm{P}(X_0 \geq v))^\top \cdot \nu_d(W_n)}{\sqrt{\nu_d(W_n)}} \xrightarrow{\mathrm{d}} \mathcal{N}(\mathbf{o}, \Sigma)$$

as $n \to \infty$. Here $\Sigma = (\sigma_{ij})_{i,j=1}^2$ with $\sigma_{12} = \int_{\mathbb{R}^d} \mathrm{cov}_X(t, u, v)\, dt$.

So, $a_n(u) = \nu_d(A_u(X, W_n))$ is the right statistic to study!

Motivation: limiting variance in FCLT

By FCLT (Meschenmoser, Shashkin, 2011) and the continuous mapping theorem, it holds for some stationary weakly dependent associated random functions X with $W_n = [0, n]^d$ that

$$\frac{\int_{\mathbb{R}} a_n(u)\mu(du) - n^d \int_{\mathbb{R}} \bar{F}_X(u)\mu(du)}{n^{d/2}} \xrightarrow{d} N(0, \sigma^2_{\mu,X})$$

as $n \to \infty$ for any finite measure μ with $\sigma^2_{\mu,X}$ as above.

So X is SRD if the asymptotic covariance $\sigma^2_{\mu,X}$ in the CLT is finite for any finite measure μ prescribing the choice of levels u.

Motivation: American options

Let $X = \{X_t, t \in \mathbb{Z}\}$ be the stock for which an American option at price $u_0 > 0$, $t \in [0, n]$, $n \in \mathbb{N}$ is issued. The customer may buy a call at price u_0 whenever $X_t > u_0$ for some $t \in [0, n]$. For $\mu = \delta_{\{u_0\}}$ we get

$$\frac{\nu_1\left(\{t \in [0, n] : X_t > u_0\}\right) - n\bar{F}_X(u_0)}{\sqrt{n}} \xrightarrow{d} N(0, \sigma^2_{\delta_{\{u_0\}}, X}).$$

Then

- X l.r.d. (i.e., $\sigma^2_{\delta_{\{u_0\}}, X} = +\infty$) $\Longrightarrow$ the amount of time within $[0, n]$ at which the option may be exercised is not asymptotically normal for large time horizons n.
- X s.r.d. $\Longrightarrow$ asymptotic normality of this time span for any price u_0 provided that X satisfies some additional conditions.

Motivation: Checking LRD

For a stationary centered Gaussian random function X with Var $X_0 = 1$ and correlation function $\rho(t)$ we have (Bulinski, S., Timmermann, 2012)

$$\mathrm{cov}_X(t,u,v) = \frac{1}{2\pi}\int_0^{\rho(t)} \frac{1}{\sqrt{1-r^2}} \exp\left\{-\frac{u^2-2ruv+v^2}{2\,(1-r^2)}\right\} dr.$$

Motivation: statistical inference of LRD

The new definition is statistically feasible. Notice that for $\mu = \delta_{\{u_0\}}$

$$\sigma^2_{\mu,X} = \int_T |F_{X_0,X_t}(u_0,u_0) - F_X(u_0)F_X(u_0)|\, dt,$$

where the bivariate d.f. $F_{X_0,X_t}(u,v) = \mathrm{P}(X_0 \le u, X_t \le v)$ and marginal d.f. $F_X(u) = \mathrm{P}(X_0 \le u)$ can be estimated from the data by their empirical counterparts.

Motivation: LRD is margin–free

Lemma (Kulik, S. 2019)

A stationary real–valued random function X is SRD if

$$\int_T \int_{[0,1]^2} |C_{0,t}(x,y) - xy| \, P_0(dx) \, P_0(dy) \, dt < +\infty$$

for any probability measure P_0 on $[0,1]$ where $C_{0,t}$ is a copula of the bivariate distribution of (X_0, X_t), $t \in T$. X is LRD if there exists a probability measure P_0 on $[0,1]$ such that the above integral is infinite.

Motivation: Checking LRD

Denote by $P_\mu(\cdot) = \mu(\cdot)/\mu(\mathbb{R})$ the probability measure associated with the finite measure μ on $\mathbb{R}$. If $X \in$ **PA** then applying Fubini–Tonelli theorem leads to

$$\sigma^2_{\mu,X} = \mu^2(\mathbb{R}) \int_T \operatorname{cov}\left(F_\mu(X_0), F_\mu(X_t)\right)\, dt,$$

where $F_\mu(x) = P_\mu((-\infty, x))$ is the (left–side continuous) distribution function of probability measure P_μ.

Mixing

Let $(\Omega, \mathcal{A}, P)$ be a probability space and $(\mathcal{U}, \mathcal{V})$ be two sub-$\sigma-$algebras of $\mathcal{A}$. α–mixing coefficient:

$$\alpha(\mathcal{U}, \mathcal{V}) = \sup \{ |P(U \cap V) - P(U)P(V)| : \; U \in \mathcal{U}, \; V \in \mathcal{V} \} .$$

Let $X = \{X_t, t \in T\}$ be a random function, and T be a normed space with distance d. Let $X_C = \{X_t, t \in C\}$, $C \subset T$, and $\mathcal{X}_\mathcal{C}$ be the $\sigma-$algebra generated by X_C. If $|C|$ is the cardinality of a finite set C, for any $z \in \{\alpha, \beta, \phi, \psi, \rho\}$ put

$$z_X(k, u, v) = \sup\{z(\mathcal{X}_\mathcal{A}, \mathcal{X}_\mathcal{B}) : \quad d(A, B) \geq k, \; |A| \leq u, |B| \leq v\},$$

where $u, v \in \mathbb{N}$ and $d(A, B)$ is the distance between subsets A and B.

SRD and mixing

Theorem (Kulik, S. 2019)

Let $X = \{X_t, t \in T\}$ be a stationary random function with $z-$mixing rate satisfying $\int_T z_X(\|t\|, 1, 1)\, dt < +\infty$ where $z \in \{\alpha, \beta, \phi, \psi, \rho\}$. Then X is SRD with

$$\int_T \int_{\mathbb{R}^2} |\mathrm{cov}_X(t, u, v)|\, \mu(du)\, \mu(dv)\, dt \leq 8 \int_T z_X(\|t\|, 1, 1)\, dt \cdot \mu^2(\mathbb{R})$$

for any finite measure μ.

Random volatility functions

Let the random function $X = \{X_t, t \in T\}$ be given by

$$X_t = F(Y_t)Z_t$$

where $Y = \{Y_t, t \in T\}$ and $Z = \{Z_t, t \in T\}$ are independent stationary random functions, Z has property

$$\text{cov}_Z(t, u, v) \geq 0 \text{ or } \leq 0 \text{ for all } t \in T,\ u, v \in \mathbb{R},$$

$F : \mathbb{R} \to \mathbb{R}_{\pm}$ and $P\big(F(Y_t) = 0\big) = 0$ for all $t \in T$.
$F(Y_t)$ is called a random volatility (being a deterministic function of a random (often LRD) function $Y = \{Y_t, t \in T\}$) scaling a heavy tailed random function $Z = \{Z_t, t \in T\}$.

Random volatility functions

Theorem (Kulik, S. 2019)

*Let the random volatility model X be given by $X_t = AZ_t$, $t \in T$, $|T| = +\infty$ where $A > 0$ a.s., A and Z are independent and $Z \in$ **PA** is stationary. Then X is LRD if there exists $u_0 \in \mathbb{R}$: $\bar{F}_Z(u_0/A) \neq const$ a.s.*

Random volatility functions

Example

The above theorem evidently holds true if e.g.

- $Z_0 \sim Exp(\lambda)$, $A \sim$ Fréchet(1) for any $\lambda > 0$.
- X is a subgaussian random function where $A = \sqrt{B}$, $B \sim S_{\alpha/2}\left(\left(\cos\frac{\pi\alpha}{4}\right)^{2/\alpha}, 1, 0\right)$, $\alpha \in (0, 2)$, and Z is a centered stationary Gaussian random function with covariance function $C(t) \geq 0$ for all $t \in T$ and a non–degenerate tail $\bar{F}_Z$.

Random volatility functions

Corollary

For the random function $X = \{X_t, t \in T\}$ given by $X_t = Y_t Z_t$, $t \in T$, assume that random functions $Y = \{Y_t, t \in T\}$ and $Z = \{Z_t, t \in T\}$ are stationary and independent. Assume that Z_0 has a regularly varying tail, that is, $P(Z_0 > x) \sim L(x)/x^\alpha$ as $x \to +\infty$ for some $\alpha > 0$ where the function L is slowly varying at $+\infty$. For $Y_0 > 0$ a.s. assume that $\mathrm{E} Y_0^\delta < \infty$ and $\mathrm{E}\left(Y_0^\delta Y_t^\delta\right) < \infty$ for some $\delta > \alpha$ and all $t \in T$. Let $Y, Z \in$ **PA** *(***NA***). Then X is LRD if $Y^\alpha = \{Y_t^\alpha, t \in T\}$ is LRD.*

Random volatility functions

Example

Assume that $X_t = e^{Y_t^2/4} Z_t$, $t \in \mathbb{Z}$, where

- Z_t is a sequence of i.i.d. random variables with finite moment of order $2+\delta$ for some $\delta > 0$,
- Y_t is a centered stationary Gaussian **PA** long memory sequence with unit variance and covariance function ρ,
- sequences Z_t and Y_t are independent .

It holds $\mathrm{E}X_0^2 = +\infty$. Choose $\mu = \delta_{\{u_0\}}$ for some $u_0 \in \mathbb{R}$. Then

$$\sum_{t=1}^{\infty} \mathrm{cov}_X(t, u_0, u_0) = \sum_{k=1}^{\infty} \frac{\langle \bar{F}_Z(u_0/G), H_k \rangle_{\varphi}^2}{k!} \sum_{t=1}^{\infty} \rho^k(t),$$

where $G(x) = e^{x^2/4}$. X is LRD if $\sum_{t=1}^{\infty} \rho^2(t) = +\infty$. In particular, if $\rho(t) \sim |t|^{-\eta}$ as $|t| \to \infty$, then LRD occurs if $\eta \in (0, 1/2]$.

LT for the volume of excursion sets

Let X be a real-valued random function on $\mathbb{Z}^d$, $d \geq 1$ and let $W \subset \mathbb{Z}^d$ be a finite subset. Let

$$A_u(X, W) := \{t \in W : X(t) \geq u\}$$

be the excursion set of X in W over the level $u \in \mathbb{R}$.

Asymptotic (non)Gaussian behavior of $|A_u(X, W)|$ as W expands to $\mathbb{Z}^d$?

Prove a more general limit theorem for sums $\sum_{t \in W} g(X_t)$ of functionals g of X!

LT for the volume of excursion sets

Let X be a random volatility function of the form

$$X_t = G(Y_t)Z_t, \quad t \in \mathbb{Z}^d,$$

where

- $\{G(Y_t), t \in \mathbb{R}^d\}$ is a subordinated measurable Gaussian random function,
- $\{Z_t, t \in \mathbb{Z}^d\}$ is a white noise,
- the random functions Y and Z are independent.

Let $W_n = [-n, n]^d$, and g be a real valued function such that $\mathrm{E}[g(X_0)] = 0, \quad \mathrm{E}[g^2(X_0)] > 0$. Introduce the function

$$\xi(y) = \mathrm{E}[g(G(y)Z_0)] \, .$$

It follows that $\xi(y) < \infty$ for ν_1–a. e. $y \in \mathbb{R}$, $\mathrm{E}[\xi(Y_0)] = 0$.

LT for the volume of excursion sets

Furthermore, set

$$m(y, Z_t) = g(G(y)Z_t) - \xi(y) , \quad \chi(y) = \mathrm{E}[m^2(y, Z_0)] .$$

Assume that

- $\mathrm{rank}\ (\xi) = q$, $\mathrm{E}[|g(X_0)|^2] < \infty$, $\mathrm{E}[\chi^3(Y_0)] < \infty$.
- Y is a homogeneous isotropic centered Gaussian random function with the covariance function $\rho(t) = \mathrm{E}[Y_0 Y_t] = |t|^{-\eta} L(|t|)$, $\eta \in (0, d/q)$ and L is slowly varying at infinity,
- Y has a spectral density $f(\lambda)$ which is continuous for all $\lambda \neq 0$ and decreasing in a neighborhood of 0.

LT for the volume of excursion sets

Theorem (Kulik, S. 2019)

1. *If* $\xi(y) \equiv 0$ *then*

$$n^{-d/2} \sum_{t \in [-n,n]^d \cap \mathbb{Z}^d} g(X_t) \xrightarrow{d} \mathcal{N}(0, \sigma^2), \quad n \to +\infty,$$

where $\sigma^2 = \mathrm{E}[g^2(X_0)]2^d > 0$.

2. *If* $\xi(y) \not\equiv 0$ *then*

$$n^{q\eta/2-d} L^{-q/2}(n) \sum_{t \in [-n,n]^d \cap \mathbb{Z}^d} g(X_t) \xrightarrow{d} R, \quad n \to +\infty,$$

where the random variable R *is a q-Rosenblatt-type random variable.*

LT for the volume of excursion sets

q–Rosenblatt-type random variable:

$$R = (\gamma(d,\eta))^{q/2} \int_{\mathbb{R}^{dq}}' \int_{[-1,1]^d} e^{i\langle \lambda_1 + \ldots + \lambda_q, u\rangle} du \frac{B(d\lambda_1)\ldots B(d\lambda_q)}{(|\lambda_1| \cdot \ldots \cdot |\lambda_q|)^{(d-\eta)/2}},$$

$$\gamma(d,\eta) = \frac{\Gamma\left((d-\eta)/2\right)}{2^\eta \pi^{d/2} \Gamma(\eta/2)},$$

and $\int_{\mathbb{R}^{dq}}'$ is the multiple Wiener–Ito integral with respect to a complex Gaussian white noise measure B (with structural measure being the spectral measure of Y).

LT for the volume of excursion sets

Example

Assume that

$$g(y) = \mathbb{1}\{y > u\} - P(G(Y_0)Z_0 > u)$$

where G is nonnegative or nonpositive ν_1–a.e. Then

$$\xi(y) = \mathrm{E}[\mathbb{1}\{G(y)Z_0 > u\}] - P(G(Y_0)Z_0 > u)\,.$$

- If $u = 0$ then $\xi(y) \equiv 0$, so the Gaussian case applies.
- If $u \neq 0$ then $\xi(y) \not\equiv 0$, so the non-Gaussian case applies. Let $uG(y) \geq 0$ for all y.
  $q = 1$: $G : \mathbb{R} \to \mathbb{R}_\pm$ is monotone right-continuous non–constant fct. with $\nu_1(\{x \in \mathbb{R} : G(x) = 0\}) = 0$.
  $q = 2$: $G(y) = G_1(|y|)$ with G_1 as above.

LT for the volume of excursion sets

Example

Let the random volatility function $X_t = G(|Y_t|)Z_t$, $t \in \mathbb{Z}^d$ be s.t.

- Y is a centered Gaussian random function with unit variance and corr. function $\rho(t) \geq 0$ as above, $\rho(t) \sim |t|^{-\eta}$ as $|t| \to +\infty$
- $G(x) \geq 0$ is continuous as above with $\mathrm{E}\,|G(|Y_0|)|^{1+\theta} < \infty$ for some $\theta \in (0,1)$.
- $\{Z_t\}$ is a heavy–tailed white noise, $\mathrm{E}Z_0^2 = +\infty$.

LT for the volume of excursion sets

For $\widetilde{G}(y) = G(|y|)$ and $\mu = \delta_{\{u_0\}}$, $u_0 > 0$ we have

$$\sum_{t\in\mathbb{Z}^d,\, t\neq 0} \mathrm{cov}_X(t, u_0, u_0) = \sum_{k=1}^{\infty} \frac{\langle \bar{F}_Z(u_0/\widetilde{G}), H_k\rangle_\varphi^2}{k!} \sum_{t\in\mathbb{Z}^d,\, t\neq 0} \rho^k(t),$$

- Since $\mathrm{rank}\,(\widetilde{G}) = 2$, X is LRD if $\sum_{t\in\mathbb{Z}^d,\, t\neq 0} \rho^2(t) = +\infty$, that is, if $\eta \in (0, d/2)$.
- For niveau $u \neq 0$, the asymptotic behavior of $|A_u(X, [-n, n]^d)|$ is of 2-Rosenblatt-type ($\mathrm{rank}\,(\xi) = q = 2$) if $\eta \in (0, d/2)$.

LT for the volume of excursion sets

Summary:
The correct statistics associated with the new definition of l.r.d. is the volume of excursion sets!!!!

Linear α–stable time series

- Let $\{Z_t, t \in \mathbb{Z}\}$ be a sequence of i.i.d. SαS random variables with characteristic function

$$\psi_Z(s) = \exp\{-\tau^\alpha |s|^\alpha\}$$

for $\tau > 0$, $\alpha \in (1, 2)$, $s \in \mathbb{R}$.

- Let $\{a_j, j \in \mathbb{Z}\}$ be a nonnegative number sequence s. t.

$$\sum_{j=-\infty}^{+\infty} a_j < \infty$$

.

- Linear SαS time series:

$$Y(t) = \sum_{j=-\infty}^{+\infty} a_j Z_{t-j}, \quad t \in \mathbb{Z}.$$

SRD/LRD for linear α–stable time series

Let $\{Y(t) = \sum_{j=-\infty}^{+\infty} a_j Z_{t-j}, \quad t \in \mathbb{Z}\}$ be as above.

Theorem (Makogin, Oesting, Rapp, S. (2019))

- *Y is SRD if $\sum_{j=-\infty}^{\infty} a_j^{\alpha/2} < \infty$.*
- *Y is LRD if $\sum_{j=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} (a_j^\alpha \wedge a_t^\alpha) = \infty$.*
- *If a_j is monotonically decreasing on $\mathbb{Z}_+$ and $a_j = a_{-j}$ for all $j \in \mathbb{Z}$ then Y is LRD whenever $\sum_{t=0}^{\infty} t\, a_t^\alpha = \infty$.*

Max–stable stationary processes

- A stochastic process $X = \{X(t),\ t \in T\}$ is called max-stable if, for all $n \in \mathbb{N}$, there exist functions $a_n : T \to (0, \infty)$ and $b_n : T \to \mathbb{R}$ such that

$$\left\{ \max_{i=1,\dots n} \frac{X_i(t) - b_n(t)}{a_n(t)},\ t \in T \right\} \stackrel{d}{=} \{X(t),\ t \in T\},$$

  where the processes X_i, $i \in \mathbb{N}$, are independent copies of X.
- Marginal distributions of a max-stable process: degenerate, Fréchet, Gumbel or Weibull law.
- α-Fréchet marginal distribution: $\mathrm{P}(X(t) \leq x) = \exp(-x^{-\alpha})$ for all $x > 0$ and some $\alpha > 0$ and all $t \in T$. Here, covariances do not exist if $\alpha \leq 2$.

Max–stable stationary processes

- Pairwise extremal coefficient: $\{\theta_t,\ t \in T\}$ defined via

$$P(X(0) \leq x,\ X(t) \leq x) = P(X(0) \leq x)^{\theta_t} \quad \text{for all } x > 0,$$

- It holds $\theta_t = 2 - \lim_{x \to \infty} P(X(t) > x \mid X(0) > x)$.
- $\theta_t \in [1, 2]$, where
  - $\theta_t = 2 \Longrightarrow X(0)$ and $X(t)$ asymptotically independent,
  - $\theta_t = 1 \Longrightarrow X(0)$ and $X(t)$ asymptotically fully dependent.

SRD/LRD for max–stable stationary processes

Theorem (Makogin, Oesting, Rapp, S. (2019))

Let $X = \{X(t),\ t \in T\}$ be a stationary max-stable process with α-Fréchet marginal distribution and pairwise extremal coefficient $\{\theta_t,\ t \in T\}$. X is LRD iff

$$\int_T (2 - \theta_t)\, dt = \infty.$$

Outlook

- Checking the new LRD definition for other classes of processes with infinite variance, e.g., for infinitely divisible moving averages
- Connection of LRD with LT for the volume of excursions of other stationary random functions

Literature: Historical and general expositions

- J. Beran, Y. Feng, S. Ghosh, R. Kulik, *Long Memory Processes: Probabilistic Properties and Statistical Methods*. Springer, 2012.
- H. E. Hurst, *Long term storage capacity of reservoirs*. Trans. Amer. Soc. Civil Engrs., 116, pp. 770-779, 1951.
- B.B. Mandelbrot, *Une classe de processus stochastiques homothétiques à soi; application à la loi climatologique de H.E. Hurst*. Comptes Rendus Acad. Sci. Paris, 260, pp. 3274-3277, 1965.
- D.B. Percival, *The statistics of long memory processes*. PhD Thesis, University of Washington, U.S.A, 1983.
- C.C. Heyde , Y. Yang, *On defining long range dependence*. Journal of Applied Probability, 34 (4), pp. 939 - 944, 1997.
- W. Hoeffding, *Scale-invariant correlation theory*. In N. I. Fisher and P. K. Sen (Eds.), The Collected Works of Wassily Hoeffding, New York: Springer-Verlag, pp. 57-107, 1940.
- F. Lavancier, *Long memory random fields*. In: Dependence in Probability and Statistics, Lecture Notes in Statist. 187, Springer, pp. 195–220, 2006.
- G. Samorodnitsky, *Stochastic processes and long range dependence*. Springer, Cham, 2016.

Literature: Limit theorems

- R. L. Dobrushin and P. Major, *Non-central limit theorems for nonlinear functionals of Gaussian fields*. Z. Wahrsch. Verw. Gebiete 50 (1979), no. 1, 27-52.
- J. Damarackas and V. Paulauskas, *Spectral covariance and limit theorems for random fields with infinite variance*. J. Multivariate Anal. 153 (2017), 156-175.
- L. Giraitis, H. L. Koul, and D. Surgailis, *Large sample inference for long memory processes*. Imperial College Press, London, 2012.
- S. N. Lahiri and Peter M. Robinson, *Central limit theorems for long range dependent spatial linear processes*. Bernoulli 22 (2016), no. 1, 345-375.
- N. Leonenko and A. Olenko, *Sojourn measures of Student and Fisher- Snedecor random fields*. Bernoulli 20 (2014), no. 3, 1454-1483.
- V. Paulauskas, *Some remarks on definitions of memory for stationary random processes and fields*. Lith. Math. J. 56 (2016), no. 2, 229-250.
- V. Pilipauskaite and D. Surgailis, *Scaling transition for nonlinear random fields with long-range dependence*. Stochastic Process. Appl. 127 (2017), no. 8, 2751-2779.
- R. Kulik, E. Spodarev, *Long range dependence of heavy tailed random functions*. Preprint, arXiv:1706.00742v3, 2019.
- V. Makogin, M. Oesting, A. Rapp, E. Spodarev, *Long range dependence for stable random processes*. Preprint, arXiv:1908.11187, 2019.

Appendix: Subordinated Gaussian random function

Let $Y = \{Y_t, t \in T\}$ be a stationary centered Gaussian real-valued random function with $Var(Y_t) = 1$ and $\rho(t) = Cov(Y_0, Y_t)$, $t \in T$. The subordinated Gaussian random function X is defined by

$$X_t = g(Y_t),\ t \in T,$$

where $g : \mathbb{R} \to Im(g) \subseteq \mathbb{R}$ is a measurable function.

Expansions in Hermite polynomials

Let $\varphi(x)$ be the density and $\Phi(x)$ the c.d.f. of the standard normal law. Hermite polynomials H_n

- are defined by $H_n(x) = (-1)^n \frac{\varphi^{(n)}(x)}{\varphi(x)}$, $n = 0, 1, 2, \ldots$
- are polynomials of degree n: $H_0(x) = 1$,

$$H_1(x) = x, \quad H_2(x) = x^2 - 1, \quad H_3(x) = x^3 - 3x, \ldots$$

- form an orthogonal basis of the Hilbert space of square integrable with $e^{-\frac{x^2}{2}}$ functions with inner product

$$\langle f, g \rangle_\varphi = \int_{-\infty}^{+\infty} f(x)g(x)\varphi(x)\,dx.$$

Hence, any function from this space has a series expansion w.r.t. Hermite polynomials.

Appendix: Expansions in Hermite polynomials

Lemma (Rozanov (1967))

Let Z_1, Z_2 be standard normal random variables with $\rho = cov(Z_1, Z_2)$, and let G be a function satisfying $E[G(Z_1)] = 0$ and $E[G^2(Z_1)] < +\infty$. Then

$$Cov(G(Z_1), G(Z_2)) = \sum_{k=1}^{\infty} \frac{\langle G, H_k \rangle_\varphi}{k!} \rho^k.$$

Assume $Y = \{Y_t,\ t \in T\}$ to be a stationary centered Gaussian real-valued random function with $Var(Y_t) = 1$ and $\rho(t) = Cov(Y_0, Y_t)$. Classical definition of LRD of $X = g(Y)$ with $C(t) = Cov(X_0, X_t) \geq 0,\ t \in T$ yields

$$\int_T |C(t)|\, dt = \int_T \sum_{k=1}^{\infty} \frac{\langle G, H_k \rangle_\varphi}{k!} \rho(t)^k\, dt = \sum_{k=1}^{\infty} \frac{\langle G, H_k \rangle_\varphi}{k!} \int_T \rho(t)^k\, dt = +\infty.$$

Appendix: Subordinated Gaussian random functions

Let $T \subseteq \mathbb{R}^d$, and ν_d be the Lebesgue measure on $\mathbb{R}^d$.

Theorem (Kulik, S., 2019)

Let X be a subordinated Gaussian random function defined by $X_t = g(Y_t)$, $t \in T$, where g is a right-continuous strictly monotone (increasing or decreasing) function. Assume $\nu_d(\{t \in T : \rho(t) = 1\}) = 0$. Let

$$b_k(\mu) = \left(\int_{Im(g)} H_k(g^-(u))\varphi(g^-(u))\, \mu(du) \right)^2$$

where g^- is the generalized inverse of g if g is increasing or of $-g$ if g is decreasing. Then X is SRD if for any finite measure μ

$$\sum_{k=1}^{\infty} \frac{b_{k-1}(\mu)}{k!} \int_T |\rho(t)| \rho(t)^{k-1}\, dt < +\infty.$$

Appendix: Subordinated Gaussian random functions, Remarks

- If $X_t = g(|Y_t|)$, $t \in T$, then the above SRD condition modifies to

$$\sum_{k=1}^{\infty} \frac{b_{2k-1}(\mu)}{(2k)!} \int_T \rho(t)^{2k}\, dt < +\infty. \qquad (2)$$

- LRD conditions can be formulated: e.g., X is LRD if $b_k(\delta_{u_0}) < +\infty$ for some $u_0 \in \mathbb{R}$ and all k, and the above series diverges to $+\infty$.

Appendix: Subordinated Gaussian random functions, Example

- Let $g(x) = e^{x^2/(2\alpha)}$, $T = \mathbb{R}^d$, $\alpha > 0$.
  For $\alpha \in (1, 2]$, $\mathrm{E}\, X_0 < \infty$, but $\mathrm{E}\, X_0^2 = +\infty$.
- One can show that $\frac{b_{2k-1}(\mu)}{(2k)!} = O\left(\frac{1}{\sqrt{k}}\right), \quad k \to +\infty.$
- For $\rho(t) \sim |t|^{-\eta}$ as $|t| \to +\infty$, $\eta > 0$, $X = e^{Y^2/(2\alpha)}$ is
  - LRD if $\eta \in (0, d/2]$, since then $\int_{\mathbb{R}^d} \rho^2(t)\, dt = +\infty$,
  - SRD if $\eta > d/2$, since

    $$\int_{\mathbb{R}^d} \rho^{2k}(t)\, dt = O(k^{-1}) \quad \text{as} \quad k \to +\infty,$$

    and the series (2) behaves as

    $$\sum_{k=1}^{\infty} \frac{1}{k^{3/2}} < +\infty.$$

- Hence, for $\eta \in (d/2, d)$ Y is LRD but $X = e^{Y^2/(2\alpha)}$ is SRD!