Bi-variate CARMA Random Fields

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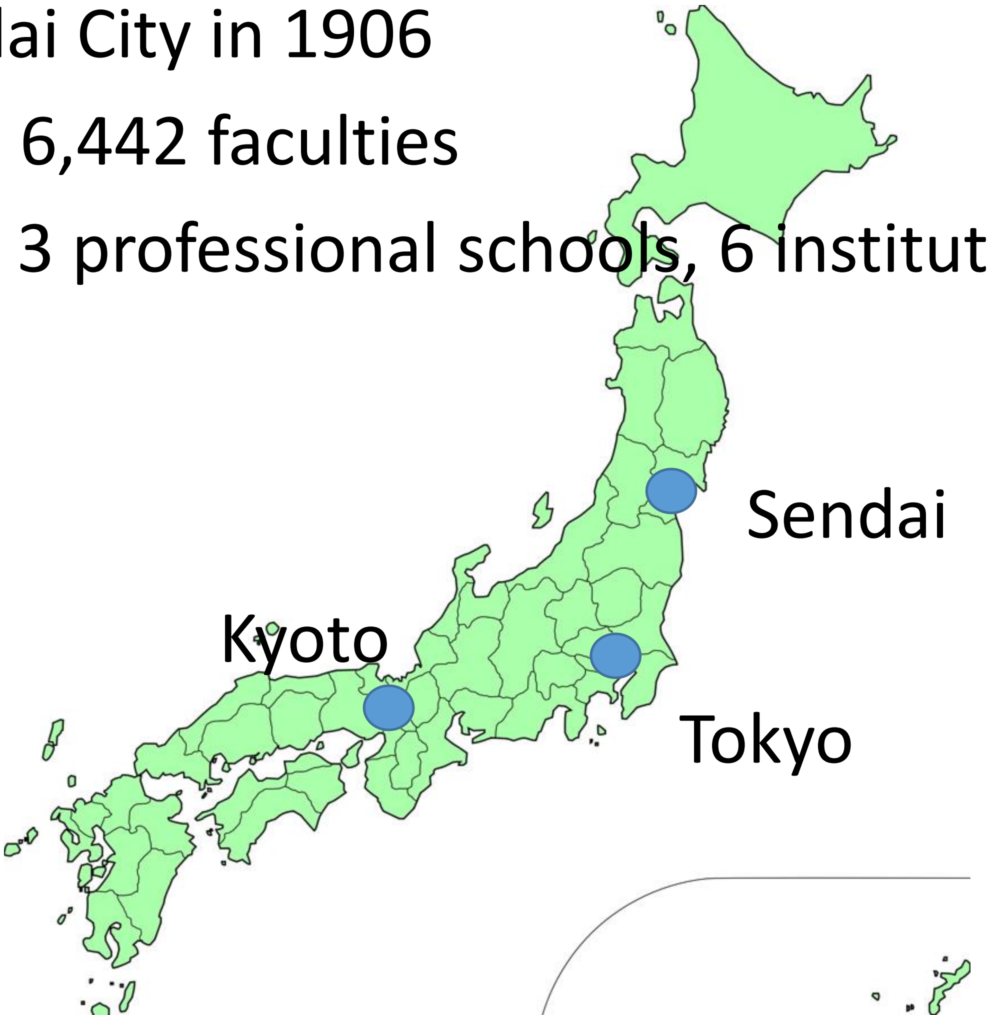
Tohoku University

- Partnership with Ulm U.
- Founded at Sendai City in 1906
- 17,865 students, 6,442 faculties
- 19 departments, 3 professional schools, 6 institutes

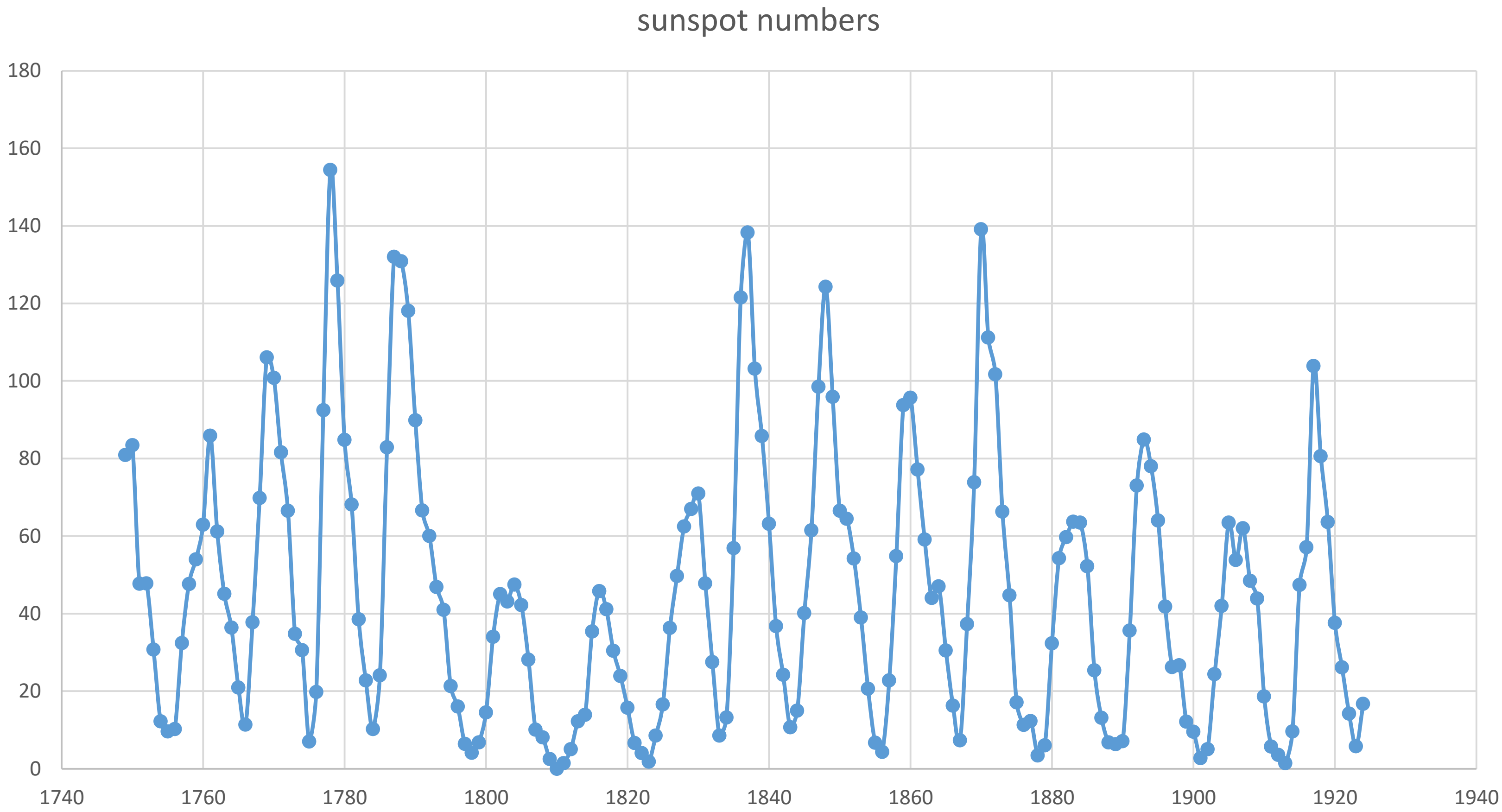


Einstein at Tohoku U. in Nov. 1922

Overview

1. Forecasting
    - Discrete time series
    - Extension to spatial data
2. CARMA random fields
3. Bivariate extension
4. Real example

Discrete time series



AR(2) model by Yule (1927)

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \varepsilon_t, \varepsilon_t \sim iid(0, \sigma^2)$$

Forecast by AR(2) model

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \varepsilon_t,$$
$$\varepsilon_t \sim iid(0, \sigma^2), t = 1, \cdots, T.$$

Estimating the parameters, we forecast x_{T+1} by
$$\hat{x}_{T+1} = \hat{\phi}_1 x_T + \hat{\phi}_2 x_{T-1}$$

Forecast by MA(2) model

$$x_t = \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2},$$
$$\varepsilon_t \sim iid(0, \sigma^2), t = 1, \cdots, T.$$

Estimating the paramers, we forecast x_{T+1} by

$$\hat{x}_{T+1} = \hat{\theta}_1 \hat{\varepsilon}_T + \hat{\theta}_2 \hat{\varepsilon}_{T-1}$$

with the estimate parameter, where

$$\hat{\varepsilon}_t = x_t - \hat{\theta}_1 \hat{\varepsilon}_{t-1} - \hat{\theta}_2 \hat{\varepsilon}_{t-2}, t = 3, \cdots, T,$$

by initializing $\hat{\varepsilon}_1 = \hat{\varepsilon}_2 = 0$

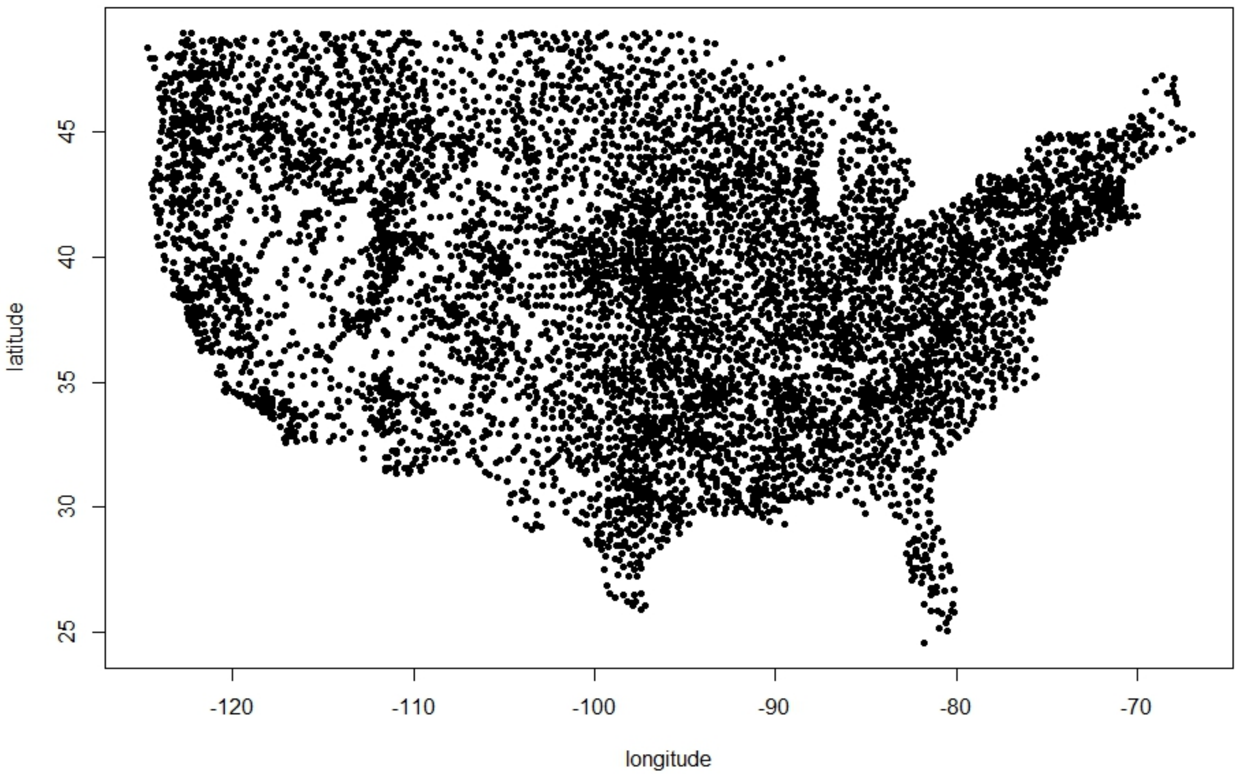
Re-interpret MA Models

$$x_t = \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2},$$

- $(\varepsilon_1, \cdots, \varepsilon_T) \sim iid N(0, \sigma^2)$: prior
- Estimate the parameter θ by the marginal likelihood
- Forecast by the posterior

US precipitation data

lon	lat	Jan	Feb
−85.25	31.57	13.3	12.4
−87.18	34.22	18.1	6.7
−87.32	34.42	NA	NA
−87.42	32.23	14.3	10.3
−86.22	34.25	NA	NA
−85.95	32.95	18.5	11
−85.87	32.98	NA	NA
−88.13	33.13	NA	NA
−88.28	33.23	26.1	6.8
−86.5	31.32	14.3	11.9
−85.85	33.58	20.3	9.1



Forecast NAs by bi-variate CARMA models

Continuous CARMA random fields

- Introduction of CARMA time series models
- Spatial extension
- bi-variate extension
- Estimation of parameters by Whittle likelihood (poster)
- Forecast (kriging) for missing components

ARMA(2,1) models

$$X_t + \phi_1 X_{t-1} + \phi_2 X_{t-2} = \varepsilon_t + \theta_1 \varepsilon_{t-1},$$
$$\varepsilon_t \sim iid(0, \sigma^2)$$

State Space Expression

$$\begin{pmatrix} Y_{t-1} \\ Y_t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\phi_2 & -\phi_1 \end{pmatrix} \begin{pmatrix} Y_{t-2} \\ Y_{t-1} \end{pmatrix} + \begin{pmatrix} 0 \\ \varepsilon_t \end{pmatrix},$$

$$X_t = Y_t + \theta_1 Y_{t-1}$$

CARMA(2,1) model

$$dY_t = \begin{pmatrix} 0 & 1 \\ -\phi_2 & -\phi_1 \end{pmatrix} Y_t dt + \begin{pmatrix} 0 \\ 1 \end{pmatrix} dL_t,$$

$$X_t = (\theta \quad 1) Y_t$$

AR polynomial

$\phi(z) = z^2 + \phi_1 z + \phi_2$ has roots wth $Re(z) < 0$

MA polynomal

$$\theta(z) = z + \theta_1$$

Lévy sheet on R^d

$\dot{L}(A), A \in B\left(R^d\right)$, is said to be a Levy measure

1. $\dot{L}(A)$ is a random variable with variance $|\mathrm{A}|$.
2. $A \bigcap B = \phi$, $A, B \in B\left(R^d\right) \Rightarrow \dot{L}(A), \dot{L}(B)$ are independent.

Levy sheet is defined as $L(t) = \dot{L}([0,t]), t \in R^d$

Example:

1. Brownian sheet
2. Compound Poisson sheet

Moving Average representation

$$X_t = \int_R g_\theta(t-u)dL(u)\,,$$

where

$$\phi(z) = (z-\lambda_1)\times\cdots\times(z-\lambda_p),$$
$$\theta(z) = (z-\mu_1)\times\cdots\times(z-\mu_q),$$

and

$$g(t) = \frac{1}{2\pi i}\int_C \frac{\theta(z)}{\phi(z)}e^{tz}dz,$$

$$= e^{\lambda t}, t \geq 0, \text{ for CAR(1)}$$

$$= \frac{\lambda_1-\mu_1}{\lambda_1-\lambda_2}e^{\lambda_1 t} + \frac{\lambda_2-\mu_1}{\lambda_2-\lambda_1}e^{\lambda_2 t}, t \geq 0, \text{ for CARMA(2,1)}$$

Spatial extension by Brcokwell and Matsuda (2017)

$$Y(s) = \int_{R^2} g_\theta(s-u)dL(u), s \in R^2$$

$$\phi(z) = (z^2 - \lambda_1^2) \times \cdots \times \left(z^2 - \lambda_p^2\right)$$
$$\theta(z) = (z^2 - \mu_1^2) \times \cdots \times \left(z^2 - \mu_q^2\right)$$

$$g_\theta(s) = \frac{1}{2\lambda} e^{\lambda\|s\|}, \text{ for CAR(1)}$$

$$= \frac{\lambda_1^2-\mu_1^2}{2\lambda_1\left(\lambda_1^2-\lambda_2^2\right)} e^{\lambda_1\|s\|} + \frac{\lambda_2^2-\mu_1^2}{2\lambda_2\left(\lambda_2^2-\lambda_1^2\right)} e^{\lambda_2\|s\|}, \text{ for CARMA(2,1)}$$

Bi-variate extension

$$Y(s) = \int_{R^2} g_\theta(s-u)dL(u), s \in R^2$$

$$\phi(z) = (z^2I - A_1) \times \cdots \times (z^2I - A_p)$$
$$\theta(z) = (z^2I - B_1) \times \cdots \times (z^2I - B_q)$$

$$g(s) = \frac{1}{2\pi i}\int_C \phi(z)^{-1}\theta(z)\, e^{\|s\|z}dz$$

Bi-variate CAR(1) kernel

$\varphi(z) = z^2 I - \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, \; \theta(z) = I.$

$$\varphi^{-1}(z)\theta(z) = \frac{1}{(z^2 - a_1)(z^2 - a_4) - a_2 a_3} \begin{pmatrix} z^2 - a_4 & -a_2 \\ -a_3 & z^2 - a_1 \end{pmatrix},$$

where we assume

$$(z^2 - a_1)(z^2 - a_4) - a_2 a_3 = (z^2 - \lambda_1^2)(z^2 - \lambda_2^2),$$

with λ_1, λ_2 negative real parts.

$$g(s) = \frac{1}{2\lambda_1(\lambda_1^2 - \lambda_2^2)} \begin{pmatrix} \lambda_1^2 - a_4 & -a_2 \\ -a_3 & \lambda_1^2 - a_1 \end{pmatrix} e^{\lambda_1 \|s\|} + \frac{1}{2\lambda_2(\lambda_2^2 - \lambda_1^2)} \begin{pmatrix} \lambda_2^2 - a_4 & -a_2 \\ -a_3 & \lambda_2^2 - a_1 \end{pmatrix} e^{\lambda_2 \|s\|},$$

where $a_1 + a_4 = \lambda_1^2 + \lambda_2^2$, and $a_1 a_4 - a_2 a_3 = \lambda_1^2 \lambda_2^2$.

1.3. CARMA random fields:

$$Y(s) = \int_{R^2} g_\theta(s-u)L(du)$$

For $k = 1, \cdots, n,$

$$Y(s_k) = \sum_j g_\theta(s_k - u_j)\varepsilon_j + Z_k,$$

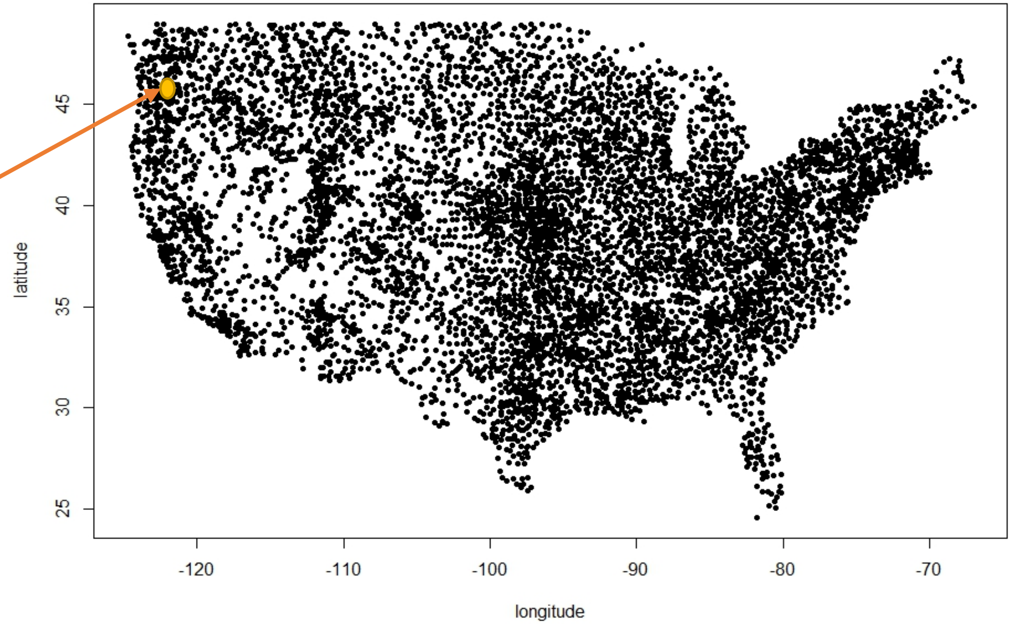
where $\{u_j, j = 1, \cdots, m\}$ random over D, and

$Z_k \sim iidN(0, \sigma^2)$

2. Forecast (kriging) by CARMA models

$$Y(s_k) = \sum_j g_\theta(s_k - u_j)\varepsilon_j + Z_k$$

Kriging for $Y(v)$



- unknown parameter θ ? ➡ MLE by marginalizing $\{u_j\}$, $\{\varepsilon_j\}$
- unobserved variables $\{u_j\}$, $\{\varepsilon_j\}$? ➡ Bayes approach

Empirical Bayes approach for CARMA models

$$Y(s_k) = \sum_j g_\theta(s_k - u_j)\varepsilon_j + Z_k$$

u_j~uniform dist. over D, $\varepsilon_j \sim \text{iid} N(0, \sigma^2), Z_k \sim \text{iid} N(0, \delta^2)$

- Marginal dist.

$$Y \sim N(0, \sigma^2 R(\theta) + \delta^2 I) \ ,$$

where $R(\theta)_{ij} = \int_{R^2} g_\theta(s_i - u) g_\theta(s_j - u) du$.

- Conditional dist. given $\{u_j\}, \{\varepsilon_j\}$,

$$Y \sim N(G_\theta(u)\varepsilon, \delta^2 I),$$

1. Estimate θ by MLE for the marginal distribution
2. Forecast by the conditional dist. by Gibbs sampling

Forecast by Gibbs Sampling

$$Y(s_k) = \sum_j g_\theta(s_k - u_j)\varepsilon_j + Z_k$$

u_j~uniform dist. over D,

ε_j~iid$N(0, \sigma^2)$, Z_k~iid$N(0, \delta^2)$

1. Simulate iid uniform $\{u_j\}$ over D
2. $\{\varepsilon_j\} \mid Y, \sigma^2, \delta^2, \{u_j\}$
3. $Y(v) \mid \{u_j\}, \{\varepsilon_j\}$

Real example:
ppt in (Dec.1996, Jan.1997) in USA

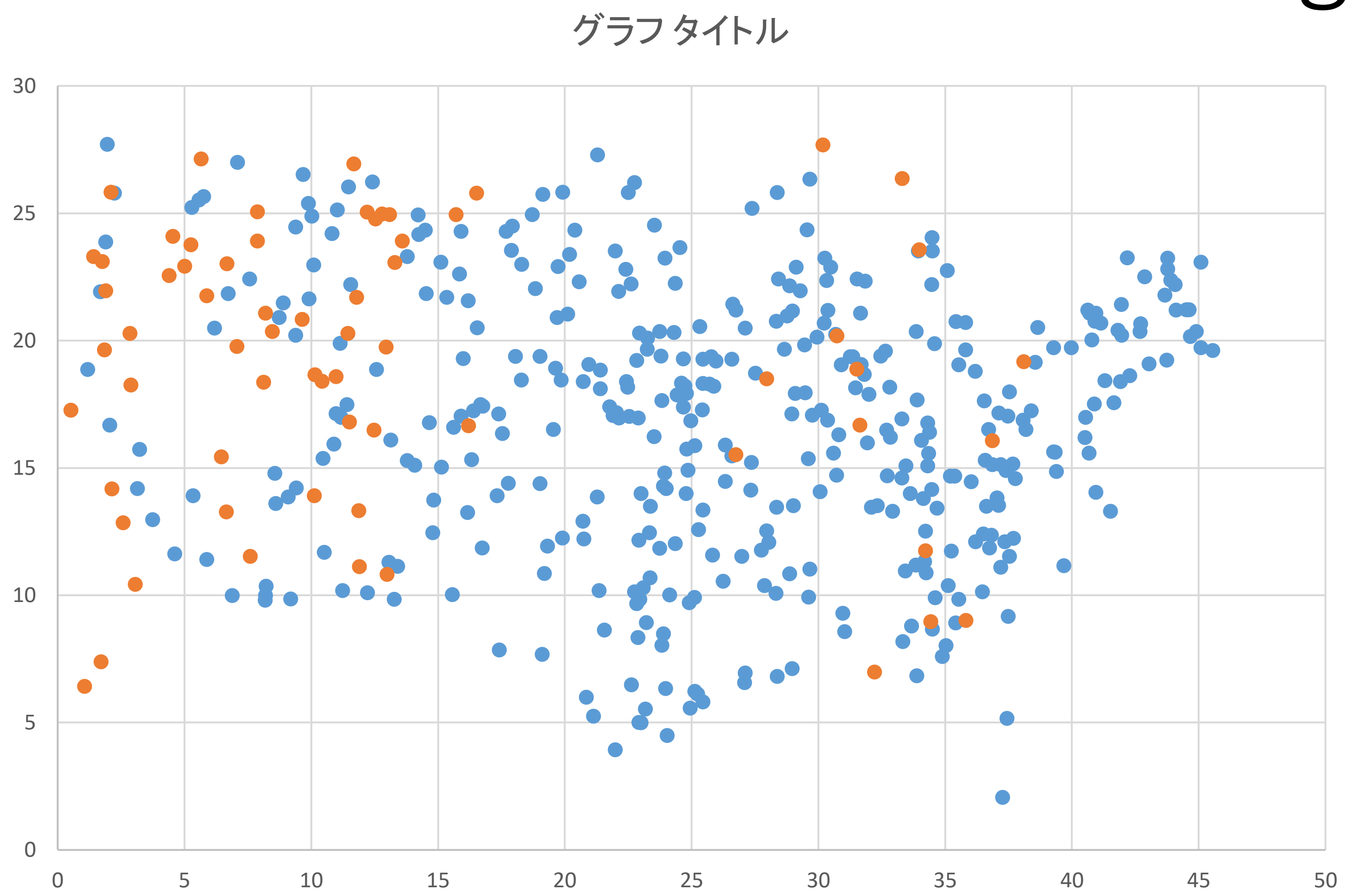
Kriging for Dec. 1996

		MSE		
	size	Bcarma	Ucarma	Loc. Ave.
in sample	6338	11.7	17.3	
out of sample	500	15.5	23.2	28.6

$$Y(s) = \int_{R^2} g_\theta(s-u)dL(u), s \in R^2$$

$$g(s) = \begin{pmatrix} 0.87 & -0.03 \\ 0.52 & 0.85 \end{pmatrix} e^{-3.82\|s\|} + \begin{pmatrix} 0.13 & 0.03 \\ 0.06 & 0.15 \end{pmatrix} e^{-0.61\|s\|}$$

Points for Better Bcarma kriging



	MSE of loc Av.
better bivariate	19.2
otherwise	6.8

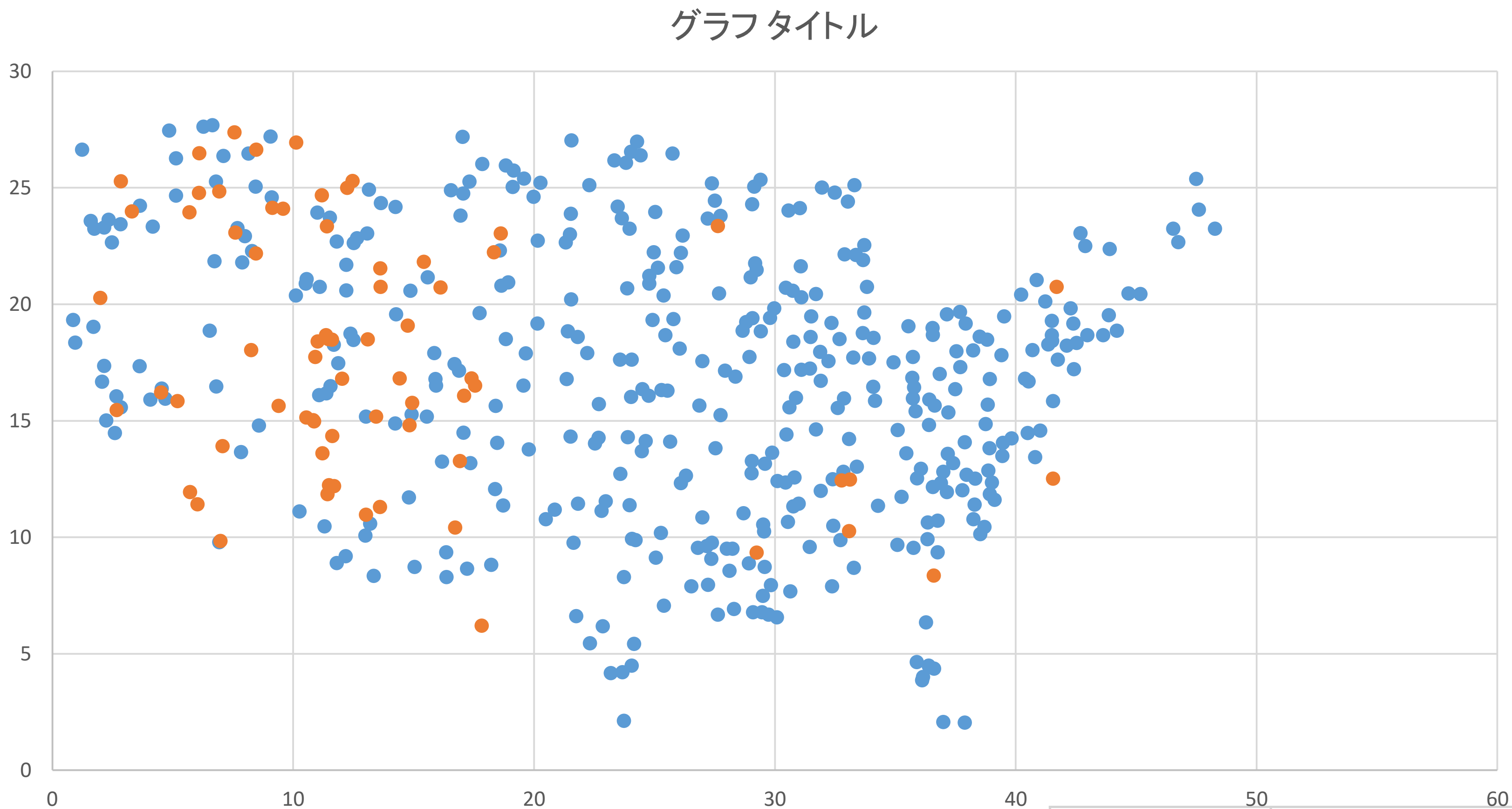
Real example 2:
(min_tmp, max_tmp) in Dec., 1996

		MSE		
	size	Bcarma	Ucarma	Loc. Ave.
in sample	4307	1.36	1.55	
out of sample	500	2.46	2.80	2.85

$$Y(s) = \int_{R^2} g_\theta(s-u)dL(u), s \in R^2$$

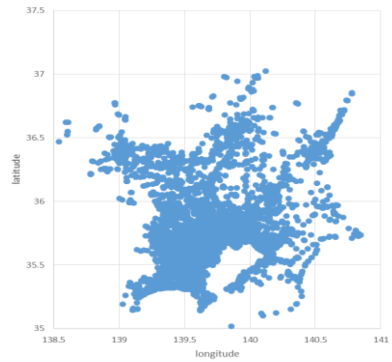
$$g(s) = \begin{pmatrix} 0.86 & -0.014 \\ 0.68 & 0.81 \end{pmatrix} e^{-2.19\|s\|} + \begin{pmatrix} 0.14 & 0.014 \\ 0.18 & 0.19 \end{pmatrix} e^{-0.25\|s\|}$$

Points for Better Bcarma kriging



	MSE_Loc.Av
bi-variate	16.53
otherwise	0.68

Summary



Discrete models for discrete samples

Continuous models for irregularly spaced samples

ARMA models by classical MLE

CARMA models by emp. Bayesian approach

Observations for $Y(s) = (Y_1(s), Y_2(s))$ are not necessarily synconized