Sampling Archimedean copulas

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Abstract

The challenge of efficiently sampling exchangeable and nested Archimedean copulas is addressed. Specific focus is put on large dimensions, where methods involving generator derivatives are not applicable. Additionally, new conditions under which Archimedean copulas can be mixed to construct nested Archimedean copulas are presented. Moreover, for some Archimedean families, direct sampling algorithms are given. For other families, sampling algorithms based on numerical inversion of Laplace transforms are suggested. For this purpose, the Fixed Talbot, Gaver Stehfest, Gaver Wynn rho, and Laguerre series algorithm are compared in terms of precision and runtime. Examples are given, including both exchangeable and nested Archimedean copulas.

Keywords

Exchangeable Archimedean copulas, nested Archimedean copulas, random number generation, inversion of Laplace transforms

1 Introduction

A distinct property of Archimedean copulas is that they are fully specified by some generator function. Important for modeling purposes is that Archimedean copulas are flexible to capture various dependence structures, e.g. concordance and tail dependence. This makes them especially suitable for the modeling of extreme events. Recently, nested Archimedean copulas gained increasing interest as they extend exchangeable Archimedean copulas to allow for asymmetries, an important property e.g. in financial applications. Besides practical applications, sampling high-dimensional copulas is also interesting from a theoretical perspective.

Different methodologies for sampling bivariate Archimedean copulas are known, e.g. the conditional distribution method or an approach based on the probability integral transformation, see Embrechts, Lindskog, McNeil (2001). The former generalizes to multivariate exchangeable Archimedean copulas, see Embrechts, Lindskog, McNeil (2001), and requires the knowledge of the first d derivatives of the generator of the d-dimensional

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Archimedean copula under consideration. Wu, Valdez, Sherris (2006) generalize the latter for sampling multivariate exchangeable Archimedean copulas. The resulting algorithm involves the first $d-1$ derivatives of the Archimedean generator. A similar approach is suggested by Whelan (2004). Recently, McNeil, Nešlehová (2007) presented a sampling algorithm for multivariate exchangeable Archimedean copulas which only involves the first $d-2$ derivatives of the generator. Still, a common drawback of all these sampling algorithms is that one has to know the involved generator derivatives. This becomes especially critical when nested Archimedean copulas are considered. The conditional distribution method is computationally impractical due to complex mixed derivatives, which is already a challenge for small dimensions, see Savu, Trede (2006) for this approach. Whelan (2004) tackles the problem of sampling nested Archimedean copulas similarly as for sampling exchangeable Archimedean copulas. His approach also requires high order derivatives, however, with respect to less variables than the conditional distribution method. In short, applicability of all these algorithms is strongly limited by the number of dimensions.

Considering the subclass of completely monotone Archimedean generators slightly simplifies the theory in that we have the well-known relation of generators to Laplace-Stieltjes transforms of distribution functions on the positive real line. Knowing the distribution corresponding to such a generator, Marshall, Olkin (1988) presented a sampling algorithm for exchangeable Archimedean copulas which does not require the knowledge of the copula density. This algorithm is therefore applicable to large dimensions. Algorithms for the few multivariate exchangeable Archimedean copulas that are straightforward to sample in large dimensions exploit the knowledge of the inverse Laplace-Stieltjes transform, see Joe (1997), page 375, for some examples. A generalization of the idea of Marshall, Olkin (1988) to nested Archimedean copulas and an elegant sampling algorithm was provided by McNeil (2007). However, for sampling in large dimensions, only one family is feasible, namely the Gumbel family. The reason for this is the lack of knowledge of the inverse Laplace-Stieltjes transforms of the involved generators. Further, no certain class of generators is known which can be nested such that sampling for its members is directly possible. Moreover, no example of generators belonging to different Archimedean families that can be mixed to build a nested Archimedean copula is known, which is interesting for practical applications, e.g. to allow for different kinds of tail dependence. The aim of this paper is to overcome some of these difficulties.

This paper is organized as follows. In Section 2 we present and discuss algorithms based on the inverse Laplace-Stieltjes transform for sampling exchangeable and nested Archimedean copulas. We also present techniques for the verification of the sufficient condition of McNeil (2007) for nested Archimedean structures to be proper copulas. We further obtain several new conditions which guarantee multivariate nested Archimedean copulas. For some families we obtain efficient sampling algorithms, including the cases of nested Ali-Mikhail-Haq, nested Frank, and nested Joe copulas. The section closes with examples of generators belonging to different Archimedean families that can be mixed to construct nested Archimedean copulas. In Section 3 we briefly introduce numerical inversion methods of Laplace transforms and present the Fixed Talbot, Gaver Stehfest, Gaver Wynn rho, and Laguerre series algorithm. Section 4 investigates and compares these algorithms in terms of precision and runtime using the Clayton family as reference. In Section 5 we present several examples that tackle unsolved problems. Finally, Section 6 concludes.

2 Sampling exchangeable and nested Archimedean copulas

2.1 Exchangeable Archimedean copulas

An *Archimedean generator* is a nonincreasing, continuous function $\psi : [0,\infty] \to [0,1]$ which satisfies $\psi(0) = 1$, $\psi(\infty) = 0$ and is strictly decreasing on $[0, \inf\{t : \psi(t) = 0\}]$. As McNeil, Nešlehová (2007) show, an Archimedean generator defines an *exchangeable Archimedean copula*, given by

$$C(\mathbf{u}) = C(u_1, \dots, u_d; \psi) = \psi(\psi^{-1}(u_1) + \cdots + \psi^{-1}(u_d)),\ \mathbf{u} \in [0,1]^d, \tag{1}$$

if and only if ψ is *d-monotone*, i.e. ψ is continuous on $[0,\infty]$, has derivatives up to the order $d-2$ satisfying $(-1)^k \frac{d^k}{dt^k}\psi(t) \ge 0$ for any $k \in \{0,\dots,d-2\}$, $t \in (0,\infty)$, and $(-1)^{d-2}\frac{d^{d-2}}{dt^{d-2}}\psi(t)$ being nonincreasing and convex on $(0,\infty)$. Unless stated otherwise we concentrate on the case where ψ is *completely monotone*, i.e. $(-1)^k \frac{d^k}{dt^k}\psi(t) \ge 0$ for any $k \in \mathbb{N}_0$, $t \in (0,\infty)$. The advantage of this condition is that we can exploit the relation of ψ to Laplace-Stieltjes transforms, see Kimberling's Theorem below. Some properties of completely monotone functions are summarized in the following lemma.

Lemma 2.1

(1) Completely monotone functions are closed under multiplications and linear combinations with positive coefficients, i.e. if f and g are completely monotone, so are fg and $\lambda f + \mu g$ for any $\lambda, \mu \ge 0$.

(2) If f is completely monotone, g nonnegative, and g' completely monotone, then $f \circ g$ is completely monotone.

(3) Let f be nonnegative, such that f' is completely monotone, then $1/f$ is completely monotone.

(4) If f is absolutely monotone, i.e. f is continuous on $[0,\infty]$ and satisfies $\frac{d^k}{dt^k} f(t) \ge 0$ for any $k \in \mathbb{N}_0$, $t \in (0,\infty)$, and g completely monotone, then $f \circ g$ is completely monotone.

(5) Let f be a Laplace-Stieltjes transform. Then f^α is completely monotone for any $\alpha \in (0,\infty)$ if and only if $(-\log f)'$ is completely monotone.

Proof

For Parts (1) and (4) of Lemma 2.1, see Widder (1946), page 145. Part (2) may be found in Feller (1971), page 441. Part (3) is a straightforward application of Part (2). Part (5) is proven in Joe (1997), page 374. □

Kimberling (1974) gave the following necessary and sufficient condition for an Archimedean generator to define a proper exchangeable Archimedean copula for any dimension larger than or equal to two.

Theorem 2.2 (Kimberling)
Let $\psi : [0, \infty] \to [0, 1]$ be continuous and strictly decreasing with $\psi(0) = 1$ and $\psi(\infty) = 0$. Then (1) is a copula for any $d \geq 2$ if and only if ψ is completely monotone.

According to the if part of Kimberling's Theorem, we may find generators ψ for exchangeable Archimedean copulas by using continuous functions $\psi : [0, \infty] \to [0, 1]$, satisfying $\psi(0) = 1$, $\psi(\infty) = 0$ and being completely monotone. In the sequel, the class of all such functions ψ will be referred to as Ψ_∞. Bernstein's Theorem, see Feller (1971), page 439, establishes the link between completely monotone functions and Laplace-Stieltjes transforms. It is the building block for efficient sampling algorithms for exchangeable and nested Archimedean copulas, especially in large dimensions.

Theorem 2.3 (Bernstein)
A function ψ on $[0, \infty]$ is the Laplace-Stieltjes transform of a distribution function if and only if ψ is completely monotone and $\psi(0) = 1$.

The following algorithm for sampling a d-dimensional exchangeable Archimedean copula with generator ψ is due to Marshall, Olkin (1988), where $\mathcal{LS}^{-1}(\psi)$ denotes the inverse Laplace-Stieltjes transform of ψ.

Algorithm 1 (Marshall, Olkin)
(1) Sample $V \sim F = \mathcal{LS}^{-1}(\psi)$.
(2) Sample i.i.d. $X_i \sim U[0, 1]$, $i \in \{1, \dots, d\}$.
(3) Return $(U_1, \dots, U_d)$, where $U_i = \psi(-\log(X_i)/V)$, $i \in \{1, \dots, d\}$.

If we know how to sample F, Algorithm 1 provides a powerful tool for sampling exchangeable Archimedean copulas. This algorithm is especially efficient in large dimensions, as only $d + 1$ random numbers are required for generating a d-dimensional observation. More precisely, only one sample $V \sim F$ is required, independent of the dimension.

For the famous cases of Clayton, Frank, Gumbel, and Joe, Joe (1997), page 375, obtained the corresponding inverse Laplace-Stieltjes transforms F as listed in Table 1, where $\Gamma(\alpha, \beta)$ denotes the Gamma distribution with shape parameter $\alpha \in (0, \infty)$, scale parameter $\beta \in (0, \infty)$, and density $\beta^\alpha x^{\alpha-1} e^{-\beta x}/\Gamma(\alpha)$, $x \in [0, \infty)$. The Stable distribution is denoted by $\mathrm{S}(\alpha, \beta, \gamma, \delta; 1)$, see Nolan (2007), page 8, with exponent $\alpha \in (0, 2]$, skewness parameter $\beta \in [-1, 1]$, scale parameter $\gamma \in [0, \infty)$, and location parameter $\delta \in \mathbb{R}$, with characteristic function $\exp(i\delta t - \gamma^\alpha |t|^\alpha (1 - i\beta \operatorname{sign}(t) w(t, \alpha)))$, $t \in \mathbb{R}$, where $\operatorname{sign}(t) = \mathbb{1}_{[0,\infty)}(t) - \mathbb{1}_{(-\infty,0]}(t)$, $w(t, \alpha) = \tan(\alpha\pi/2)$ for $\alpha \neq 1$, and $w(t, \alpha) = -2\log(|t|)/\pi$ for $\alpha = 1$. The parameter ranges listed in Table 1 correspond to where the generators are completely monotone. In the case of Ali-Mikhail-Haq's, Frank's, and Joe's family, F is discrete with given probability mass function $(y_k)_{k \in \mathbb{N}}$ at $k \in \mathbb{N}$.

Family	ϑ	$\psi(t)$	F
Ali-Mikhail-Haq	$[0,1)$	$\frac{1-\vartheta}{e^t-\vartheta}$	$y_k=(1-\vartheta)\vartheta^{k-1},\ k\in\mathbb{N}$
Clayton	$(0,\infty)$	$(1+t)^{-1/\vartheta}$	$\Gamma(1/\vartheta,1)$
Frank	$(0,\infty)$	$-(\log(e^{-t}(e^{-\vartheta}-1)+1))/\vartheta$	$y_k=\frac{(1-e^{-\vartheta})^k}{k\vartheta},\ k\in\mathbb{N}$
Gumbel	$[1,\infty)$	$\exp(-t^{1/\vartheta})$	$\mathrm{S}(1/\vartheta,1,(\cos(\frac{\pi}{2\vartheta}))^{\vartheta},0;1)$
Joe	$[1,\infty)$	$1-(1-e^{-t})^{1/\vartheta}$	$y_k=(-1)^{k+1}\binom{1/\vartheta}{k},\ k\in\mathbb{N}$

Table 1 Archimedean families with corresponding parameter ranges, generators, and inverse Laplace-Stieltjes transforms.

Unfortunately, it is not known how to find F explicitly. For a given Laplace-Stieltjes transform ψ, one might check tables of Laplace transforms to find F, e.g. Oberhettinger, Badii (1973). The case where F is a step function is covered by the following theorem.

Theorem 2.4
Let $\psi\in\Psi_\infty$ with $F=\mathcal{LS}^{-1}(\psi)$ and let $G(x)=\sum_{k=0}^\infty y_k\mathbb{1}_{[x_k,\infty)}(x)$, with $0<x_0<x_1<\dots$ and $y_k>0$, $k\in\mathbb{N}_0$, with $\sum_{k=0}^\infty y_k=1$. Then

$$F\equiv G\Leftrightarrow\psi(t)=\sum_{k=0}^\infty y_ke^{-x_kt},\ t\in[0,\infty].$$

Proof
The only if part of the claim follows from the definition of Laplace-Stieltjes transforms. For the if part of the statement we note that the Laplace-Stieltjes transform of G at $t\in[0,\infty]$ equals $\sum_{k=0}^\infty y_ke^{-x_kt}$, which in turn equals ψ, the Laplace-Stieltjes transform of F. The uniqueness theorem of inverse Laplace-Stieltjes transforms, see Doetsch (1970), page 32, implies that $F\equiv G$ up to a set of Lebesgue measure zero, which implies $F\equiv G$.□

An application of Theorem 2.4 leads to the probability mass functions of F for the families of Ali-Mikhail-Haq, Frank, and Joe as listed in Table 1. The computation involves a geometric series expansion for Ali-Mikhail-Haq's, a logarithmic series expansion for Frank's, and a binomial series expansion for Joe's generator.

Now assume as given an Archimedean generator $\psi\in\Psi_\infty$, for which we know $F=\mathcal{LS}^{-1}(\psi)$ explicitly. An interesting question is, if we can transform ψ to a generator ψ_0 in such a way that we still know $F_0=\mathcal{LS}^{-1}(\psi_0)$ in terms of F. The following theorem concerns such a result, where we note that the involved generator ψ_0 is simply a shifted and appropriately scaled ψ. In the sequel, we therefore refer to the resulting Archimedean copulas simply as *shifted Archimedean copulas*.

Theorem 2.5
Let $\psi\in\Psi_\infty$ with $F=\mathcal{LS}^{-1}(\psi)$. For a given $h\in[0,\infty)$, let $\psi_0(t)=\psi(t+h)/\psi(h)$ for any $t\in[0,\infty]$. Then

(a) $\psi_0(t)$, $t \in [0,\infty]$, is completely monotone and $\psi_0(0) = 1$.

(b) $F_0 = \mathcal{LS}^{-1}(\psi_0)$ is given by $F_0(x) = (F(0) + \int_0^x e^{-hu}\,dF(u))/\psi(h)$, $x \in [0,\infty)$.

(c) If F admits a density f, then F_0 admits the exponentially tilted density $f_0(x) = e^{-hx} f(x)/\psi(h)$, $x \in [0,\infty)$.

Proof

As $\psi \in \Psi_\infty$, Part (a) directly follows. For Part (b), let $g_1(x) = e^{-hx}F(x)$, $x \in [0,\infty)$. From (6) at the beginning of Section 3, we know that the Laplace transform of F_0 at t is given by

$$\begin{aligned}(\mathcal{L}(F_0))(t) = \frac{\psi_0(t)}{t} &= \frac{\psi(t+h)}{t+h}\,\frac{t+h}{t\psi(h)} = (\mathcal{L}(F))(t+h)\,(\mathcal{L}(g_2))(t)\\ &= (\mathcal{L}(g_1))(t)\,(\mathcal{L}(g_2))(t),\ t \in [0,\infty],\end{aligned}$$

with $g_2(x) = (\delta(x) + h)/\psi(h)$, $x \in [0,\infty)$, where δ denotes the Dirac delta function. Hence, $\mathcal{L}(F_0) = \mathcal{L}(g_1 * g_2)$, which implies $F_0 = g_1 * g_2$. Solving this convolution using partial integration for Riemann-Stieltjes integrals leads to the desired result. Part (c) follows from (b). □

The following rejection algorithm, see Devroye (1986), page 42, can be applied to sample the exponentially tilted density given in Part (c) of Theorem 2.5. However, we note that this approach can be slow, as the expected number of iterations in the algorithm is $1/\psi(h)$.

Algorithm 2

Repeatedly sample $V \sim f$ and $U \sim U[0,1]$, until $U \le e^{-hV}$, then return V.

2.2 Nested Archimedean copulas

The symmetry inherent in exchangeable Archimedean copulas is a strong restriction, especially for large dimensions, as the dependence among all components is identical; in other words, all common marginal distribution functions of the same dimension are equal. One way to extend exchangeable Archimedean copulas to allow for asymmetry is to use an important property of Archimedean copulas, namely associativity.

Extending the notation introduced in (1), we write

$$C(u_1,\dots,u_d;\psi_0,\dots,\psi_{d-2}) = \psi_0(\psi_0^{-1}(u_1) + \psi_0^{-1}(C(u_2,\dots,u_d;\psi_1,\dots,\psi_{d-2}))) \quad (2)$$

for any $d \ge 3$, $u_i \in [0,1]$, $i \in \{1,\dots,d\}$. The structure recursively defined via (2) is called *fully nested Archimedean copula*. For $d = 3$, this corresponds to

$$C(\mathbf{u}) = C(u_1, C(u_2,u_3;\psi_1);\psi_0) = \psi_0(\psi_0^{-1}(u_1) + \psi_0^{-1}(\psi_1(\psi_1^{-1}(u_2) + \psi_1^{-1}(u_3)))). \quad (3)$$

Note that a d-dimensional fully nested Archimedean copula is able to capture $d-1$ different pairwise dependencies, i.e. it possesses $d-1$ different bivariate margins. In real-world applications, it is often sufficient and convenient to combine structures (1) and (2). The resulting copulas are called *partially nested Archimedean copulas.* Although nesting is possible at any level, for notational convenience, we only consider the case

$$\begin{aligned} C(\mathbf{u}) &= C(C(u_{11},\dots,u_{1d_1};\psi_1),\dots,C(u_{s1},\dots,u_{sd_s};\psi_s);\psi_0) \\ &= \psi_0(\psi_0^{-1}(\psi_1(\psi_1^{-1}(u_{11})+\dots+\psi_1^{-1}(u_{1d_1})))+\dots \\ &\quad + \psi_0^{-1}(\psi_s(\psi_s^{-1}(u_{s1})+\dots+\psi_s^{-1}(u_{sd_s})))) \\ &= \psi_0\bigg(\sum_{i=1}^{s}\psi_0^{-1}\bigg(\psi_i\bigg(\sum_{j=1}^{d_i}\psi_i^{-1}(u_{ij})\bigg)\bigg)\bigg), \end{aligned} \tag{4}$$

$u_{ij}\in I$, $i\in\{1,\dots,s\}$, $j\in\{1,\dots,d_i\}$, where s is the number of *sectors* and $d=\sum_{i=1}^{s}d_i$ is the dimension. This copula allows for modeling $s+1$ different pairwise dependencies. The four-dimensional example with two sectors is given by

$$\begin{aligned} C(\mathbf{u}) &= C(C(u_{11},u_{12};\psi_1),C(u_{21},u_{22};\psi_2);\psi_0) \\ &= \psi_0(\psi_0^{-1}(\psi_1(\psi_1^{-1}(u_{11})+\psi_1^{-1}(u_{12})))+\psi_0^{-1}(\psi_2(\psi_2^{-1}(u_{21})+\psi_2^{-1}(u_{22})))). \end{aligned}$$

According to their structures, fully and partially nested Archimedean copulas are simply referred to as *nested* (or *hierarchical*) *Archimedean copulas.* The following theorem, see McNeil (2007), gives a sufficient condition for (2) being a proper copula. Similarly, for (4) being a proper copula, a sufficient condition is that nodes of the form $\psi_0^{-1}\circ\psi_i$ for any $i\in\{1,\dots,s\}$ have completely monotone derivatives.

Theorem 2.6 (McNeil)
Let $\psi_i\in\Psi_\infty$ for $i\in\{0,\dots,d-2\}$ such that $\psi_k^{-1}\circ\psi_{k+1}$ have completely monotone derivatives for any $k\in\{0,\dots,d-3\}$, then $C(u_1,\dots,u_d;\psi_0,\dots,\psi_{d-2})$ is a copula.

Based on the idea of Algorithm 1, McNeil (2007) proposed a sampling strategy for nested Archimedean copulas. The principal idea of this algorithm is to apply Algorithm 1 iteratively. In each step, one samples the distribution functions associated with the Laplace-Stieltjes transforms of certain generators. These generators are, due to the nodes, of the form $\psi_{i,j}(t;V_0)=\exp(-V_0\psi_i^{-1}\circ\psi_j(t))$, where V_0 is a given sample of the distribution function F_0 from an earlier step, or outer structure, and i and j refer to the node under consideration. For (2), a recursive algorithm is given as follows.

Algorithm 3 (McNeil)

(1) Sample $V_0\sim F_0=\mathcal{LS}^{-1}(\psi_0)$.

(2) Sample $(X_2,\dots,X_d)\sim C(u_2,\dots,u_d;\psi_{0,1}(\cdot;V_0),\dots,\psi_{0,d-2}(\cdot;V_0))$.

(3) Sample $X_1\sim U[0,1]$.

(4) Return $(U_1,\dots,U_d)$, where $U_i=\psi_0(-\log(X_i)/V_0)$, $i\in\{1,\dots,d\}$.

The following algorithm of McNeil (2007) is for sampling the partially nested Archimedean copula (4).

Algorithm 4 (McNeil)

(1) Sample $V_0 \sim F_0 = \mathcal{LS}^{-1}(\psi_0)$.

(2) For $i \in \{1,\dots,s\}$, sample $(X_{i1},\dots,X_{id_i}) \sim C(u_{i1},\dots,u_{id_i};\psi_{0,i}(\cdot;V_0))$ using Algorithm 1.

(3) Return $(U_{11},\dots,U_{sd_s})$, where $U_{ij} = \psi_0(-\log(X_{ij})/V_0)$, $i \in \{1,\dots,s\}$, $j \in \{1,\dots, d_i\}$.

Assume as given Archimedean generators ψ_i belonging to the Gumbel family with parameters ϑ_i, $i \in \{0,\dots,d-2\}$, satisfying $1 \le \vartheta_0 \le \dots \le \vartheta_{d-2}$. By Joe (1997), page 375, it follows that (2) and (4) are proper copulas. It is readily verified that the generators $\psi_{0,j}(t;V_0)$, $j \in \{1,\dots,d-2\}$, are given by $\psi_{0,j}(t;V_0) = \exp(-V_0 t^{\alpha_j})$, $\alpha_j = \vartheta_0/\vartheta_j$, which corresponds to a $\mathrm{S}(\alpha_j, 1, (\cos(\frac{\pi}{2}\alpha_j)V_0)^{1/\alpha_j}, 0; 1)$ distribution. For Step (2) of Algorithm 3 or 4, one may alternatively sample the copula generated by $\psi_{0,j}(t;1) = \exp(-t^{\alpha_j})$, corresponding to a $\mathrm{S}(\alpha_j, 1, (\cos(\frac{\pi}{2}\alpha_j))^{1/\alpha_j}, 0; 1)$ distribution, since for an Archimedean generator $\psi(t)$, $\psi(ct)$ generates the same copula for any $c \in (0,\infty)$. Hence for Gumbel's family, the algorithm works efficiently. Unfortunately, this is the only known case where sampling a nested Archimedean copula of large dimension is feasible.

Verifying the condition of Theorem 2.6 for a nested Archimedean copula with generators belonging to the same family is often unproblematic. Joe (1997), page 375, already gave sufficient conditions for the families of Clayton, Frank, Gumbel, and Joe. For either of these, if ψ_0 and ψ_1 are generators for two members of the involved family with parameters ϑ_0 and ϑ_1, respectively, then $(\psi_0^{-1} \circ \psi_1)'$ is completely monotone if $\vartheta_0 \le \vartheta_1$. Table 2 lists the Archimedean generators whose inverses are completely monotone on the given parameter ranges that may be found in Nelsen (1998), pages 94-97, with corresponding numbering. For Clayton's family, we use a different generator. Further, this table contains conditions on the parameters ϑ_0 and ϑ_1 under which $(\psi_0^{-1} \circ \psi_1)'$ is completely monotone, where ϑ_i denotes the parameter of ψ_i as before. All conditions may be obtained by applying Lemma 2.1 and checking no. 14 additionally involves the binomial theorem.

For each of the families of Ali-Mikhail-Haq, Frank, and Joe with generator $\psi = \mathcal{LS}(F)$, F is discrete. In the following theorem, we show that for the corresponding nested Archimedean copulas, the inverse Laplace-Stieltjes transforms of the involved inner generators are also discrete.

Theorem 2.7

(a) If $\psi_i(t) = (1-\vartheta_i)/(e^t - \vartheta_i)$, $t \in [0,\infty]$, with $\vartheta_i \in [0,1)$ for $i \in \{0,1\}$ such that $\vartheta_0 \le \vartheta_1$, then $\psi_{0,1}(t;V_0)$, with $V_0 \in \mathbb{N}$, has inverse Laplace-Stieltjes transform $F_{0,1}(x) = \sum_{k=V_0}^{\infty} y_k \mathbb{1}_{[x_k,\infty)}(x)$ with

$$x_k = k,\ y_k = \frac{c_2^{k-V_0}}{c_1^k}\binom{k-1}{k-V_0},\ k \in \mathbb{N}\backslash\{1,\dots,V_0-1\},$$

Family	ϑ_i	$\psi_i(t)$	$(\psi_0^{-1} \circ \psi_1)'$ c.m.
1	$(0,\infty)$	$(1+t)^{-1/\vartheta_i}$	$\vartheta_0, \vartheta_1 \in (0,\infty) : \vartheta_0 \le \vartheta_1$
3	$[0,1)$	$(1-\vartheta_i)/(e^t - \vartheta_i)$	$\vartheta_0, \vartheta_1 \in [0,1) : \vartheta_0 \le \vartheta_1$
4	$[1,\infty)$	$\exp(-t^{1/\vartheta_i})$	$\vartheta_0, \vartheta_1 \in (0,\infty) : \vartheta_0 \le \vartheta_1$
5	$(0,\infty)$	$-(\log(e^{-t}(e^{-\vartheta_i}-1)+1))/\vartheta_i$	$\vartheta_0, \vartheta_1 \in [1,\infty) : \vartheta_0 \le \vartheta_1$
6	$[1,\infty)$	$1-(1-e^{-t})^{1/\vartheta_i}$	$\vartheta_0, \vartheta_1 \in [1,\infty) : \vartheta_0 \le \vartheta_1$
12	$[1,\infty)$	$(1+t^{1/\vartheta_i})^{-1}$	$\vartheta_0, \vartheta_1 \in [1,\infty) : \vartheta_0 \le \vartheta_1$
13	$[1,\infty)$	$\exp(1-(1+t)^{1/\vartheta_i})$	$\vartheta_0, \vartheta_1 \in [1,\infty) : \vartheta_0 \le \vartheta_1$
14	$[1,\infty)$	$(1+t^{1/\vartheta_i})^{-\vartheta_i}$	$\vartheta_0, \vartheta_1 \in [1,\infty) : \vartheta_0 \in \mathbb{N}, \vartheta_1/\vartheta_0 \in \mathbb{N}$
19	$(0,\infty)$	$\vartheta_i/\log(t+e_i^{\vartheta})$	$\vartheta_0, \vartheta_1 \in (0,\infty) : \vartheta_0 \le \vartheta_1$
20	$(0,\infty)$	$(\log(t+e))^{-1/\vartheta_i}$	$\vartheta_0, \vartheta_1 \in (0,\infty) : \vartheta_0 \le \vartheta_1$

Table 2 Completely monotone Archimedean generators of Nelsen (1998), pages 94-97, with parameter ranges such that the condition of Theorem 2.6 holds.

where $c_1 = (1-\vartheta_0)/(1-\vartheta_1)$ and $c_2 = (\vartheta_1 - \vartheta_0)/(1-\vartheta_1)$. Hence, for the nested Archimedean family of Ali-Mikhail-Haq, $F_{0,1}$ is discrete with probability density function y_k at $k \in \mathbb{N}\backslash\{1,\dots,V_0-1\}$.

(b) If $\psi_i(t) = -(\log(e^{-t}(e^{-\vartheta_i}-1)+1))/\vartheta_i$, $t \in [0,\infty]$, with $\vartheta_i \in (0,\infty)$ for $i \in \{0,1\}$ such that $\vartheta_0 \le \vartheta_1$, then $\psi_{0,1}(t;V_0)$, with $V_0 \in \mathbb{N}$, has inverse Laplace-Stieltjes transform $F_{0,1}(x) = \sum_{k=V_0}^{\infty} y_k \mathbb{1}_{[x_k,\infty)}(x)$ with

$$x_k = k,\ y_k = \frac{c_2^k}{c_1^{V_0}} \sum_{j=0}^{V_0} \binom{V_0}{j}\binom{j\vartheta_0/\vartheta_1}{k}(-1)^{j+k},\ k \in \mathbb{N}\backslash\{1,\dots,V_0-1\},$$

where $c_i = 1 - e^{-\vartheta_i}$ for $i \in \{1,2\}$. Hence, for the nested Archimedean family of Frank, $F_{0,1}$ is discrete with probability density function y_k at $k \in \mathbb{N}\backslash\{1,\dots,V_0-1\}$.

(c) If $\psi_i(t) = 1-(1-e^{-t})^{1/\vartheta_i}$, $t \in [0,\infty]$, with $\vartheta_i \in [1,\infty)$ for $i \in \{0,1\}$ such that $\vartheta_0 \le \vartheta_1$, then $\psi_{0,1}(t;V_0)$, with $V_0 \in \mathbb{N}$, has inverse Laplace-Stieltjes transform $F_{0,1}(x) = \sum_{k=V_0}^{\infty} y_k \mathbb{1}_{[x_k,\infty)}(x)$ with

$$x_k = k,\ y_k = \sum_{j=0}^{V_0} \binom{V_0}{j}\binom{j\vartheta_0/\vartheta_1}{k}(-1)^{j+k},\ k \in \mathbb{N}\backslash\{1,\dots,V_0-1\}.$$

Hence, for the nested Archimedean family of Joe, $F_{0,1}$ is discrete with probability density function y_k at $k \in \mathbb{N}\backslash\{1,\dots,V_0-1\}$.

Proof

For Part (a), an application of the binomial series theorem leads to

$$\psi_{0,1}(t;V_0) = \sum_{k=V_0}^{\infty} \frac{c_2^{k-V_0}}{c_1^k}\binom{k-1}{k-V_0} e^{-kt}.$$

The conditions of Theorem 2.4 are easily verified. For Part (b), apply the binomial theorem and the binomial series theorem to observe that

$$\psi_{0,1}(t;V_0)=\sum_{k=0}^{\infty}\left(\frac{c_2^k}{c_1^{V_0}}\sum_{j=0}^{V_0}\binom{V_0}{j}\binom{j\vartheta_0/\vartheta_1}{k}(-1)^{j+k}\right)e^{-kt},$$

which we interpret as $\sum_{k=0}^{\infty} y_k e^{-kt}$. Checking $y_k \geq 0$ for $k \in \mathbb{N}_0$ is done by computing the corresponding generating function and by showing that it is absolutely monotone. Showing $y_k = 0$ for $k \in \{0, \dots, V_0 - 1\}$ is done by an inductive argument. Part (c) works similarly to Part (b). □

Instead of considering each generator individually, the following general result is shown under which we can construct a parametric family that can be nested, starting from any given completely monotone Archimedean generator.

Theorem 2.8
Let $\psi \in \Psi_\infty$. If $\psi_i(t) = \psi((c^{\vartheta_i} + t)^{1/\vartheta_i} - c)$, $t \in [0,\infty]$, with $\vartheta_i \in [1,\infty)$ for $i \in \{0,1\}$ and $c \in [0,\infty)$, then

(a) ψ_i is completely monotone and $\psi_i(0) = 1$ for $i \in \{0,1\}$.

(b) $(\psi_0^{-1} \circ \psi_1)(t) = (c^{\vartheta_1} + t)^{\vartheta_0/\vartheta_1} - c^{\vartheta_0}$, $t \in [0,\infty]$, so $(\psi_0^{-1} \circ \psi_1)'$ is completely monotone if $\vartheta_0 \leq \vartheta_1$.

Proof
Both statements are applications of Lemma 2.1. □

Remark 2.9
Based on the generator $\psi(t) = 1/(1+t)$ we obtain families no. 1, for $c = 1$, i.e. Clayton's family, and no. 12, for $c = 0$, from Theorem 2.8. Further, with $\psi(t) = 1/(1 + \log(1 + t))$ we obtain an equivalent generator for family no. 19, for $c = 1$. We notice the frequent appearance of the generator $\psi(t) = 1/(1 + t)$, a fact which was already addressed by Nelsen (2005) in his open question no. 5. Based on the generator of the independent copula, $\psi(t) = e^{-t}$, we obtain families no. 4, for $c = 0$, i.e. Gumbel's family, and no. 13, for $c = 1$. Also note that all generators of the form $\psi_i(t) = \psi(\frac{1}{\vartheta_i}\log(1 + t))$ for a $\psi \in \Psi_\infty$ and $\vartheta_i \in (0,\infty)$ are special cases of Theorem 2.8.

Sampling in the framework of Theorem 2.8 involves the inner distribution function $F_{0,1} = \mathcal{LS}^{-1}(\psi_{0,1})$, where $\psi_{0,1}(t;V_0) = \exp(-V_0((h + t)^\alpha - h^\alpha))$, $\alpha = \vartheta_0/\vartheta_1$, and $h = c^{\vartheta_1}$. Note that this is a shifted Archimedean generator, so we may apply Theorem 2.5, where the involved density f is the density of a $\mathrm{S}(\alpha, 1, (\cos(\frac{\pi}{2}\alpha)V_0)^{1/\alpha}, 0; 1)$ distribution as for Gumbel's family. As before, instead of sampling the copula generated by $\psi_{0,1}$, we may as well sample the copula generated by $\psi_{0,1}(V_0^{-1/\alpha}t;V_0) = \exp(-((\tilde{h}+t)^\alpha - \tilde{h}^\alpha))$, where $\tilde{h} = V_0{}^{1/\alpha}h$. In this case, the involved density f is the density of a $\mathrm{S}(\alpha, 1, (\cos(\frac{\pi}{2}\alpha))^{1/\alpha}, 0; 1)$ distribution and sampling the exponentially tilted Stable density $f_{0,1}$ corresponding to $F_{0,1}$ with Algorithm 2 requires $\exp(V_0 c^{\vartheta_0})$ iterations on average.

For a generator ψ of a two-dimensional Archimedean copula, it is known that $\psi_0(t) = (\psi(t))^{1/\alpha}$ for any $\alpha \in (0,1]$ as well as $\psi_1(t) = \psi(t^{1/\beta})$ for any $\beta \in [1,\infty)$ generate two-dimensional Archimedean copulas again. Families generated by ψ_0 and ψ_1 are referred to as *inner* and *outer power families*, respectively, where the unintuitive use of "inner" and "outer" relates to the fact that they were named with reference to generator inverses. An interesting question is, if and how these copula families generalize to our multivariate setting. For outer power families, Theorem 2.8 answers both questions, choose $c = 0$. Further, the simulation of nested Archimedean copulas based on an outer power family of a given generator ψ involves generators of the form $\psi_{0,1}(t; V_0) = \exp(-V_0 t^\alpha)$, $\alpha = \vartheta_0/\vartheta_1$. Hence, one can sample a Stable distribution as for Gumbel's family. As an example, consider generator no. 12, which is an outer power family based on Clayton's generator no. 1 with parameter equal to one. For inner power families, note that all generators of Table 2 fulfill that $\psi_0(t) = (\psi(t))^{1/\alpha}$ is completely monotone for any $\alpha \in (0,\infty)$. If our sole knowledge is that ψ is completely monotone, we still have that ψ_0 is completely monotone if $1/\alpha \in \mathbb{N}$, by Lemma 2.1 (4). Unfortunately, no general result on when we can construct nested Archimedean copulas based on inner power families is known. Note that the sufficient condition of Theorem 2.6 is usually not fulfilled, e.g. consider the generator $\psi(t) = (2e^t - 1)^{-1}$ of Ali-Mikhail-Haq's copula with parameter equal to one half and let $\psi_i(t) = (\psi(t))^{1/\vartheta_i}$ with $\vartheta_i \in (0,1]$ for $i \in \{0,1\}$ such that $\vartheta_0/\vartheta_1 = 1/2$. This implies $(\psi_0^{-1} \circ \psi_1)''(\log 5) > 0$, so $(\psi_0^{-1} \circ \psi_1)'$ is not completely monotone.

Generators ψ which are zero at some finite point can also be brought into play, as the following result shows, where $\psi^{-1}(0)$ is defined as $\inf\{t : \psi(t) = 0\}$.

Theorem 2.10
For $i \in \{0,1\}$, let $\psi_i : [0,\infty] \to [0,1]$ with $\psi_i(0) = 1$ and $\psi_i^{-1}(0) < \infty$ be completely monotone on $[0, \psi_i^{-1}(0)]$. If $\tilde{\psi}_i(t) = \psi_i(\psi_i^{-1}(0)(1 - e^{-t}))$, $t \in [0,\infty]$, for $i \in \{0,1\}$, then

(a) $\tilde{\psi}_i$ is completely monotone and $\tilde{\psi}_i(0) = 1$ for $i \in \{0,1\}$.

(b) $(\tilde{\psi}_0^{-1} \circ \tilde{\psi}_1)(t) = -\log(1 - \psi_0^{-1}(\psi_1((1-e^{-t})\psi_1^{-1}(0)))/\psi_0^{-1}(0))$, $t \in [0,\infty]$, so $(\tilde{\psi}_0^{-1} \circ \tilde{\psi}_1)'$ is completely monotone if $(1 - \psi_0^{-1}(\psi_1((1 - e^{-t})\psi_1^{-1}(0)))/\psi_0^{-1}(0))^\alpha$ is completely monotone for any $\alpha \in (0,\infty)$.

Proof
Both parts are applications of Lemma 2.1. □

Remark 2.11
The families of Frank, no. 5, and Joe, no. 6, fit in the framework of Theorem 2.10, as we can see by choosing $\psi_i(t) = (-\log(e^{-\vartheta_i} + t))/\vartheta_i$ for $t \in [0, 1 - e^{-\vartheta_i}]$, zero else, with $\vartheta_i \in (0,\infty)$ for $i \in \{0,1\}$, and $\psi_i(t) = 1 - t^{1/\vartheta_i}$ for $t \in [0,1]$, zero else, with $\vartheta_i \in [1,\infty)$ for $i \in \{0,1\}$, respectively. For these families, Part (b) of Theorem 2.10 is verified in Joe (1997), page 375. Another application of Theorem 2.10 is given by family no. 3, i.e. Ali-Mikhail-Haq's family. In this case, we use $\psi_i(t) = (1-t)(1-\vartheta_i)/(1-\vartheta_i(1-t))$ for $t \in [0,1]$, zero else, with $\vartheta_i \in [0,1)$ for $i \in \{0,1\}$. Further note that Part (b) of Theorem 2.10 is fulfilled by Lemma 2.1 Parts (1) and (3).

If generators of different Archimedean families are mixed, the sufficient condition of Theorem 2.6 does not always hold. For example, if ψ_0 and ψ_1 are generators for Clayton's and Gumbel's copula, respectively, a simple calculation shows that the condition $(\psi_0^{-1} \circ \psi_1)'$ being completely monotone is not fulfilled for any parameter choice. Further, $(\psi_1^{-1} \circ \psi_0)'$ being completely monotone only holds if the parameter of Gumbel's copula equals one, i.e. if ψ_1 generates the independent copula. Table 3 lists all combinations of the completely monotone generators of Nelsen (1998), pages 94-97, which result in proper nested Archimedean copulas according to the sufficient condition of Theorem 2.6.

Family combination	ϑ_0	ϑ_1	$(\psi_0^{-1} \circ \psi_1)'(t)$ c.m.
(1,12)	$(0,\infty)$	$[1,\infty)$	$\vartheta_0 \in (0,1]$
(1,14)	$(0,\infty)$	$[1,\infty)$	$\vartheta_0\vartheta_1 \in (0,1]$
(1,19)	$(0,\infty)$	$(0,\infty)$	$\vartheta_0 \in (0,1]$
(1,20)	$(0,\infty)$	$(0,\infty)$	$\vartheta_0 \leq \vartheta_1$
(3,1)	$[0,1)$	$(0,\infty)$	$\vartheta_1 \in [1,\infty)$
(3,19)	$[0,1)$	$(0,\infty)$	any ϑ_0, ϑ_1
(3,20)	$[0,1)$	$(0,\infty)$	any ϑ_0, ϑ_1

Table 3 Proper family combinations and corresponding parameter ranges for the Archimedean families of Nelsen (1998), pages 94-97.

3 Numerical inversion of Laplace transforms

In this section, we present numerical algorithms for inverting Laplace transforms to sample exchangeable and nested Archimedean copulas. This is especially encouraged by the fact that runtime for sampling Archimedean copulas primarily depends on the number of sectors and hardly on the dimension of the copula, since uniform random numbers are easily generated. Assume as given an Archimedean generator $\psi \in \Psi_\infty$ with

$$\psi(t) = \int_0^\infty e^{-tx}\, dF(x),\ t \in [0,\infty), \tag{5}$$

and let the distribution corresponding to F be denoted by $\mathbb{P}$. For $(\mathcal{L}(F))(t)$, i.e. the Laplace transform of F at t, an application of Tonelli's Theorem leads to

$$\begin{aligned} t(\mathcal{L}(F))(t) &= t\int_0^\infty e^{-tx} F(x)\,dx = t\int_0^\infty e^{-tx}\int_0^x d\mathbb{P}(u)\,dx \\ &= \int_0^\infty \int_u^\infty te^{-tx}\,dx\,d\mathbb{P}(u) = \int_0^\infty e^{-tu}\,d\mathbb{P}(u) = \int_0^\infty e^{-tu}\,dF(u) \\ &= \psi(t),\ t \in [0,\infty). \end{aligned}$$

Hence, in terms of Laplace transforms, F can be written as

$$F(x) = (\mathcal{L}^{-1}(\psi(t)/t))(x),\ x \in [0, \infty). \tag{6}$$

Our suggestion in what follows is to compute (6) numerically. For the required inversion of Laplace transforms, many algorithms are known, see Valkó, Vojta (2001). We briefly review the Fixed Talbot, Gaver Stehfest, Gaver Wynn rho, and Laguerre series algorithm, however, we note that these algorithms should rather serve as examples, as numerical inversion of Laplace transforms is an ill-posed problem, meaning that there is no uniformly best algorithm for all Laplace transforms.

3.1 Fixed Talbot, Gaver Stehfest, Gaver Wynn rho, and Laguerre series algorithm

The Fixed Talbot algorithm is an improved version of the original algorithm of Talbot (1979). The principal idea of this algorithm is to numerically evaluate the Bromwich inversion integral for Laplace transforms and to overcome some technical difficulties by using a certain choice for the contour; an excellent reference is Abate, Valkó (2004). In terms of our problem (6), the Fixed Talbot algorithm is efficiently implemented as follows, where we compute as much as possible only once, and as little as required for every value x where we would like to evaluate F. The approximation of F is denoted by $\tilde{F}$ and still depends on the choice of the parameter M of this algorithm, where Abate, Valkó (2004) suggest an implementation in which the number of decimal digits that can be represented in the mantissa should be at least M in order to prevent round-off errors.

Algorithm 5 (Fixed Talbot)

(1) Choose $M \in \mathbb{N}$.

(2) Compute $c = \frac{e^{0.4M}}{2M}$.

(3) For $k \in \{1, \ldots, M-1\}$, compute $\tilde{\vartheta}_k = \frac{k\pi}{M}$, $\tilde{s}_k = 0.4M\tilde{\vartheta}_k(\cot(\tilde{\vartheta}_k) + i)$, $\tilde{\sigma}_k = 1 + i(\tilde{\vartheta}_k + (\tilde{\vartheta}_k \cot(\tilde{\vartheta}_k) - 1)\cot(\tilde{\vartheta}_k))$ and finally $w_k = \frac{e^{\tilde{s}_k}\tilde{\sigma}_k}{\tilde{s}_k}$.

(4) For any x, return $\tilde{F}(x) = c\psi(0.4M/x) + \frac{0.4}{x}\sum_{k=1}^{M-1} \mathrm{Re}(w_k x \psi(\tilde{s}_k/x))$.

Note that the Fixed Talbot algorithm requires ψ at complex arguments. The extension of ψ to the complex plane is usually quite simple, as the Laplace-Stieltjes transform of F is a holomorphic function on the right half-plane U defined by the abscissa of convergence. It coincides with $\psi(s)$ (ψ considered as a function with domain $\mathbb{C}$) on the nonnegative real line intersected with U. Therefore, it suffices to find $c \in \mathbb{R}$ such that $\psi(s)$ is holomorphic for any $s \in \mathbb{C}$ with $\mathrm{Re}(s) > c$, as $\psi(s)$ is then the unique extension of $\psi(t)$ to $\mathbb{C}$ by the identity theorem for holomorphic functions.

Another theoretical inversion formula for Laplace transforms is the Post-Widder formula, see Widder (1946), page 277, which uses derivatives of the given Laplace transform.

Gaver (1966) obtained a similar formula, where derivatives are replaced by finite differences. For approximating the inverse Laplace transform F, he started with the delta convergent sequence

$$\delta_n(x,t) = \frac{(2n)!}{n!(n-1)!}\frac{\log 2}{x}2^{-n\frac{t}{x}}(1-2^{-\frac{t}{x}})^n$$

and computed the so-called *Gaver functionals*

$$\tilde{F}(x) = \int_0^\infty \delta_n(x,t)F(t)\,dt = \frac{(2n)!}{n!(n-1)!}\frac{\log 2}{x}\sum_{k=0}^{n}\binom{n}{k}(-1)^k(\mathcal{L}(F))\Big((n+k)\frac{\log 2}{x}\Big)$$

as an approximation of $F(x)$ for large n. The Gaver functionals are efficiently evaluated recursively, see Abate, Valkó (2004), however, they converge logarithmically to F. Based on the first $M \in \mathbb{N}$ Gaver functionals, Stehfest (1970) proposed to use the linear Salzer summation as convergence accelerator, which is known to be a regular sequence transformation for logarithmically convergent sequences, see Toeplitz's Theorem in Wimp (1981), page 27. The resulting Gaver Stehfest algorithm is widely used and known to be fast. According to detailed numerical experiments, Valkó, Abate (2004) suggest as choice for M that the number of decimal digits that can be represented in the mantissa should be at least $2M$. The Gaver Wynn rho algorithm uses a nonlinear convergence accelerator, given by a recursive scheme, to obtain a faster convergence for the Gaver functionals. According to Valkó, Abate (2004), this accelerator provides the best results among several accelerators. In our case, we can efficiently implement the Gaver Wynn rho algorithm as follows, where as precision requirement, Abate, Valkó (2004) suggest to use at least $2.1M$ as the number of decimal digits that can be represented in the mantissa, according to the chosen parameter M.

Algorithm 6 (Gaver Wynn rho)

(1) Choose $M \in 2\mathbb{N}$ and set $c = \log 2$.

(2) Allocate a matrix $A = (a_{k,j})$, $k \in \{0,\dots,M\}$, $j \in \{1,\dots,2M\}$ and a matrix $B = (b_{k,j})$, $k \in \{0,\dots,M+1\}$, $j \in \{0,\dots,M+1\}$ with $b_{0,j} = 0$ for $j \in \{0,\dots,M+1\}$ and $b_{0,0} = 0$.

(3) For any x, do

   (i) Compute $a_{k,k}$ for $k \in \{1,\dots,M\}$ via $a_{0,j} = \psi(cj/x)$ for $j \in \{1,\dots,2M\}$ and $a_{k,j} = (1+j/k)a_{k-1,j} - (j/k)a_{k-1,j+1}$ for $j \in \{k,\dots,2M-k\}$.

   (ii) Set $b_{1,j} = a_{j,j}$ for $j \in \{1,\dots,M\}$.

   (iii) For $k \in \{1,\dots,M\}$, set $b_{k+1,j} = b_{k-1,j+1} + \frac{k}{b_{k,j+1}-b_{k,j}}$ for $j \in \{0,\dots,M-k\}$.

   (iv) Return $\tilde{F}(x) = b_{M+1,0}$.

The principal idea of the Laguerre series algorithm is to expand the given Laplace transform of F in a series of the form $\sum_{k=0}^{\infty} q_k l_k(x)$, where $(l_k)_{k\in\mathbb{N}_0}$ are the Laguerre functions, see Abate, Choudhury, Whitt (1996). The Laguerre coefficients $(q_k)_{k\in\mathbb{N}_0}$ are then approximated by $(\tilde{q}_k)_{k\in\mathbb{N}_0}$ using the Lattice-Poisson algorithm and the infinite series is approximated by a finite sum. As a truncation point for this series, Abate, Choudhury, Whitt (1996) suggest to use $M = \min\{k_0, M_0\}$ where $k_0 = \min\{k \in \mathbb{N}_0 : |\tilde{q}_j| < \varepsilon \text{ for any } j \geq k\}$ with ε sufficiently small, say 10^{-8}, and $M_0 \in \mathbb{N}$ is a cut-off point, say 100. For further details about this algorithm, we refer to Abate, Choudhury, Whitt (1996). Note that one usually assumes bounded Laguerre coefficients. This condition is guaranteed if we consider the function $e^{-x}F(x)$, which has corresponding Laplace transform $\psi(1+t)/(1+t)$.

4 Comparison of the algorithms

In this section, we apply the Fixed Talbot, Gaver Stehfest, Gaver Wynn rho, and Laguerre series algorithm to evaluate F for Clayton's family with parameter $\vartheta = 0.8$, where we have a closed-form solution as reference. This corresponds to a Kendall's tau of 0.2857, which seems realistic for many applications. We compare the algorithms according to precision and runtime.

4.1 Precision and runtime

In order to compare the precision of the different algorithms, we evaluated the involved Gamma distribution function F at 10,000 equally spaced points, ranging from 0.001 to 10, the set of these points is denoted by P. As $F(0.001)$ is close to zero and $F(10)$ is close to one, these values represent the overall approximation of F quite well. Precision is measured using the *maximal relative error* MXRE. We consider a numerical procedure for inverting Laplace transforms to be accurate enough for our sampling problem (6) if MXRE satisfies

$$\text{MXRE}(x) = \max_{x\in P} \left| \frac{F(x) - \tilde{F}(x)}{F(x)} \right| < 0.0001. \tag{7}$$

We further report the *mean relative error* MRE, defined as

$$\text{MRE} = \frac{1}{|P|} \sum_{x\in P} \left| \frac{F(x) - \tilde{F}(x)}{F(x)} \right|.$$

Table 4 presents the results of our investigation for all four algorithms. The parameter choices for the algorithms are reported, which is M for the Fixed Talbot, Gaver Stehfest, and Gaver Wynn rho algorithm (chosen such that the precision requirements of Abate, Valkó (2004) hold) and ε for the Laguerre series algorithm. For the latter we do not use the bound M_0 in order to obtain a better insight in the precision of this algorithm. We further report eventual numerical difficulties if observed, i.e. values $\tilde{F}(x)$ which are not in the unit interval for at least one $x \in P$. Finally, MXRE and MRE, as well as runtimes for 1,000 evaluations of $\tilde{F}(x)$ for all points $x \in P$, are reported. In our study, the Fixed Talbot and Gaver Wynn rho algorithms perform best. For both algorithms, the smallest M that meets criterion (7) is $M = 6$ and the corresponding runtimes for the 10^7 function evaluations are comparably small. Especially the Fixed Talbot algorithm achieves excellent precision in small runtime.

Algorithm	Parameter	Difficulties	MXRE	MRE	Runtime
Fixed Talbot	1	no	1.64990123	0.30362935	2.61s
	2	yes	0.07726105	0.01880226	5.50s
	3	no	0.01039537	0.00384815	9.31s
	4	yes	0.00228519	0.00104870	13.00s
	5	no	0.00050258	0.00019882	16.93s
	6	no	0.00008386	0.00002621	22.19s
	7	no	0.00001308	0.00000516	29.90s
	8	no	0.00000280	0.00000119	55.41s
	9	no	0.00000064	0.00000020	63.29s
	10	no	0.00000008	0.00000004	70.49s
Gaver Stehfest	1	no	1.82544975	0.11408466	10.14s
	2	no	0.14920303	0.02117223	19.99s
	3	yes	0.01475319	0.00479624	30.15s
	4	yes	0.00309887	0.00145811	29.75s
	5	yes	0.00082358	0.00047443	48.32s
	6	no	0.00027974	0.00012669	61.31s
	7	no	0.00008557	0.00003468	73.95s
Gaver Wynn rho	2	no	0.36213746	0.03422636	10.37s
	4	yes	3.72470932	0.00352945	21.45s
	6	no	0.00007830	0.00002987	33.48s
Laguerre series	10^{-6}	yes	0.62016824	0.00072673	35.26s
	10^{-7}	yes	0.02712199	0.00008886	96.36s
	10^{-8}	yes	0.02291828	0.00001404	266.99s
	10^{-9}	no	0.00095077	0.00000158	746.33s
	10^{-10}	no	0.00015965	0.00000022	2111.75s

Table 4 Precision (implemented in `double` precision) and runtime in seconds for 1,000 evaluations of $\tilde{F}(x)$ for all $x \in P$.

4.2 A word concerning the implementation

All numerical experiments are run on a node containing two AMD Opteron 252 processors (2.6 GHz) with 8 GB RAM as part of a Linux cluster. The algorithms are implemented in C/C++ and compiled using GCC 3.3.3 (SuSE Linux) with option `-O2` for code optimization. Under `double` precision, the number of base-10 digits that can be represented in the mantissa is 15, reported by the function `digits10` of the C/C++ library `limits`. The command `gettimeofday` is used to measure runtime as wall-clock time. For generating uniform random numbers we use an implementation of the Mersenne Twister by Wagner (2003).

In order to find the quantile of a realization $U \sim U[0,1]$ close to one, we proceed as follows: If a numerical algorithm is not able to find the corresponding quantile, as the computed values $\tilde{F}(x)$ for reasonable large x do not reach U, we use truncation. For this, we specify a maximal value $x_{\max}$, where $\tilde{F}$ is evaluated. $\tilde{F}(x_{\max})$ is then used as truncation point. If $U \in [0, \tilde{F}(x_{\max})]$, a root finding procedure is used to find the U-quantile, otherwise $\tilde{F}(x_{\max})$ is returned. For numerical root finding, we use the function `nag_zero_cont_func_bd_1` of the Numerical Algorithms Group (NAG) and for each numerical experiment, we give our choices of the corresponding parameters `xtol` and `ftol`. Beside truncation, more sophisticated methods, e.g. transformations or asymptotics in the tail, may be applied.

5 Examples

In this section, we present several examples of how exchangeable and nested Archimedean copulas can be sampled based on the studied techniques. For all examples, we use three, ten, and one hundred dimensions, involving only one level of nesting, however, we note that all algorithms are applicable to sample more complex hierarchies, due to the recursive character of Algorithm 3. We further remark that for small dimensions, the conditional distribution method or other sampling algorithms may be faster; the reason why we investigate the three-dimensional case is to give justification for correctness of the generated random numbers. For this task, each dimension is subdivided into five equally-spaced bins. The corresponding grid partitions the three-dimensional unit cube into 125 cubes. For every three-dimensional example, 1,000 sets of samples, each of size 100,000, are taken and the p-value of the Kolmogorov-Smirnov test based on the 1,000 χ^2 test statistics is reported. For each example, all bins have expected number of observations greater than or equal to ten, being a rule of thumb for the χ^2 test. We further report pairwise sample versions of Kendall's tau. Note that plots may also serve as quality checks for detecting major flaws of numerical inversion algorithms. Whenever we sample an exchangeable Archimedean copula with generator $\psi = \mathcal{LS}(F)$, respectively a nested Archimedean copula involving $\psi_0 = \mathcal{LS}(F_0)$ and generators of the form $\psi_{0,1} = \mathcal{LS}(F_{0,1})$, we plot the approximation $\tilde{F}(x)$, respectively $\tilde{F}_0(x)$ and $\tilde{F}_{0,1}(x)$, based on all $x \in P$. For plotting $\tilde{F}_{0,1}$, reasonable realizations V_0 of F_0 are chosen. The plots are presented for the nested (Ali-Mikhail-Haq,Clayton) copula. We further report runtimes based on samples of size 100,000.

5.1 An exchangeable power Clayton copula

Consider the generator $\psi_0(t) = (\psi(t^{1/\beta}))^{1/\alpha}$, where $\psi(t) = 1/(1+t)$, i.e. ψ generates the Clayton copula with parameter equal to one. In this case, $\psi_0(t)$ is completely monotone for any $\alpha \in (0,1]$ and $\beta \in [1,\infty)$. For our example, we choose $\alpha = 0.5$ and $\beta = 1.5$. For accessing F_0, we use the Fixed Talbot algorithm with $M = 6$, $\texttt{xtol} = 10^{-8}$, and $\texttt{ftol} = 10^{-8}$. As truncation point, we choose $x_{\max} = 10^{16}$, resulting in $\tilde{F}_0(x_{\max}) = 0.999996$. We obtain 0.51 as p-value of the Kolmogorov-Smirnov test for the three-dimensional copula and pairwise sample versions of Kendall's tau are given by $\hat{\tau}_{1,2} = 0.4667$, $\hat{\tau}_{1,3} = 0.4650$, and $\hat{\tau}_{2,3} = 0.4685$, which are close to the theoretical value 0.4666. Runtimes for generating 100,000 observations of the three-, ten-, and one hundred-dimensional exchangeable power Clayton copulas are given by 16.16s, 16.76s, and 21.54s, respectively. We note that runtime is mainly determined by the evaluation of $\tilde{F}_0$.

5.2 A nested outer power Clayton copula

Again, we consider ψ from Clayton's family with parameter equal to one. We then construct an outer power family based on this generator, i.e. we use the generators $\psi_i(t) = \psi(t^{1/\vartheta_i})$, $i \in \{0,1\}$, with $\vartheta_0 = 1.1$, $\vartheta_1 = 1.5$, and construct nested Archimedean copulas with dimensions (1,2), (5,5), and (50,50), where the notation (1,2) corresponds to the three-dimensional structure as given in (3), and the cases (5,5) and (50,50) are constructed accordingly, involving only larger dimensions for the nonsectorial and the sector part. For these nested outer power copulas, we apply the Fixed Talbot algorithm with $M = 6$, $x_{\max} = 10^8$ (resulting in $\tilde{F}_0(x_{\max}) = 0.999996$), $\texttt{xtol} = 10^{-8}$, and $\texttt{ftol} = 0$ for sampling F_0. As the inner distribution function $F_{0,1}$ corresponds to a known Stable distribution, we do not need to apply a numerical inversion algorithm for sampling $F_{0,1}$. As p-value of the Kolmogorov-Smirnov test for the three-dimensional copula we obtain 0.81 and sample versions of Kendall's tau are given by $\hat{\tau}_{1,2} = 0.3945$, $\hat{\tau}_{1,3} = 0.3924$, and $\hat{\tau}_{2,3} = 0.5561$. The corresponding theoretical values are $\tau_{1,2} = \tau_{1,3} = 0.3939$ and $\tau_{2,3} = 0.5556$, where the former reflect the concordance of pairs of random variables with bivariate Archimedean copula generated by ψ_0 with parameter ϑ_0 and the latter reflects the concordance of a pair of random variables with bivariate Archimedean copula generated by ψ_1 with parameter ϑ_1. Runtimes for generating 100,000 observations of the three-, ten-, and one hundred-dimensional nested outer power Clayton copulas are given by 10.89s, 11.39s, and 17.57s, respectively.

5.3 A nested Ali-Mikhail-Haq copula

We now examine nested Archimedean copulas of types (1,2), (5,5), and (50,50), as before, constructed using generators of the Archimedean family of Ali-Mikhail-Haq with parameters $\vartheta_0 = 0.5$ and $\vartheta_1 = 0.9$, where F_0 is a geometric distribution. For sampling $F_{0,1}$, we precompute its values until $F_{0,1}(x) \geq 1 - \varepsilon$, where ε is chosen as 10^{-8}. The p-value of the Kolmogorov-Smirnov test for the three-dimensional copula is reported as 0.76 and sample versions of Kendall's tau are given by $\hat{\tau}_{1,2} = 0.1264$, $\hat{\tau}_{1,3} = 0.1292$, and $\hat{\tau}_{2,3} = 0.2792$, corresponding to the theoretical values $\tau_{1,2} = \tau_{1,3} = 0.1288$ and $\tau_{2,3} = 0.2782$. As runtimes for the three-, ten-, and one hundred-dimensional nested Ali-Mikhail-Haq copulas, we obtain 0.17s, 0.32s, and 2.40s, even outperforming the runtimes for nested Gumbel copulas of the same types (0.51s, 0.85s, and 5.46s), for which one only has to sample Stable distributions.

5.4 A nested (Ali-Mikhail-Haq,Clayton) copula

As before, consider nested Archimedean copulas of the form (1,2), (5,5), and (50,50), where ψ_0 is now a generator of Ali-Mikhail-Haq's copula with parameter $\vartheta_0 = 0.8$ and ψ_1 is a generator of Clayton's copula with parameter $\vartheta_1 = 2$. For the resulting nested mixed Archimedean copulas, we sample the known distribution function F_0 for Step (1) of Algorithm 3, and use the Fixed Talbot algorithm with $M = 6$, $x_{\max} = 10^4$, $\tilde{F}_{0,1}(x_{\max}) = 0.999996$, $\texttt{xtol} = 10^{-8}$, and $\texttt{ftol} = 0$ for sampling $F_{0,1}$. The p-value of the Kolmogorov-Smirnov test for the three-dimensional copula is reported as 0.53 and sample versions of Kendall's tau are given by $\hat{\tau}_{1,2} = 0.2349$, $\hat{\tau}_{1,3} = 0.2358$, and $\hat{\tau}_{2,3} = 0.5029$, corresponding to the theoretical values $\tau_{1,2} = \tau_{1,3} = 0.2337$ and $\tau_{2,3} = 0.5$. Runtimes for generating 100,000 observations of the three-, ten-, and one hundred-dimensional nested (Ali-Mikhail-Haq,Clayton) copulas are given by 7.54s, 7.78s, and 11.58s, respectively. Figure 1 shows plots of $\tilde{F}_{0,1}(x)$ for the samples $V_0 \in \{1, 2, 5, 10\}$ of F_0 and a scatter plot matrix for the three-dimensional nested (Ali-Mikhail-Haq,Clayton) copula based on 1,000 observations. We especially emphasize the different tail behavior of the generated data.

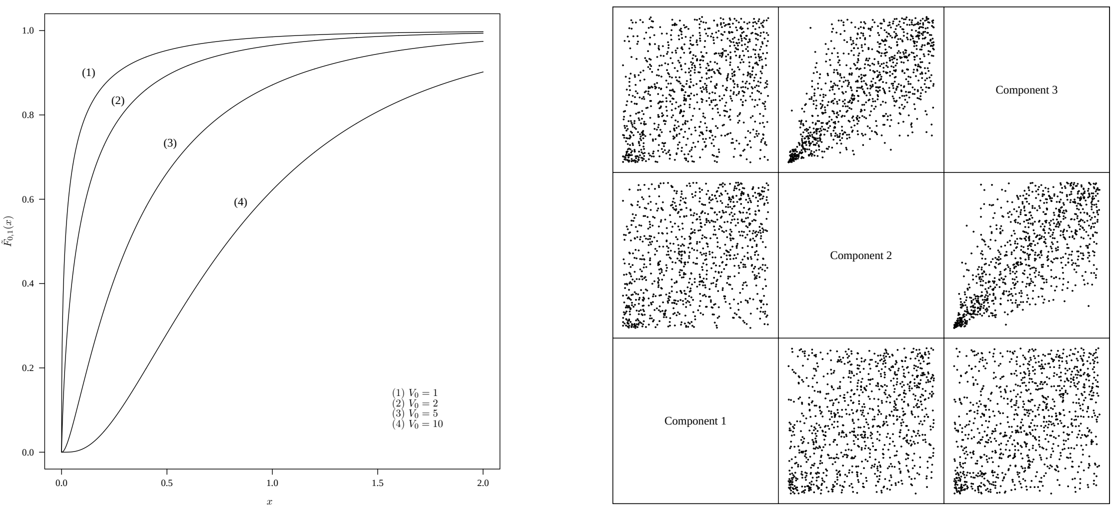

Figure 1 $\tilde{F}_{0,1}(x)$ with $V_0 \in \{1, 2, 5, 10\}$ and a scatter plot matrix for the nested (Ali-Mikhail-Haq,Clayton) copula.

We close this section by advising the reader to carefully use numerical inversion algorithms for Laplace transforms. Usually, these algorithms involve assumptions on the underlying functions, which are not easily checked or which can not be checked at all, as these functions are not directly given. Further, there is no guarantee that algorithms which worked well in the presented examples, are applicable to others. It is comparably easy to check the quality of an approximation $\tilde{F}$ to a distribution function F for an exchangeable Archimedean copula, as well as an approximation $\tilde{F}_0$ to an outer distribution function F_0 for a nested Archimedean copula. However, sampling an inner distribution function $F_{0,1}$ for a nested Archimedean copula using a numerical inversion procedure for Laplace transforms requires the algorithm to work accurately for all samples V_0 from F_0.

6 Conclusion

In this paper, we gave sufficient conditions for the parameters of the completely monotone generators listed in Nelsen (1998), pages 94-97, such that nested Archimedean copulas can be constructed. Many of these generators fall into two categories, for which Theorems 2.8 and 2.10 addressed conditions for nesting. We could generally relate the inner distribution function $F_{0,1}$ involved in Theorem 2.8 to an exponentially tilted Stable distribution by introducing shifted Archimedean generators. As a special case, we further obtained that outer power families extend to multivariate Archimedean copulas and that the inner distribution function $F_{0,1}$ corresponds to a Stable distribution, independent of the family under consideration. We also obtained probability mass functions for the inner distribution functions $F_{0,1}$ for the families of Ali-Mikhail-Haq, Frank, and Joe. These results allow to directly sample the corresponding nested Archimedean copulas. For sampling exchangeable and nested Archimedean copulas where F, respectively F_0 or $F_{0,1}$, can not be directly sampled, we suggested an approach based on numerical inversion of Laplace transforms. Using such algorithms for sampling exchangeable and nested Archimedean copulas in large dimensions is motivated by the fact that we only need a comparably small amount of random variates from the inverse Laplace-Stieltjes transforms of the involved generators. We investigated the Fixed Talbot, Gaver Stehfest, Gaver Wynn rho, and Laguerre series algorithm according to precision and runtime. Especially the Fixed Talbot algorithm turned out to be fast and highly accurate for the examples we considered. We further presented several examples of nested Archimedean copulas based on generators of different families. As a special case we sampled a nested (Ali-Mikhail-Haq,Clayton) copula showing different kinds of tail dependence. Our results are encouraging in that we are now able to sample both exchangeable and nested Archimedean copulas in many different cases. The short runtimes for sampling large dimensions allow to draw large amounts of multivariate random numbers of these copulas, which is required for many simulation studies based on Monte Carlo estimations.

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