The Solvency II Square-Root Formula for Systematic Biometric Risk

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The Solvency II square-root formula for systematic biometric risk

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Abstract

In this paper, we develop a model supporting the so-called square-root formula used in Solvency II to aggregate the modular life SCR. Describing the insurance policy by a Markov jump process, we can obtain expressions similar to the square-root formula in Solvency II by means of limited expansions around the best estimate. Numerical illustrations are given, based on German population data. Even if the square-root formula can be supported by theoretical considerations, it is shown that the QIS correlation matrix is highly questionable.

Key Words: Solvency capital requirement (SCR), Markov jump process (MJP), generalized life insurance, disability insurance, systematic risk, correlation.

1 Introduction and motivation

In line with the Basel II requirements for banks, the European Commission has established the Solvency II Directive for insurance companies. Solvency II is based on a three-pillar framework similar to Basel II. Pillar I focuses on capital requirements, such as market-consistent valuation of the balance sheet, including insurance liabilities and assets. Under Solvency II, two capital levels are determined: the minimum capital requirement, a threshold at which companies are no longer permitted to sell policies, and a solvency capital requirement (SCR) below which companies may need to discuss remedies with their regulator. The SCR is computed by means of a 99.5% Value-at-Risk.

The standard approach to the SCR can be applied by all insurers, irrespective of size, portfolio mix and geographical location. The standard formula given by the directive for the SCR calculation uses a modular approach with modules for

- non-life underwriting risk
- life underwriting risk
- health underwriting risk
- market risk
- counterparty default risk
- operational risk
- intangible assets risk.

The biometric risks considered in this paper are contained in the life underwriting risk module. The risk modules are in turn built on submodules. The life underwriting risk module consists of 7 submodules for

1. mortality risk (risk of increasing mortality)
2. longevity risk (risk of falling mortality)
3. disability or morbidity risk
4. lapse risk
5. life expense risk
6. revision risk
7. life catastrophe risk.

The first 6 categories are first aggregated and the result is then aggregated with the last life catastrophe risk. Modules 1-6 are aggregated to $SCR_\bullet$ by means of the so-called 'square-root formula'

$$SCR_\bullet = \sqrt{\sum_{i,j} Corr_{ij}\, SCR_i\, SCR_j}\,, \tag{1.1}$$

where SCR_i and SCR_j refer to the 6 sub-modules listed above and the pairwise correlation coefficients $Corr_{ij}$ come from a given correlation matrix. For example, the 5th Quantitative Impact Study (QIS5) uses the following correlation matrix:

	SCR_{mort}	SCR_{long}	SCR_{dis}	SCR_{lapse}	SCR_{exp}	SCR_{rev}
SCR_{mort}	1					
SCR_{long}	-0.25	1				
SCR_{dis}	0.25	0	1			
SCR_{lapse}	0	0.25	0	1		
SCR_{exp}	0.25	0.25	0.5	0.5	1	
SCR_{rev}	0	0.25	0	0	0.5	1

Finally, $SCR_{\bullet}$ and SCR_{cat} are aggregated by using again the square-root formula with correlation 0.25. In this paper, we only consider the first step, and more specifically, the aggregation of the three first categories SCR_{mort}, SCR_{long}, and SCR_{dis} using (1.1).

The technical specifications of QIS5 define the life submodules as 'the change in the net value of assets minus liabilities' due to shock scenarios. For the submodules SCR_{mort}, SCR_{long}, and SCR_{dis} the following scenarios are used:

1. (permanent) 15% increase in mortality rates for each age for SCR_{mort}
2. (permanent) 25% decrease in mortality rates for each age for SCR_{long}, and
3. 50% increase in disability rates for the next year, together with a (permanent) 25% increase (over best estimate) in disability rates at each age in following years plus, when applicable, a (permanent) 20% decrease in morbidity/disability recovery rates.

The standard formulas suffer from the two following shortcomings:

- The Solvency II directive defines SCR by means of the 99.5% Value-at-Risk but the standard formula for the life risk module SCR_{life} neither uses stochastic modeling nor are we aware of an equivalence result to a stochastic background model.
- The standard formula models dependencies between mortality, disability, lapsation, etc. not at the roots but on the level of the submodules SCR_{mort}, SCR_{long}, SCR_{dis}, etc. Because of this fact and the fact that the same correlation matrix is used for any kind of insurance portfolio, the standard formula does not cover the diversity of risk structures that insurance portfolios can have.

The standard formula developed by experienced actuaries with a lot of expertise may yield in practice a good approximation to the Value-at-Risk. However, we are not aware of any sound study supporting the standard formula. In the present paper, we propose a stochastic model that allows to theoretically substantiate (1.1). Moreover, we introduce an alternative concept for the calculation of SCR_{life} that models dependencies between mortality, disability, etc. directly on the level of the transition rates.

The paper proceeds as follows. In Section 2, we describe the multistate model for general insurance policies used in the present work. This allows us to quantify the effect of a departure of future transition rates (mortality rates, disability rates, and recovery rates, for instance) from best estimates. In Section 3, we fit a multivariate Lee-Carter model to German mortality, disability and recovery rates. This model serves as the basis to explore the impact of systematic biometric risk to determine the numerical values for the correlations between the different sources for this risk. Section 4 proposes a formal justification to the square-root formula (1.1) by means of expansion formulas derived in Section 2. Sections 5 and 6 discuss alternatives to the square-root formula, based on asymptotic approximations and upper bounds. The final Section 7 concludes.

2 Basic model of life insurance policies

Consider an insurance policy p that is driven by a Markovian jump process $(X_t^p)_t$ with finite state space $\mathcal{S}$ and transition space $J \subset \{(j,k) \in \mathcal{S}^2 \,|\, j \neq k\}$. We write x_p for the age of the policyholder at time zero (usually the beginning of the contract period) and $\omega_p = \omega - x_p$ where ω stands for the ultimate age.

The cash-flows of the contract are described by the following functions:

1. The lump sum $b_{jk}^p(x)$ is payable upon a transition of policyholder p from state j to state k at age x. We assume that the functions b_{jk}^p, $(j,k) \in J$, are well-behaved (i.e., have bounded variation on $[0, \omega_p]$).

2. The function $B_j^p(x)$ gives the accumulated annuity benefits minus accumulated premiums for a sojourn of policy p in state j during $[0, x]$. We assume that the functions B_j^p, $j \in \mathcal{S}$, are well-behaved (i.e., have bounded variation on $[0, \omega_p]$ and are right-continuous).

We write $v(s,t)$ for the value at time s of a unit payable at time $t > s$ and assume that it has a representation of the form

$$v(s,t) = e^{-\int_{(s,t]} \varphi(u)\,\mathrm{d}u}$$

with φ being the interest intensity (or short interest rate).

We assume that there exists a general transition intensity matrix $\mu(x;t) = (\mu_{jk}(x;t))_{(j,k)\in\mathcal{S}^2}$, where the first and second argument of μ are the age and the calendar time at which a transition takes place, such that the Markovian jump process $(X_t^p)_t$ has the transition intensity matrix $t \mapsto \mu(x_p + t; t)$. We write

$$p(x_p; s, t) = \Big(P(X_t^p = k \,|\, X_s^p = j) \Big)_{(j,k)\in\mathcal{S}^2}, \qquad 0 \leq s \leq t \leq \omega_p\,,$$

for the corresponding transition probability matrix, which is uniquely determined by the Kolmogorov forward equation

$$\frac{\mathrm{d}}{\mathrm{d}t} p(x_p; s, t) = p(x_p; s, t)\, \mu(x_p + t; t)\,, \quad s \leq t\,,$$

and starting values $p(x_p; s, s) = \mathbb{I}$ for all s, where $\mathbb{I}$ denotes the identity matrix. In the present paper, we build our model on log transition intensities, which means that the μ_{jk} are exponentials of auxiliary γ_{jk}, that is,

$$\mu_{jk}(x;t) = \exp\big(\gamma_{jk}(x;t)\big)\,, \quad (j,k) \in J\,. \tag{2.1}$$

Example 2.1 (Disability model)**.** In the present paper we focus on disability insurance for the numerical illustrations. The disability model has state space $\mathcal{S} = \{\mathrm{a}, \mathrm{i}, \mathrm{d}\}$, where a=active, i=invalid/disabled, d=dead, and transition space $J = \{(\mathrm{a}, \mathrm{i}), (\mathrm{a}, \mathrm{d}), (\mathrm{i}, \mathrm{a}), (\mathrm{i}, \mathrm{d})\}$. We consider the following insurance products:

(a) **Disability insurance.** A yearly disability annuity of 1000 is paid as long as the policyholder is in state disabled. A constant premium is paid yearly in advance as long as the insured is in state active.

(b) **Disability & pure endowment insurance.** Additionally to the disability benefits in (a), an endowment benefit of 10 000 is paid if the policyholder is still alive at termination of the contract.

(c) **Disability & temporary life insurance.** Additionally to the disability benefits in (a), a lump sum benefit of 10 000 is paid if the policyholder dies before termination of the contract.

(d) **Disability & endowment insurance.** Additionally to the disability benefits in (a), a lump sum benefit of 10 000 is paid in case of policyholder's death or termination of the contract, whichever occurs first.

(e) **Pure endowment insurance with disability waiver.** An endowment benefit of 10 000 is paid if the policyholder is still alive at termination of the contract. A constant premium is paid yearly in advance as long as the insured is in state active.

(f) **Temporary life insurance with disability waiver.** A lump sum benefit of 10 000 is paid if the policyholder dies before termination of the contract. A constant premium is paid yearly in advance as long as the insured is in state active.

(g) **Endowment insurance with disability waiver.** A lump sum benefit of 10 000 is paid in case of policyholder's death or termination of the contract, whichever occurs first. A constant premium is paid yearly in advance as long as the insured is in state active.

The prospective reserve of policy p at time s in state i is obtained as the expected present value of future benefits minus the expected present value of future premiums given that the policyholder is in state i at time s,

$$\begin{aligned} V_i^p(s) = &\sum_{j\in\mathcal{S}} \int_{(s,\omega_p]} v(s,t)\, p_{ij}(x_p; s,t)\, B_j^p(x_p + \mathrm{d}t) \\ &+ \sum_{(j,k)\in J} \int_{(s,\omega_p]} v(s,t)\, b_{jk}^p(x_p+t)\, p_{ij}(x_p; s,t)\, \mu_{jk}(x_p+t; t)\, \mathrm{d}t\,. \end{aligned} \tag{2.2}$$

The family of prospective reserves $V_i^p(s)$, $i \in \mathcal{S}$, $s \in [0, \omega_p]$, can be obtained as the unique solution of Thiele's integral equation system

$$V_i^p(s) = B_i^p(x_p+\omega_p) - B_i^p(x_p+s) - \int_{(s,\omega_p]} V_i^p(t-)\, \varphi(t)\, \mathrm{d}t + \sum_{j:j\neq i} \int_{(s,\omega_p]} R_{ij}^p(t)\, \mu_{ij}(x_p+t,t)\, \mathrm{d}t\,, \tag{2.3}$$

with starting values $V_i^p(\omega_p) = 0$ for all $i \in \mathcal{S}$, where

$$R_{ij}^p(t) := b_{ij}^p(x_p + t) + V_j^p(t-) - V_i^p(t-)$$

is the so-called sum-at-risk associated with a possible transition of policy p from state i to state j at time t. According to Christiansen (2008) the prospective reserve $V_i^p(s)$ – seen as a mapping of the general transition intensity matrix μ – has a first-order Taylor expansion of the form

$$\begin{aligned} V_i^p(s, \widetilde{\mu}) = V_i^p(s, \mu) + \sum_{(j,k) \in J} \int_{(s,\omega_p]} v(s,t)\, p_{ij}(x_p; s, t)\, R_{jk}^p(t) \left(\widetilde{\mu}_{jk}(x_p + t; t) - \mu_{jk}(x_p + t; t) \right) \mathrm{d}t \\ + o(\|\widetilde{\mu}(x_p + \cdot; \cdot) - \mu(x_p + \cdot; \cdot)\|)\,, \end{aligned} \tag{2.4}$$

where $\|\cdot\|$ means the L_1-norm of the maximum row sum. We refer the reader to Christiansen (2008) for more details.

If the transition intensities μ_{jk} are modeled as exponentials of the form (2.1), it can be more convenient to regard the prospective reserve $V_i^p(s)$ not as a mapping of μ but as a mapping of $\gamma = (\gamma_{jk})_{(j,k) \in \mathcal{S}^2}$. Using (2.4), the property

$$\begin{aligned} e^{\widetilde{\gamma}_{jk}(x_p+t;t)} &= e^{\gamma_{jk}(x_p+t;t)} + \left(\widetilde{\gamma}_{jk}(x_p + t; t) - \gamma_{jk}(x_p + t; t) \right) e^{\gamma_{jk}(x_p+t;t)} \\ &\quad + o(\|\widetilde{\gamma}_{jk}(x_p + \cdot; \cdot) - \gamma_{jk}(x_p + \cdot; \cdot)\|)\,, \end{aligned}$$

and the chain rule of Fréchet differentiation, we get a first-order Taylor expansion in γ,

$$\begin{aligned} & V_i^p\big(s, \exp(\widetilde{\gamma})\big) \\ & = V_i^p\big(s, \exp(\gamma)\big) \\ & \quad + \sum_{(j,k) \in J} \int_{(s,\omega_p]} v(s,t)\, p_{ij}(x_p; s, t)\, R_{jk}^p(t)\, \mu_{jk}(x_p + t; t) \big(\widetilde{\gamma}_{jk}(x_p + t; t) - \gamma_{jk}(x_p + t; t) \big) \mathrm{d}t \\ & \quad + o(\|\widetilde{\gamma}(x_p + \cdot; \cdot) - \gamma(x_p + \cdot; \cdot)\|)\,. \end{aligned} \tag{2.5}$$

The next section explains how γ_{jk} can be estimated from yearly transition data. Between integer ages and integer times, we assume that the log intensities $\gamma_{jk}(x;t)$ are constant. More precisely,

$$\gamma_{jk}(x; t) = \gamma_{jk}([x]; [t])\,, \quad (j,k) \in J, \tag{2.6}$$

where $[\xi]$ denotes the integer part of the real ξ.

3 Stochastic modeling of the transition rates

In the numerical part of the present paper, we consider the classical 3-state Markov model for disability insurance policies described in Example 2.1.

Very few studies investigated time trends in transition rates for multistate actuarial models. A noticeable exception is Renshaw and Haberman (2000) who considered the sickness recovery and inception transition rates, together with the mortality rates when sick, which form the basis of the UK continuous mortality investigation Bureau's multistate model. These authors identified the underlying time trends from the observation period 1975-1994 using separate Poisson GLM regression models for each transition. In this paper, we allow for possible correlations by means of a multivariate Lee-Carter type model fitted as described in Hyndman and Ullah (2007), that is, by means of a functional data approach. The methodology developed by Hyndman and Ullah (2007) combines functional data analysis, nonparametric smoothing and robust statistics. It is a convenient generalization of the Lee-Carter model for mortality rates: compared with Lee and Carter (1992), Hyndman and Ullah (2007) use smooth rather than crude mortality rates, replace conventional principal component analysis with its functional counterpart and allow for more than one principal component.

This methodology is used here to reduce the dimensionality of the problem. For each of the four transitions (a, i), (a, d), (i, a), and (i, d), the Hyndman-Ullah approach is used as a way to summarize the information contained in the age-specific transition rates multivariate time series. Specifically, the first step consists in smoothing the observed log-transition rates $\gamma_{jk}(x; t)$ across age x for each calendar year t and given transition $(j, k) \in J$, using a nonparametric smoothing method. In a second step, the smoothed set of transition rates is decomposed into

$$\gamma_{jk}(x; t) = m_{jk}(x) + \phi_{jk}(x)\, \beta_{jk}(t) \tag{3.1}$$

where the m_{jk} and ϕ_{jk} are deterministic functions of age and the vectors

$$\boldsymbol{\beta}(t) = \begin{pmatrix} \beta_{(a,i)}(t) \\ \beta_{(a,d)}(t) \\ \beta_{(i,a)}(t) \\ \beta_{(i,d)}(t) \end{pmatrix}$$

form a 4-dimensional times series. This expression for $\gamma_{jk}(x; t)$ is similar to the model introduced for mortality rates by Lee and Carter (1992). A two-step procedure for robust functional principal component analysis is applied to estimate the age functions m_{jk} and ϕ_{jk}, to avoid problems with outlying years.

To perform numerical illustrations, the model is fitted to annual German age-specific transition data. The sources of the data used in this paper are as follows. First, there is the data set of the Statistisches Bundesamt Deutschland (German Federal Statistical Office, www.destatis.de) that contains mortality rates for the general population in Germany. In order to reduce inconsistency in the time series due to reunification of Eastern and Western Germany in 1990, we use only the data for Western Germany.

Second, there is the data set of the Deutsche Rentenversicherung (German statutory pension insurance, www.deutsche-rentenversicherung.de) that gathers data about mortality, disability, and reactivation rates for all persons employed in the private or public sector (about 90% of the entire population). For consistency reasons, we use only data for Western Germany, and we stop the time series at the end of 2000, because since 2001 the definition of disability is reduced to unemployability. Based on data provided by all its 26 member institutes, the Federation of German Pension Insurance Institutes compiles data and publishes several statistical reports annually. In this paper, we used the following statistical reports:

- 'Statistikband Rentenzugang/Rentenwegfall' for the years 1984–2000 (report no. 66, 71, 75, 81, 86, 91, 96, 99, 104, 109, 113, 117, 121, 125, 129, 133, 137)
  - number of transitions from active to disabled: table 40 RV Z from 1984 to 1991, table 201.10 Z RV from 1992 to 2000
  - number of transitions from disabled to active: table 23 RV W from 1984 to 1991, table 204.10 W RV from 1992 to 2000
  - number of transitions from disabled to dead: 23 RV W from 1984 to 1991, table 204.10 W RV from 1992 to 2000 (in fact, the number of transitions is given jointly for disability and old-age pensions, but before age 60 – the minimum age for old-age pensions – all transitions come from disability pensions)
- 'Statistikband Rentenbestand' for the years 1985–2000 (report no. 65, 70, 74, 80, 85, 90, 95, 100, 105, 110, 111, 116, 120, 124, 128, 132)
  - exposure to risk in state disabled: table 13 G RV from 01.01.1985 to 01.01.1991, table 201.10 G RV from 31.12.1991 to 31.12.1999
- 'Statistikband Versicherte' for the years 1984–2000 (report no. 73, 78, 83, 88, 93, 98, 103, 106, 108, 115, 119, 123, 127, 131, 135, 139)
  - exposure to risk in state active: table 181 RV from 1984 to 1990, table 081 RV for 1991, table 6.11 V RV from 1992 to 2000.

The Deutsche Rentenversicherung also provides data about the number of transitions from active to dead. However, for the years 1992–2000 these data are only available in age groups of five years. Therefore, we estimate the mortality in state active from the data of the Statistisches Bundesamt Deutschland.

Thus, statistics are available for period 1984-2000 for (a, d) and (a, i) transitions and for period 1985-2000 for (i, a) and (i, d) transitions. Transitions within the calendar year and exposure to risk were available for each age from 21 to 64, except for the (i, d) transition where data cover each age from 21 to 59.

The estimation is carried on with the help of the function `fdm` from the `demography` package, implementing the Hyndman-Ullah methodology in the `R` software. The model explains 82.1% of the variance for transition (a, d), 65.9% of the variance for transition (a, i), 67.3% of the variance for transition (i, a), and 49.5% of the variance for transition (i, d). We acknowledge the relatively low percentage of explained variance for the transition (i, d) but we nevertheless keep the simple first component model for the numerical illustrations proposed in this paper[1].

[1] Incorporating the second principal components allows to explain 90.7% of the variance for transition (a, d), 90.4% of the variance for transition (a, i), 83.3% of the variance for transition (i, a), and 70.4% of the variance for transition (i, d). Incorporating the third component increases the explained variance for transition (i, a) up to 88.6% and for transition (i, d) up to 95.7%.

The coefficients time series $\boldsymbol{\beta}(t)$ for each of the four transitions are considered as the components of a 4-dimensional time series. A multivariate random walk with drift appears to appropriately describe the dynamic of this time series. Indeed, the differences $\boldsymbol{\beta}(t) - \boldsymbol{\beta}(t-1)$ appear to be non correlated and multivariate Normally distributed. Several multivariate residual tests have been performed:

- the null hypothesis of no serial correlation up to 4 lags is not rejected by the multivariate Ljung-Box Q-statistics (the Portmanteau autocorrelation test statistic is $Q = 54.8$ with a p-value equal to 0.78);

- the autocorrelation LM test does not detect serial correlation in residuals (the LM test statistic is $LM = 21.54$, with a p-value 0.15).

The multivariate Jarque-Bera Normality test statistic is $JB = 2.36$, with corresponding p-value 0.78 so that Normality is not rejected. Hence, we consider that the differences $\boldsymbol{\beta}(t) - \boldsymbol{\beta}(t-1)$ are independent and identically distributed, obeying the Normal law with mean vector

$$\begin{pmatrix} -0.072526 \\ -0.037238 \\ 0.012203 \\ -0.039883 \end{pmatrix}$$

and variance-covariance matrix

$$\boldsymbol{\Sigma} = \begin{pmatrix} 0.008808 & 0.000770 & 0.014397 & 0.010440 \\ 0.000770 & 0.001392 & -0.000844 & 0.003532 \\ 0.014397 & -0.000844 & 0.053968 & 0.008771 \\ 0.010440 & 0.003532 & 0.008771 & 0.029803 \end{pmatrix}.$$

4 Solvency II standard formula

4.1 Value-at-Risk principle

The Solvency II directive defines the solvency capital requirement by means of a Value-at-Risk at probability level 99.5%. However, the Solvency II standard formula for the life risk module SCR_{life} does not use probabilistic arguments, and it is not clear if the formula really corresponds to the intended Value-at-Risk. In this section, we present a stochastic model in which the standard formula for SCR_{life} is (asymptotically) equivalent to the intended Value-at-Risk.

According to the Solvency II framework, the life risk module SCR_{life} is roughly defined by

$$SCR_{life} = VaR_{0.995}\begin{pmatrix} \text{'change in net value of assets minus liabilities} \\ \text{due to changes of transition rates'} \end{pmatrix},$$

where 'change in net value' means deviations from best estimates. We assume here that changes in mortality rates, disability rates, etc. have no effect on assets.

4.2 Single policy

For a single policy p, the change in the net value of liabilities is given by the change in its prospective reserve. Therefore, given that the policyholder is at present time s in state i, we rewrite the definition of SCR_{life} to

$$SCR_{life} = VaR_{0.995}\Big(V_i^p(s, \mu^{BE}) - V_i^p(s, \mu) \Big), \tag{4.1}$$

where μ^{BE} is a deterministic best estimate transition intensity matrix, and where μ is the actual transition intensity matrix applying to this policy. Since these intensities relate to the future, they are only forecasts and, thus, stochastic.

A possible explanation for (1.1) being equal to (4.1) is a model where $V_i^p(s, \mu^{BE}) - V_i^p(s, \mu)$ has a decomposition into a sum with terms corresponding to the different life risk submodules being Normally distributed with zero mean. Usually we do not have such a linear decomposition because the prospective reserve $V_i^p(s, \mu) = V_i^p(s, \exp(\gamma))$ is nonlinear in the argument μ or γ. But with the help of (2.5) we can at least approximate $V_i^p(s, \exp(\gamma^{BE})) - V_i^p(s, \exp(\gamma))$ by the sum

$$\sum_{(j,k)\in J} \int g_{jk}^p(t) \left(\gamma_{jk}^{BE}(x_p + t; t) - \gamma_{jk}(x_p + t; t)\right) \mathrm{d}t\,, \tag{4.2}$$

where $g_{jk}^p(t) := \mathbf{1}_{(s,\omega_p]}(t)\, v(s,t)\, p_{ij}^{BE}(x_p; s, t)\, R_{jk}^{p,BE}(t)\, \mu_{jk}^{BE}(x_p + t; t)$. With the best estimate γ^{BE} being deterministic and only γ being stochastic, each term involved in the sum (4.2) uniquely corresponds to the fluctuation risk of one transition rate. If furthermore the γ_{jk} are Gaussian with $\mathbb{E}[\gamma_{jk}(x;t)] = \gamma_{jk}^{BE}(x;t)$, then (4.2) is Normally distributed with zero mean and its Value-at-Risk can in fact be calculated by a square root formula similar to (1.1):

$$\begin{aligned} &VaR_{0.995}\Big(V_i^p(s, \mu^{BE}) - V_i^p(s, \mu) \Big) \\ &\approx VaR_{0.995}\Big(\sum_{(j,k)\in J} \int g_{jk}^p(t) \left(\gamma_{jk}^{BE}(x_p + t; t) - \gamma_{jk}(x_p + t; t)\right) \mathrm{d}t \Big) \\ &= \sqrt{\sum_{(j,k),(l,m)} Corr_{(j,k),(l,m)}\, SCR_{(j,k)}\, SCR_{(l,m)}} \end{aligned} \tag{4.3}$$

with $Corr_{(j,k),(l,m)}$ giving the pairwise correlations and $SCR_{(j,k)}$ being defined by

$$SCR_{(j,k)} = VaR_{0.995}\Big(\int g_{jk}^p(t) \left(\gamma_{jk}^{BE}(x_p + t; t) - \gamma_{jk}(x_p + t; t)\right) \mathrm{d}t \Big).$$

Solvency II ignores the fact that mortality can evolve differently in different states (mortality in state active, mortality in state disabled, etc.). In order to take respect of that, we aggregate all mortality terms into a single quantity and define

$$SCR_{(\cdot,d)} = VaR_{0.995}\Big(\sum_{j:j\neq d} \int g_{jd}^p(t) \left(\gamma_{jd}^{BE}(x_p + t; t) - \gamma_{jd}(x_p + t; t)\right) \mathrm{d}t \Big).$$

(Recall that 'd' means state 'dead'.) Furthermore, Solvency II does not see the risk of changing mortality rates as a single risk factor, but splits the risk up into an upward risk SCR_{mort} and a downward risk SCR_{long}. However, usually one of the two quantities SCR_{mort} and SCR_{long} is zero and can be ignored. In such a case we can identify $SCR_{(\cdot,d)}$ with the one of these two quantities that is not zero.

The standard formula of Solvency II combines the risk of changing disability rates and the risk of changing reactivation rates into the single submodule SCR_{dis}. In order to replicate this submodule we define

$$SCR_{(a,i)\&(i,a)} = VaR_{0.995}\Bigg(\sum_{(j,k)\in\{(a,i),(i,a)\}} \int g^p_{jk}(t)\, \big(\gamma^{BE}_{jk}(x_p+t;t) - \gamma_{jk}(x_p+t;t)\big)\, \mathrm{d}t \Bigg).$$

4.3 Application to disability insurance

For the disability insurance with state space $\mathcal{S} = \{a, i, d\}$ and no death benefits, we have $SCR_{mort} = 0$, and thus we can identify $SCR_{(\cdot,d)}$ and $SCR_{(a,i)\&(i,a)}$ with the Solvency II submodules SCR_{long} and SCR_{dis}. The Solvency II correlation $Corr_{long,dis}$ should here be equal to

$$\begin{aligned}
&Corr_{(\cdot,d),(a,i)\&(i,a)} \\
&= \mathbb{C}\mathrm{orr}\Bigg(\int \Big(g^p_{ad}(t)\, \big(\gamma^{BE}_{ad}(x_p+t;t) - \gamma_{ad}(x_p+t;t)\big) + g^p_{id}(t)\, \big(\gamma^{BE}_{id}(x_p+t;t) - \gamma_{id}(x_p+t;t)\big)\Big)\mathrm{d}t, \\
&\qquad \int \Big(g^p_{ai}(t)\, \big(\gamma^{BE}_{ai}(x_p+t;t) - \gamma_{ai}(x_p+t;t)\big) + g^p_{ia}(t)\, \big(\gamma^{BE}_{ia}(x_p+t;t) - \gamma_{ia}(x_p+t;t)\big)\Big)\mathrm{d}t \Bigg).
\end{aligned}$$

In order to calculate $Corr_{(\cdot,d),(a,i)\&(i,a)} = Cov_{(\cdot,d),(a,i)\&(i,a)} \big/ \sqrt{Cov_{(\cdot,d),(\cdot,d)}\, Cov_{(a,i)\&(i,a),(a,i)\&(i,a)}}$, we just need to know the covariances between all $\gamma_{jk}(x;t)$. The covariance $Cov_{(\cdot,d),(a,i)\&(i,a)}$ can be rewritten to

$$\begin{aligned}
Cov_{(\cdot,d),(a,i)\&(i,a)} = \int\!\!\int \Big(& g^p_{ad}(t)\, g^p_{ai}(u)\, \mathbb{C}\mathrm{ov}(\gamma_{ad}(x_p+t;t), \gamma_{ai}(x_p+u;u)) \\
&+ g^p_{id}(t)\, g^p_{ai}(u)\, \mathbb{C}\mathrm{ov}(\gamma_{id}(x_p+t;t), \gamma_{ai}(x_p+u;u)) \\
&+ g^p_{ad}(t)\, g^p_{ia}(u)\, \mathbb{C}\mathrm{ov}(\gamma_{ad}(x_p+t;t), \gamma_{ia}(x_p+u;u)) \\
&+ g^p_{id}(t)\, g^p_{ia}(u)\, \mathbb{C}\mathrm{ov}(\gamma_{id}(x_p+t;t), \gamma_{ia}(x_p+u;u))\Big)\, \mathrm{d}t\, \mathrm{d}s
\end{aligned} \tag{4.4}$$

where we assumed $\mathbb{E}[\gamma_{jk}(x;t)] = \gamma^{BE}_{jk}(x;t)$. Analogous formulas can be obtained for $Cov_{(\cdot,d),(\cdot,d)}$ and $Cov_{(a,i)\&(i,a),(a,i)\&(i,a)}$. Applying the results of Section 3, the covariances between every pair of $\gamma_{jk}(x;t)$ can be obtained as follows:

$$\begin{aligned}
&\mathbb{C}\text{ov}\big(\gamma_{ij}(x;t), \gamma_{kl}(y;s)\big) \\
&= \mathbb{C}\text{ov}\Big(\phi_{ij}(x) \sum_{\tau=1}^{t} \big(\beta_{ij}(\tau) - \beta_{ij}(\tau-1)\big), \phi_{kl}(y) \sum_{\sigma=1}^{s} \big(\beta_{kl}(\sigma) - \beta_{kl}(\sigma-1)\big)\Big) \\
&= \phi_{ij}(x)\, \phi_{kl}(y) \sum_{\tau=1}^{t} \sum_{\sigma=1}^{s} \mathbb{C}\text{ov}\Big(\big(\beta_{ij}(\tau) - \beta_{ij}(\tau-1)\big), \big(\beta_{kl}(\sigma) - \beta_{kl}(\sigma-1)\big)\Big) \\
&= \phi_{ij}(x)\, \phi_{kl}(y) \sum_{\tau=1}^{\min\{t,s\}} \mathbb{C}\text{ov}\Big(\big(\beta_{ij}(\tau) - \beta_{ij}(\tau-1)\big), \big(\beta_{kl}(\tau) - \beta_{kl}(\tau-1)\big)\Big) \\
&= \phi_{ij}(x)\, \phi_{kl}(y)\, \min\{t,s\}\, \mathbf{\Sigma}_{((i,j),(k,l))}\,.
\end{aligned} \tag{4.5}$$

Consider a 30-year-old male who contracts insurance policies according to Example 2.1 (a)-(g). All policies start at the beginning of year 2001 and terminate at age 60. The yearly constant premiums are calculated on the basis of the forecasts (best estimates) of Section 3 and an interest intensity of ln(1.0225). We then get the following results:

(a) **Disability insurance.** The equivalence principle leads to a constant yearly premium of 27.58. The policy has to a large extent a survival character, and, hence, we can identify $Corr_{(\cdot,d),(a,i)\&(i,a)} \approx -0.658$ with $Corr_{long,dis}$.

(b) **Disability & pure endowment insurance.** The equivalence principle leads to a constant yearly premium of 247.28. The policy has a pure survival character, and, hence, we can identify $Corr_{(\cdot,d),(a,i)\&(i,a)} \approx -0.563$ with $Corr_{long,dis}$.

(c) **Disability & temporary life insurance.** The equivalence principle leads to a constant yearly premium of 57.93. The policy has to a large extent an occurrence character, and, hence, we can identify $Corr_{(\cdot,d),(a,i)\&(i,a)} \approx 0.375$ with $Corr_{mort,dis}$.

(d) **Disability & endowment insurance.** The equivalence principle leads to a constant yearly premium of 275.58. The policy has a mixed character, and $Corr_{(\cdot,d),(a,i)\&(i,a)} \approx -0.354$ corresponds to a combination of $Corr_{mort,dis}$ and $Corr_{long,dis}$.

(e) **Pure endowment insurance with disability waiver.** The equivalence principle leads to a constant yearly premium of 219.70. The policy has a pure survival character, and, hence, we can identify $Corr_{(\cdot,d),(a,i)\&(i,a)} \approx 0.407$ with $Corr_{long,dis}$.

(f) **Temporary life insurance with disability waiver.** The equivalence principle leads to a constant yearly premium of 30.35. The policy has a pure occurrence character, and, hence, we can identify $Corr_{(\cdot,d),(a,i)\&(i,a)} \approx 0.485$ with $Corr_{mort,dis}$.

(g) **Endowment insurance with disability waiver.** The equivalence principle leads to a constant yearly premium of 250.05. The policy has a pure occurrence character, and, hence, we can identify $Corr_{(\cdot,d),(a,i)\&(i,a)} \approx 0.447$ with $Corr_{mort,dis}$.

The examples (a)-(g) show that the correlation $Corr_{(\cdot,d),(a,i)\&(i,a)}$ varies significantly with the type of insurance contract. In (e) we even obtain a sign that is opposite to the one contained in the QIS5 correlation matrix. This indicates that Solvency II presumably over-simplifies the risk structures of life insurance portfolios by using the same correlation factors $Corr_{long,dis}$ and $Corr_{mort,dis}$ for all kinds of products.

4.4 Extension to portfolios

If we have not only a single policy but a portfolio of policies $p \in N$ that are at present in state i_p, we naturally extend definition (4.1) to

$$SCR_{life} = VaR_{0.995}\bigg(\sum_{p\in N}\Big(V^p_{i_p}(s,\mu^{BE}) - V^p_{i_p}(s,\mu)\Big)\bigg). \tag{4.6}$$

(Here all policies use the same calendar time. We take the earliest start of a contract as time zero. For policies that start at a later calendar time, the payments functions $b^p_{jk}(x_p+t)$ and $B^p_j(x_p+t)$ are constantly zero at the beginning.) Analogously to (4.3), we can derive the decomposition

$$\begin{aligned} &VaR_{0.995}\bigg(\sum_{p\in N}\Big(V^p_{i_p}(s,\mu^{BE}) - V^p_{i_p}(s,\mu)\Big)\bigg)\\ &\approx VaR_{0.995}\bigg(\sum_{(j,k)\in J}\Big(\sum_{p\in N}\int g^p_{jk}(t)\,\big(\gamma^{BE}_{jk}(x_p+t;t) - \gamma_{jk}(x_p+t;t)\big)\mathrm{d}t\Big)\bigg)\\ &= \sqrt{\sum_{(j,k),(l,m)} Corr_{(j,k),(l,m)}\,SCR_{(j,k)}\,SCR_{(l,m)}} \end{aligned} \tag{4.7}$$

with $Corr_{(j,k),(l,m)}$ giving the correlations between the addends of the sum $\sum_{(j,k)\in J}$ and the $SCR_{(j,k)}$ being defined by

$$SCR_{(j,k)} = VaR_{0.995}\bigg(\sum_{p\in N}\int g^p_{jk}(t)\,\big(\gamma^{BE}_{jk}(x_p+t;t) - \gamma_{jk}(x_p+t;t)\big)\mathrm{d}t\bigg).$$

Also completely analogous to the single policy case, we can define a general mortality risk quantity

$$SCR_{(\cdot,d)} = VaR_{0.995}\bigg(\sum_{j:j\neq d}\sum_{p\in N}\int g^p_{jd}(t)\,\big(\gamma^{BE}_{jd}(x_p+t;t) - \gamma_{jd}(x_p+t;t)\big)\,\mathrm{d}t\bigg)$$

and identify it with either SCR_{mort} or SCR_{long} (given that one of the two risk modules is negligible). In the same way, we generalize $SCR_{(a,i)\&(i,a)}$ by

$$SCR_{(a,i)\&(i,a)} = VaR_{0.995}\bigg(\sum_{(j,k)\in\{(a,i),(i,a)\}}\sum_{p\in N}\int g^p_{jk}(t)\,\big(\gamma^{BE}_{jk}(x_p+t;t) - \gamma_{jk}(x_p+t;t)\big)\,\mathrm{d}t\bigg).$$

5 Alternative calculation of SCR_{life}

5.1 Asymptotic method

The Solvency II directive says that each risk module shall be calibrated using a Value-at-Risk measure with a confidence level of 99.5%. In particular this holds for the life risk module SCR_{life}. We mathematically formalized this approach in (4.1) and (4.6). Unfortunately, it is usually very difficult to calculate the Value-at-Risks in (4.1) or (4.6) analytically. In this section, we assume that the decomposition (3.1) applies and that the $\boldsymbol{\beta}(n)$ vectors obey a random walk with drift.

One idea to obtain at least an approximation of (4.1) and (4.6) is to use the first-order Taylor approximation concept of (4.3). But instead of seeing the prospective reserve as a mapping of the log intensities γ_{jk} as done in (4.2), we now see the prospective reserve as a mapping of the differences $\Delta\beta_{jk}(n) := \beta_{jk}(n) - \beta_{jk}(n-1)$ with $\beta_{jk}(-1) := 0$. Here, we assume that each β_{jk} evolves according to (3.1). Using (2.6) and the linearity of the γ_{jk} with respect to the $\Delta\beta_{jk}$, we can transform (2.5) into

$$\begin{aligned} &V_i^p(s, \exp(\mathbb{E}[\gamma])) - V_i^p(s, \exp(\gamma)) + o\Big(\sum_{n=0}^{[\omega_p]} \|\mathbb{E}[\Delta\boldsymbol{\beta}(n)] - \Delta\boldsymbol{\beta}(n)\| \Big) \\ &= \sum_{(j,k)\in J} \int_{(s,\omega_p]} g_{jk}^p(t)\, \phi_{jk}([x_p + t]) \sum_{n=0}^{[t]} \big(\mathbb{E}[\Delta\beta_{jk}(n)] - \Delta\beta_{jk}(n)\big)\, \mathrm{d}t \qquad (5.1) \\ &= \sum_{(j,k)\in J} \sum_{n=0}^{[\omega_p]} \big(\mathbb{E}[\Delta\beta_{jk}(n)] - \Delta\beta_{jk}(n)\big) \int_{(s\vee n,\omega_p]} g_{jk}^p(t)\, \phi_{jk}([x_p + t])\, \mathrm{d}t\,, \end{aligned}$$

where the $g_{jk}^p(t)$ are defined as in (4.2). Under our assumptions, the random vectors $\mathbb{E}[\Delta\boldsymbol{\beta}(n)] - \Delta\boldsymbol{\beta}(n)$ are mutually independent and admit the representation

$$\mathbb{E}[\Delta\boldsymbol{\beta}(n)] - \Delta\boldsymbol{\beta}(n) = \boldsymbol{\Sigma}^{1/2}\boldsymbol{X}(n)\,,$$

where the $\boldsymbol{X}(n)$ are independent column random vectors with a multivariate standard Normal distribution and $\boldsymbol{\Sigma}^{1/2}$ is the Cholesky decomposition of $\boldsymbol{\Sigma}$. With defining the row vectors

$$\boldsymbol{G}(n) = \Big(\int_{(s\vee n,\omega_p]} g_{jk}^p(t)\, \phi_{jk}([x_p + t])\, \mathrm{d}t \Big)_{(j,k)\in J}\,,$$

we may write

$$V_i^p(s, \exp(\mathbb{E}[\gamma])) - V_i^p(s, \exp(\gamma)) + o\Big(\sum_{n=0}^{[\omega_p]} \|\mathbb{E}[\Delta\boldsymbol{\beta}(n)] - \Delta\boldsymbol{\beta}(n)\| \Big) = \sum_{n=0}^{[\omega_p]} \boldsymbol{G}(n)\boldsymbol{\Sigma}^{1/2}\boldsymbol{X}(n)\,. \qquad (5.2)$$

The right-hand side is a sum of independent standard Normal random variables. Thus, it is Normally distributed with zero mean and variance

$$\sum_{n=0}^{[\omega_p]} \boldsymbol{G}(n)\, \boldsymbol{\Sigma}\, (\boldsymbol{G}(n))^T\,.$$

Given that $\mu^{BE} = \exp(\mathbb{E}[\gamma]))$ and given that $o(\sum_{n=0}^{[\omega_p]} \|\mathbb{E}[\Delta\boldsymbol{\beta}(n)] - \Delta\boldsymbol{\beta}(n)\|)$ is negligible, we can approximate (4.1) by

$$\begin{aligned} SCR_{life} &= VaR_{0.995}\Big(V_i^p(s, \exp(\mathbb{E}[\gamma])) - V_i^p(s, \exp(\gamma)) \Big) \\ &\approx VaR_{0.995}\Big(\sum_{n=0}^{[\omega_p]} \boldsymbol{G}(n)\boldsymbol{\Sigma}^{1/2}\boldsymbol{X}(n) \Big) \\ &= u_{0.995}\Big(\sum_{n=0}^{[\omega_p]} \boldsymbol{G}(n)\,\boldsymbol{\Sigma}\,(\boldsymbol{G}(n))^T \Big)^{1/2} \end{aligned} \tag{5.3}$$

with $u_{0.995}$ being the 0.995-quantile of the standard Normal distribution. So far we only studied a single policy. In the portfolio case we get the same formula but the vectors $\boldsymbol{G}(n)$ are of the form

$$\boldsymbol{G}(n) = \Big(\sum_{p\in N} \int_{(s\vee n, \omega_p]} g_{jk}^p(t)\, \phi_{jk}([x_p + t])\, \mathrm{d}t \Big)_{(j,k)\in J}. \tag{5.4}$$

Comparing (5.3) with the Solvency II concept (see the previous section), we see that both approaches are based on the same first-order Taylor approximation, but (5.3) has the advantage that it much better reflects the various risk structures of life insurance portfolios when it comes to dependencies. Unfortunately, the approximation error in formula (5.3) is difficult to control. Therefore, in the next section we give a concept that allows to calculate upper bounds of SCR_{life}.

5.2 Upper bound

Using the same model setting as in the previous section, we now see the prospective reserve $V_i^p(s) = V_i^p(s, \mu(\Delta\boldsymbol{\beta}))$ as a mapping of the $\Delta\boldsymbol{\beta}(n)$, $n = 0, ..., [\omega_p]$. If $M_{0.995}$ is an arbitrary but fixed set with $P(\Delta\boldsymbol{\beta} \in M_{0.995}) \geq 0.995$, then we have

$$\begin{aligned} SCR_{life} &= VaR_{0.995}\Big(V_i^p(s, \mu(\mathbb{E}[\Delta\boldsymbol{\beta}])) - V_i^p(s, \mu(\Delta\boldsymbol{\beta})) \Big) \\ &\leq - \inf_{\Delta\widetilde{\boldsymbol{\beta}}\in M_{0.995}} \Big(V_i^p(s, \mu(\mathbb{E}[\Delta\boldsymbol{\beta}])) - V_i^p(s, \mu(\Delta\widetilde{\boldsymbol{\beta}})) \Big) \\ &= \sup_{\Delta\widetilde{\boldsymbol{\beta}}\in M_{0.995}} V_i^p(s, \mu(\Delta\widetilde{\boldsymbol{\beta}})) - V_i^p(s, \mu(\mathbb{E}[\Delta\boldsymbol{\beta}])) \end{aligned} \tag{5.5}$$

in case of a single policy and

$$\begin{aligned} SCR_{life} &= VaR_{0.995}\Big(\sum_{p\in N} \Big(V_i^p(s, \mu(\mathbb{E}[\Delta\boldsymbol{\beta}])) - V_i^p(s, \mu(\Delta\boldsymbol{\beta})) \Big) \Big) \\ &\leq \sup_{\widetilde{\mu}\in M_{0.995}} \sum_{p\in N} V_i^p(s, \mu(\Delta\widetilde{\boldsymbol{\beta}})) - \sum_{p\in N} V_i^p(s, \mu(\mathbb{E}[\Delta\boldsymbol{\beta}])) \end{aligned} \tag{5.6}$$

for an inhomogeneous portfolio of policies $p \in N$ that are at present in state i_p.

In order to compute the upper bounds for SCR_{life} in (5.5) and (5.6), we need to

- find suitable scenario sets $M_{0.995}$ with $P(\Delta\boldsymbol{\beta} \in M_{0.995}) \geq 0.995$,
- solve the optimization problems $\sup_{\Delta\widetilde{\boldsymbol{\beta}} \in M_{0.995}} V_i^p(s, \mu(\Delta\widetilde{\boldsymbol{\beta}}))$ or $\sup_{\widetilde{\mu} \in M_{0.995}} \sum_{p\in N} V_i^p(s, \mu(\Delta\widetilde{\boldsymbol{\beta}$

There are lots of ways to choose the scenario set $M_{0.995}$. For a single policy, the optimal choice is the scenario set for which the upper bound (5.5) is sharp:

$$\left\{\Delta\widetilde{\boldsymbol{\beta}} \,\middle|\, V_i^p(s, \mu(\mathbb{E}[\Delta\boldsymbol{\beta}])) - V_i^p(s, \mu(\Delta\widetilde{\boldsymbol{\beta}})) \geq -VaR_{0.995}\Big(V_i^p(s, \mu(\mathbb{E}[\Delta\boldsymbol{\beta}])) - V_i^p(s, \mu(\Delta\boldsymbol{\beta}))\Big)\right\}. \tag{5.7}$$

However, we have no method to find this optimal set. Let us now show how to find a scenario set that leads to an upper bound (5.5) that is at least close to the true Value-at-Risk. By using the first-order Taylor approximation (5.1) on both hand sides of the inequality in (5.7), we obtain

$$M_{0.995} = \left\{\Delta\widetilde{\boldsymbol{\beta}} \,\middle|\, \sum_{n=0}^{[\omega_p]} \boldsymbol{G}(n)\left(\mathbb{E}[\Delta\boldsymbol{\beta}(n)] - \Delta\widetilde{\boldsymbol{\beta}}(n)\right) \geq -u_{0.995}\left(\sum_{n=0}^{[\omega_p]} \boldsymbol{G}(n)\,\boldsymbol{\Sigma}\,(\boldsymbol{G}(n))^T\right)^{1/2}\right\}. \tag{5.8}$$

By following the arguments of the previous section, we see that this scenario set indeed satisfies $P(\Delta\boldsymbol{\beta} \in M_{0.995}) \geq 0.995$. In the portfolio case we get the same scenario set but with the portfolio version (5.4) of $\boldsymbol{G}(n)$. The examples given later on will show that (5.8) is indeed a good choice for $M_{0.995}$.

The second step is now to calculate the suprema in (5.5) and (5.6). A solution can be found by using a gradient ascent method. From (5.1) we get

$$\frac{\partial\left(\sum_{p\in N} V_i^p(s, \mu(\Delta\widetilde{\boldsymbol{\beta}}))\right)}{\partial(\Delta\widetilde{\beta}_{jk}(n))} = \boldsymbol{G}_{jk}(n)$$

with $\boldsymbol{G}$ as defined in (5.4). The partial derivatives form the gradient vector of $V_i^p(s, \mu(\Delta\widetilde{\boldsymbol{\beta}})))$ with respect to the argument $\Delta\widetilde{\boldsymbol{\beta}}$. The gradient gives locally the direction with the steepest incline. The gradient ascent method follows that direction in small steps:

Algorithm 5.1 (calculation of the supremum in (5.5) or (5.6))**.**

1. Choose as starting value $\Delta\boldsymbol{\beta}^{(0)} = \mathbb{E}[\Delta\boldsymbol{\beta}]$ and calculate the corresponding $\boldsymbol{G} = \boldsymbol{G}^{(0)}$.
2. Calculate a new $\Delta\boldsymbol{\beta}^{(m+1)}$ by using the iteration
$$\Delta\boldsymbol{\beta}^{(m+1)}(n) = \Delta\boldsymbol{\beta}^{(m)}(n) + K\,(\boldsymbol{G}^{(m)}(n))^T\,, n = 0, ..., [\omega_p]\,, \tag{5.9}$$
where $\boldsymbol{G}^{(m)}(n)$ is defined by (5.4) with the $g_{jk}^p(t)$ on the right hand side being calculated on the basis of $\mu(\Delta\boldsymbol{\beta}^{(m)})$. The constant $K > 0$ controls the stepsize.
3. Repeat step 2 as long as $\Delta\boldsymbol{\beta}^{(m+1)}$ is an element of (5.8), that is, as long as $\Delta\boldsymbol{\beta}^{(m+1)}$ satisfies the inequality in (5.8) for $\boldsymbol{G} = \boldsymbol{G}^{(0)}$.

5.3 Numerical illustration

We continue with Examples 2.1 (a)-(g) and calculate the corresponding

- asymptotic Value-at-Risks according to (5.3),
- upper bound Value-at-Risks according to (5.5) by using Algorithm 5.1.

The results are displayed in the next table

$VaR_{0.995}$ calculated by ...	asymptotic formula	upper bound formula
Disability ins.	231.29	97.27
Disability & pure endowment ins.	267.09	89.32
Disability & temporary life ins.	500.05	204.22
Disability & endowment ins.	277.00	119.50
Pure endowment ins. with disability waiver	238.22	67.30
Temporary life ins. with disability waiver	347.73	146.79
Endowment ins. with disability waiver	118.43	57.09

We also conducted 10 000 Monte Carlo simulations to evaluate the true Value-at-Risk for each (a)-(g) product. This shows that the upper bound (5.5) is pretty accurate if we use the scenario set (5.8), overestimating the true value by less than 5% in all the cases considered in this paper. We also see that in all the examples the asymptotic formula (5.3) overestimates the true Value-at-Risk to a large extent. That means that we are always on the safe side, but the approximation error is quite large.

6 Conclusions

Solvency II is the new regulatory regime for EU countries to be mandatory for insurance companies by 2012. Its three pillar structure is similar to Basel II except for the additional insurance risk. Under the standard approach, SCR is approximated with the help of the square-root formula. In this paper, we have provided a theoretical foundation for this formula using limited expansion around the best estimate. Numerical illustrations performed on the basis of German data have suggested that the QIS correlation matrix was not appropriate and that the correlations greatly varied from one product to another. Therefore, the problem is not so much with the square-root formula itself but well in using the same correlation values for all types of products.

The numerical illustrations provided in the present paper constitute only a first step. First, they are based on a single German data set, only. The correlations may vary among EU member countries and the data set is not necessarily representative of the German insurance market. Second, it could be interesting to relax the Markovian assumption and to extend the results derived in this paper to semi-Markov processes: bivariate transition rates could be specified for states where duration effects are significant (typically, disability states). Despite these limitations, this paper is a first attempt to legitimate the standard formula under Solvency II, questioning the QIS correlation matrix.

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