ulm university universität uulm

Prof. Dr. Evgeny Spodarev **Schuljahr 2019/20**

Begabtenseminar

Gipfel und Nordwände der Schulmathematik oder Aufstieg bis zur Mathe-Olympia

Algebraische, Exponentielle und logarithmische (Un)Gleichungen

Formelsammlung (18.10.2019)

- **Umgang mit Beträgen:**

  - Definition des Betrags: $|a| = \begin{cases} a, & \text{wenn } a \geq 0, \\ -a, & \text{wenn } a < 0. \end{cases}$
  - Gleichung $|f(x)| + g(x) = 0$ ist äquivalent zu 2 Systemen

$$\begin{cases} f(x) + g(x) = 0, \\ f(x) \geq 0 \end{cases} \qquad \begin{cases} -f(x) + g(x) = 0, \\ f(x) < 0. \end{cases}$$

- **(Un)gleichungen mit Wurzeln:**

(1)
$$\sqrt{f(x)} = g(x) \iff \begin{cases} f(x) = (g(x))^2 \\ g(x) \geq 0 \end{cases}$$

(2)
$$\sqrt{f(x)} \geq g(x) \iff \left[\begin{array}{l} \begin{cases} f(x) \geq (g(x))^2 \\ g(x) \geq 0 \end{cases} \\ \begin{cases} f(x) \geq 0 \\ g(x) < 0 \end{cases} \end{array} \right.$$

(3)
$$\sqrt{f(x)} > g(x) \iff \left[\begin{array}{l} \begin{cases} f(x) > (g(x))^2 \\ g(x) \geq 0 \end{cases} \\ \begin{cases} f(x) \geq 0 \\ g(x) < 0 \end{cases} \end{array} \right.$$

(4)
$$\sqrt{f(x)} < g(x) \iff \begin{cases} f(x) < (g(x))^2 \\ f(x) \geq 0 \\ g(x) \geq 0 \end{cases}$$

(5) $$\sqrt{f(x)} \le g(x) \iff \begin{cases} f(x) \le (g(x))^2 \\ f(x) \ge 0 \\ g(x) \ge 0 \end{cases}$$

- **Umgang mit Potenzen:**

(1) $$a^x > 0 \text{ für alle } a > 0 \text{ und } x \in \mathbb{R}.$$

(2) $$a^x = b \iff x = \begin{cases} \log_a b, & \text{wenn } a > 0, b > 0, a \neq 1 \\ \emptyset, & \text{wenn } a = 1, b \neq 1 \\ \text{beliebig}, & \text{wenn } a = b = 1. \end{cases}$$

(3) Falls m und $n-$ natürliche Zahlen, dann gilt $a^{\frac{m}{n}} = \sqrt[n]{a^m}$.

(4) $$a^{-x} = \frac{1}{a^x}$$

(5) $$a^{x+y} = a^x a^y, \quad a^{x-y} = \frac{a^x}{a^y}, \quad a^{xy} = (a^x)^y = (a^y)^x$$

(6) $$(ab)^x = a^x b^x, \quad \left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$$

(7) $$a^0 = 1, a^1 = a$$

- **(Un)gleichungen mit Potenzen:**

(1) $$a^{g(x)} = a^{h(x)}, a > 0, a \neq 1 \iff g(x) = h(x)$$

(2) $$f(x)^{g(x)} = f(x)^{h(x)} \iff \left[\begin{array}{l} \begin{cases} g(x) = h(x) \\ f(x) > 0 \end{cases} \\ \begin{cases} f(x) = 1 \\ g(x), h(x) \text{ beliebig} \end{cases} \end{array} \right.$$

(3) $$f(x)^{g(x)} > f(x)^{h(x)} \Longleftrightarrow \left[\begin{array}{l} \begin{cases} g(x) > h(x) \\ f(x) > 1 \end{cases} \\ \begin{cases} g(x) < h(x) \\ 0 < f(x) < 1 \end{cases} \end{array}\right.$$

(4) $$f(x)^{g(x)} \geq f(x)^{h(x)} \Longleftrightarrow \left[\begin{array}{l} \begin{cases} f(x) \geq 1 \\ g(x) \geq h(x) \end{cases} \\ \begin{cases} 0 < f(x) \leq 1 \\ g(x) \leq h(x) \end{cases} \end{array}\right.$$

(5) $$f(x)^{g(x)} < f(x)^{h(x)} \Longleftrightarrow \left[\begin{array}{l} \begin{cases} g(x) < h(x) \\ f(x) > 1 \end{cases} \\ \begin{cases} g(x) > h(x) \\ 0 < f(x) < 1 \end{cases} \end{array}\right.$$

(6) $$f(x)^{g(x)} \leq f(x)^{h(x)} \Longleftrightarrow \left[\begin{array}{l} \begin{cases} f(x) \geq 1 \\ g(x) \leq h(x) \end{cases} \\ \begin{cases} 0 < f(x) \leq 1 \\ g(x) \geq h(x) \end{cases} \end{array}\right.$$

- **Umgang mit Logarithmen:**

(1) $$a^{\log_a b} = b,\ a > 0, \quad b > 0,\ a \neq 1.$$

(2) $$\log_a 1 = 0,\ \log_a a = 1, \quad a > 0,\ a \neq 1$$

(3) $$\log_a(bc) = \log_a b + \log_a c, \quad a, b, c, > 0,\ a \neq 1$$

(4) $$\log_a \left(\frac{b}{c}\right) = \log_a b - \log_a c, \quad a, b, c, > 0,\ a \neq 1$$

(5) $$\log_a(b^x) = x \log_a b, \quad a, b > 0,\ a \neq 1$$

(6) $$\log_{(a^x)} b = \frac{1}{x} \log_a b, \quad a, b > 0,\ a \neq 1, x \neq 0$$

(7) $$\log_a(b^{2n}) = 2n\log_a\ |b|, \quad a>0,\ a\neq 1, b\neq 0,\ n\in\mathbb{N}$$

(8) $$\log_a\frac{1}{b} = -\log_a b, \quad a,b>0,\ a\neq 1$$

(9) $$\log_a b = \frac{\log_c b}{\log_c a}, \quad a,b,c>0,\ a,c\neq 1$$

(10) $$\log_a b = \frac{1}{\log_b a}, \quad a,b>0,\ a,b\neq 1$$

(11) $$\log_a(bc) = \log_a|b| + \log_a|c|, \quad a>0,\ b\cdot c>0, a\neq 1$$

(12) $$\log_a\left(\frac{b}{c}\right) = \log_a|b| - \log_a|c|, \quad a>0,\ b\cdot c>0, a\neq 1$$

- **(Un)gleichungen mit Logarithmen:**

(13) $$\log_{f(x)} g(x) = \log_{f(x)} h(x) \Longleftrightarrow \begin{cases} g(x)=h(x)\\ f(x)>0\\ g(x)>0\\ f(x)\neq 1\end{cases}$$

(14) $$\log_a g(x) = \log_a h(x) \Longleftrightarrow \begin{cases} g(x)=h(x),\\ f(x)>0,\end{cases} \quad \text{falls } a>0,\ a\neq 1.$$

(15) $$\log_{f(x)} g(x) > \log_{f(x)} h(x) \Longleftrightarrow \left[\begin{array}{l} \begin{cases} g(x)>h(x)\\ h(x)>0\\ f(x)>1\end{cases}\\ \begin{cases} h(x)>g(x)\\ g(x)>0\\ 0<f(x)<1\end{cases}\end{array}\right.$$

(16) $$\log_{f(x)} g(x) \geq \log_{f(x)} h(x) \Longleftrightarrow \left[\begin{array}{l} \begin{cases} g(x)\geq h(x)\\ h(x)>0\\ f(x)>1\end{cases}\\ \begin{cases} h(x)\geq g(x)\\ g(x)>0\\ 0<f(x)<1\end{cases}\end{array}\right.$$

$$\log_{f(x)} g(x) < \log_{f(x)} h(x) \Longleftrightarrow \left[\begin{array}{l} \left\{ \begin{array}{l} g(x) < h(x) \\ g(x) > 0 \\ f(x) > 1 \end{array} \right. \\ \left\{ \begin{array}{l} h(x) < g(x) \\ h(x) > 0 \\ 0 < f(x) < 1 \end{array} \right. \end{array} \right. \tag{17}$$

$$\log_{f(x)} g(x) \leq \log_{f(x)} h(x) \Longleftrightarrow \left[\begin{array}{l} \left\{ \begin{array}{l} g(x) \leq h(x) \\ g(x) > 0 \\ f(x) > 1 \end{array} \right. \\ \left\{ \begin{array}{l} h(x) \leq g(x) \\ h(x) > 0 \\ 0 < f(x) < 1 \end{array} \right. \end{array} \right. \tag{18}$$

Falls a, b sind Zahlen mit $a > 0, a \neq 1,$ dann

$$\log_a f(x) = b \Longleftrightarrow f(x) = a^b, \tag{19}$$

$$\log_a f(x) > b \Longleftrightarrow \left\{ \begin{array}{ll} f(x) > a^b, & a > 1 \\ 0 < f(x) < a^b, & a < 1, \end{array} \right. \tag{20}$$

$$log_a f(x) \geq b \Longleftrightarrow \left\{ \begin{array}{ll} f(x) \geq a^b, & a > 1 \\ 0 < f(x) \leq a^b, & a < 1, \end{array} \right. \tag{21}$$

$$\log_a f(x) < b \Longleftrightarrow \left\{ \begin{array}{ll} 0 < f(x) < a^b, & a > 1 \\ f(x) > a^b, & a < 1, \end{array} \right. \tag{22}$$

$$\log_a f(x) \leq b \Longleftrightarrow \left\{ \begin{array}{ll} 0 < f(x) \leq a^b, & a > 1 \\ f(x) \geq a^b, & a < 1. \end{array} \right. \tag{23}$$