A Sturmian Separation Theorem for Symplectic Difference Systems

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A STURMIAN SEPARATION THEOREM FOR SYMPLECTIC DIFFERENCE SYSTEMS

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Abstract. We establish a Sturmian separation theorem for conjoined bases of $2n$-dimensional symplectic difference systems. In particular, we show that the existence of a conjoined basis without focal points in a discrete interval $(0, N+1]$ implies that any other conjoined basis has at most n focal points (counting multiplicities) in this interval.

1. Introduction

In this paper we deal with the symplectic difference system

$$x_{k+1} = \mathcal{A}_k x_k + \mathcal{B}_k u_k, \quad u_{k+1} = \mathcal{C}_k x_k + \mathcal{D}_k u_k. \tag{1}$$

for $k = 0, \dots, N$, where $\mathcal{A}_k, \mathcal{B}_k, \mathcal{C}_k, \mathcal{D}_k$ are real $n \times n$ matrices, $x_k, u_k \in \mathbb{R}^n$, and $N \in \mathbb{N}$. It is supposed that the $2n \times 2n$ matrices

$$\mathcal{S}_k := \begin{pmatrix} \mathcal{A}_k & \mathcal{B}_k \\ \mathcal{C}_k & \mathcal{D}_k \end{pmatrix}$$

are *symplectic*, i.e.,

$$\mathcal{S}_k^T \mathcal{J} \mathcal{S}_k = \mathcal{J} \quad \text{with} \quad \mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

where I is the $n \times n$ identity matrix. Symplecticity of S_k in terms of the matrices $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ reads

$$\mathcal{A}_k^T \mathcal{C}_k = \mathcal{C}_k^T \mathcal{A}_k, \quad \mathcal{B}_k^T \mathcal{D}_k = \mathcal{D}_k^T \mathcal{B}_k, \quad \mathcal{A}_k^T \mathcal{D}_k - \mathcal{C}_k^T \mathcal{B}_k = I.$$

The symplectic difference system (1) is closely related to the discrete quadratic functional

$$\mathcal{F}(x, u) = \sum_{k=0}^{N} \left\{ x_k^T \mathcal{A}_k^T \mathcal{C}_k x_k + 2 x_k^T \mathcal{C}_k^T \mathcal{B}_k u_k + u_k^T \mathcal{D}_k^T \mathcal{B}_k u_k \right\}. \tag{2}$$

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A pair of n-dimensional sequences $z = (z_k)_{k=0}^{N+1} = (x_k, u_k)_{k=0}^{N+1}$ is said to be *admissible* for $\mathcal{F}$ if it satisfies the first equation in (1), the so-called *equation of motion*

$$x_{k+1} = \mathcal{A}_k x_k + \mathcal{B}_k u_k \quad \text{for } k = 0, \dots, N. \tag{3}$$

We will use the following *notation.* By $M^\dagger$ we denote the *Moore-Penrose inverse* of a matrix M (cf. [5]). For a real and symmetric matrix P we write $P \geq 0$ if P is *nonnegative definite* and $\operatorname{ind} P$ denotes its *index*, i.e., the number of negative eigenvalues (including multiplicities) of P. By $\operatorname{Ker} M$, $\operatorname{Im} M$, $\operatorname{rank} M$, M^T, and M^{-1} we denote the *kernel, image, rank, transpose*, and *inverse* of a matrix M, respectively.

Together with (1) we will consider its matrix version (labeled again by (1))

$$X_{k+1} = \mathcal{A}_k X_k + \mathcal{B}_k U_k, \quad U_{k+1} = \mathcal{C}_k X_k + \mathcal{D}_k U_k,$$

where X, U are real $n \times n$ matrices. A matrix solution (X, U) of (1) is said to be a *conjoined basis* if

$$\operatorname{rank}(X_k^T, U_k^T) = n \quad \text{and} \quad X_k^T U_k = U_k^T X_k \quad \text{for } k = 0, \dots, N+1. \tag{4}$$

The *principal solution at* $k = 0$ is the conjoined basis (X, U) which satisfies the initial condition $X_0 = 0$, $U_0 = I$.

The following matrices were introduced in [18]

$$\begin{cases} M_k = (I - X_{k+1} X_{k+1}^\dagger)\mathcal{B}_k, \\ T_k = I - M_k^\dagger M_k \\ P_k = T_k^T X_k X_{k+1}^\dagger \mathcal{B}_k T_k, \end{cases} \tag{5}$$

for $k \in \{0, \dots, N\}$. Then obviously $M_k T_k = 0$ and it can be shown (see, e.g., [18]) that the matrix P_k is symmetric.

We say that a conjoined basis (X, U) has no *focal point* in the interval $(k, k+1]$ if

$$\operatorname{Ker} X_{k+1} \subseteq \operatorname{Ker} X_k \quad \text{and} \quad X_k X_{k+1}^\dagger \mathcal{B}_k \geq 0 \tag{6}$$

holds. Note that if the first condition in (6) holds then the matrix $X_k X_{k+1}^\dagger \mathcal{B}_k$ is really symmetric, (cf. [8]), and it equals the matrix P_k given by (5) since $T_k = I$ in this case (cf. [18]). The *multiplicity* of a focal point in the interval $(k, k+1]$ is defined as the number (cf. [18])

$$\operatorname{rank} M_k + \operatorname{ind} P_k.$$

Now we are in a position to formulate the main result of our paper, its proof is postponed to Section 3.

Theorem 1. *Suppose that there exists a conjoined basis of* (1) *with no focal point in* $(0, N+1]$. *Then any other conjoined basis of this system has at most* n

focal points in $(0, N+1]$*, each focal point counted a number of times equal to its multiplicity.*

Remark 1. (i) The previous statement can be regarded as a discrete version of [19, Lemma 7.1, p. 357] which concerns the linear Hamiltonian differential system

$$\dot{x} = A(t)x + B(t)u, \quad \dot{u} = C(t)x - A^T(t)u, \tag{7}$$

where A, B, C are real $n \times n$ matrices, B, C are symmetric, and B is nonnegative definite. It is supposed that (7) is *identically normal* on the interval $[a, b]$, i.e., if (x, u) is a solution of this system such that $x(t) = 0$ on a nondegenerate subinterval of $[a, b]$ then $(x, u) \equiv (0, 0)$ on $[a, b]$. The above mentioned Lemma 7.1 of [19] claims: *If* (7) *is disconjugate on* $[a, b]$ *then the matrix* $X(t)$ *of any conjoined basis* (X, U) *of* (7) *is noninvertible for at most* n *points* $t \in (a, b]$. Recall that the *conjoined basis* of (7) is defined in the same way as in (4) (only with $(X(t), U(t))$ instead of (X_k, U_k)). Note also that under the assumption of identical normality, the disconjugacy of (7) in $[a, b]$ is equivalent to the existence of a conjoined basis $(\tilde{X}, \tilde{U})$ of this system with $\tilde{X}(t)$ invertible for $t \in (a, b]$.

(ii) The basic facts of the oscillation theory of (1) (especially, the concept of a focal point of a conjoined basis defined by (6)) were established in the paper [8] which appeared in 1997. Since that time, a relatively great effort has been made to define the *multiplicity* of a focal point (which plays a crucial role in Theorem 1). This problem was successfully solved in the recent paper [18] and this enabled to formulate Theorem 1 in the form presented here.

(iii) Symplectic difference systems cover a large variety of difference equations and systems. Let us recall at least the linear Hamiltonian difference system

$$\Delta x_k = A_k x_{k+1} + B_k u_k, \quad \Delta u_k = C_k u_{k+1} - A_k^T u_k, \tag{8}$$

where $\Delta x_k = x_{k+1} - x_k$ is the usual forward difference, A, B, C are real $n \times n$ matrices, B, C are symmetric and $I - A$ is invertible, the $2n^{\text{th}}$ order Sturm-Liouville difference equation

$$\sum_{\nu=0}^{n} (-1)^\nu \Delta^\nu \left(r_k^{[\nu]} \Delta^\nu y_{k+n-\nu} \right) = 0, \quad \Delta^\nu = \Delta(\Delta^{\nu-1}), \quad r_k^{[n]} \neq 0, \tag{9}$$

(which can be written in the form (8), see, e.g., [1]), and the special case $n = 1$ in (9) – the classical second order Sturm-Liouville difference equation

$$\Delta(r_k \Delta y_k) + p_k y_{k+1} = 0. \tag{10}$$

While the Sturmian theory is deeply developed for (10) and the separation theorem is well known (see, e.g., [2, 13, 16]), the statement presented in Theorem 1 *is new* even for the special cases of symplectic systems (8) and (9). Finally note that a kind of separation theorem for conjoined bases of (8) is given in [7] but this statement does not consider multiplicities of focal points. It claims that if there exists a conjoined basis of (8) without focal points in $(0, N+1]$ then the principal solution of this system in $k = 0$ has no focal point in $(0, N+1]$ as well.

2. Preliminaries

In this preparatory section we collect basic facts concerning relationships between the symplectic difference system (1) and the discrete quadratic functional (2) which we will need in the proof of our main result.

We start with a statement proved in [8, Theorem 1], which relates oscillatory properties of (1) (which are defined via (non)existence of focal points) to the positivity of the discrete quadratic functional (2). We formulate this statement in a slightly modified form here, we consider any conjoined basis instead of the principal solution at $k = 0$ (as formulated in [8]), but the proof of this modified statement is the same as that given in [8].

Recall that the functional $\mathcal{F}$ is said to be *positive* if $\mathcal{F}(x, u) \geq 0$ for every admissible (x, u) with $x_0 = 0 = x_{N+1}$, and equality $F(x, u) = 0$ happens if and only if $x \equiv 0$ (i.e., $x_k = 0$, $k = 0, \dots, N+1$).

Proposition 1. *The functional $\mathcal{F}$ is positive if and only there exists a conjoined basis of* (1) *which has no focal point in* $(0, N+1]$.

In the next statement (which can be found e.g. in [10] or [12]) we recall the construction of an admissible pair for which the functional (2) is nonpositive when the principal solution of (1) does have a focal point in $(0, N+1]$.

Proposition 2. *Let (X, U) be a conjoined basis of* (1).

(i) *If there exists $m \in \{0, \dots, N\}$ such that $\operatorname{Ker} X_{m+1} \not\subseteq \operatorname{Ker} X_m$, i.e., there exists $\alpha \in \operatorname{Ker} X_{m+1} \setminus \operatorname{Ker} X_m$, then the pair (x, u) defined by*

$$x_k = \begin{cases} X_k\alpha & 0 \leq k \leq m, \\ 0 & m+1 \leq k \leq N+1, \end{cases}$$

$$u_k = \begin{cases} U_k\alpha & 0 \leq k \leq m, \\ 0 & m+1 \leq k \leq N+1, \end{cases}$$

is admissible, and we have $\mathcal{F}(x, u) = -\alpha^T X_0 U_0 \alpha$.

(ii) *If there exists $m \in \{0, \dots, N\}$ such that $P_m \not\geq 0$, i.e., there exists $c \in \mathbb{R}^n$ such that $c^T P_m c < 0$, then the pair (x, u) defined by*

$$x_k = \begin{cases} X_k d & 0 \leq k \leq m, \\ 0 & m+1 \leq k \leq N+1, \end{cases}$$

$$u_k = \begin{cases} U_k d & 0 \leq k \leq m-1, \\ U_k d - T_k c & k = m, \\ 0 & m+1 \leq k \leq N+1, \end{cases}$$

where $d = X_{m+1}^{\dagger}\mathcal{B}_m T_m c$*, is admissible, and we have* $\mathcal{F}(x,u) = -\alpha^T X_0 U_0 \alpha + c^T P_m c$.

(iii) *In particular, if* (X,U) *is the principal solution at* $k=0$*, it holds* $\mathcal{F}(x,u) = 0$ *in case* (i) *and* $\mathcal{F}(x,u) < 0$ *in case* (ii).

Next we recall the concept of the *bilinear form* associated with (2).

Lemma 1. *Let* $\hat{z} = (\hat{x},\hat{u})$, $\tilde{z} = (\tilde{x},\tilde{u})$ *be two admissible pairs for* $\mathcal{F}$*. Then we have*

$$\begin{aligned}
\mathcal{F}(\hat{z};\tilde{z}) &:= \sum_{k=0}^{N}\left\{\hat{x}_k^T\mathcal{A}_k^T\mathcal{C}_k\tilde{x}_k + \hat{x}_k^T\mathcal{C}_k^T\mathcal{B}_k\tilde{u}_k + \hat{u}_k^T\mathcal{B}_k^T\mathcal{C}_k\tilde{x}_k + \hat{u}_k^T\mathcal{D}_k^T\mathcal{B}_k\tilde{u}_k\right\} \\
&= \hat{x}_k^T\tilde{u}_k\Big|_0^{N+1} + \sum_{k=0}^{N}\hat{x}_{k+1}^T\left\{\mathcal{C}_k\tilde{x}_k + \mathcal{D}_k\tilde{u}_k - \tilde{u}_{k+1}\right\} \\
&= \tilde{x}_k^T\hat{u}_k\Big|_0^{N+1} + \sum_{k=0}^{N}\tilde{x}_{k+1}^T\left\{\mathcal{C}_k\hat{x}_k + \mathcal{D}_k\hat{u}_k - \hat{u}_{k+1}\right\}.
\end{aligned}$$

In particular, if one of $\hat{z} = (\hat{x},\hat{u})$, $\tilde{z} = (\tilde{x},\tilde{u})$ *is a solution of* (1) *satisfying* $\hat{x}_0 = 0 = \hat{x}_{N+1}$ *or* $\tilde{x}_0 = 0 = \tilde{x}_{N+1}$*, then* $\mathcal{F}(\hat{z};\tilde{z}) = 0$.

In the proof of the last two auxiliary results of this section we will need the following consequence of Lemma 3.1.5 and Lemma 3.1.6 of [17], see also [18, p. 142].

Lemma 2. *Let* (X,U) *be a conjoined basis of* (1), M_k *be given by* (5)*, and* $k \in \{0,\dots,N\}$*. Then there exists an* $n\times n$ *matrix* S_k *such that*

(11) $$\operatorname{rank} S_k = \operatorname{rank} M_k, \quad X_{k+1}S_k = 0, \quad \operatorname{Ker} X_k \cap \operatorname{Im} S_k = \{0\}.$$

In the next two lemmas, the matrices M_k, P_k, T_k are defined by (5).

Lemma 3. *Let* $\operatorname{rank} M_k = p$*. Then there exist linearly independent vectors* $\alpha_1,\dots,\alpha_p \in \mathbb{R}^n$ *such that*

$$X_{k+1}\alpha_j = 0, \quad X_k\alpha_j \neq 0, \quad j = 1,\dots,p.$$

Proof. Let S_k be the $n\times n$ matrix for which (11) hold, and let $\alpha_1,\dots,\alpha_p$ be a basis of $\operatorname{Im} S_k$. Then $\operatorname{Im} S_k \subseteq \operatorname{Ker} X_{k+1}$ implies $X_{j+1}\alpha_j = 0$ and $\operatorname{Ker} X_k \cap \operatorname{Im} S_k = \{0\}$ implies $X_j\alpha_j \neq 0$, $j = 1,\dots,p$. □

Lemma 4. *Let* $(k,k+1]$ *contain a focal point of multiplicity* $p+q \le n$ *of a conjoined basis* (X,U) *of* (1), $p = \operatorname{rank} M_k$, $q = \operatorname{ind} P_k$*. Further, let* $\alpha_1,\dots,\alpha_p$ *be the same as in the Lemma 3 and* $\beta_1,\dots,\beta_q$ *be orthogonal vectors corresponding to the negative eigenvalues of* P_k*, i.e.,* $\beta_j^T P_k\beta_j < 0$, $j = 1,\dots,q$*. Denote* $\gamma_j = X_{k+1}^{\dagger}\mathcal{B}_k T_k\beta_j$*. Then the vectors* $\alpha_1,\dots\alpha_p,\gamma_1,\dots,\gamma_q$ *are linearly independent.*

Proof. First we prove that $\gamma_1, \dots, \gamma_q$ are linearly independent. Suppose that this is not the case, i.e., there exists a nontrivial linear combination $\sum_{j=1}^{q} \mu_j \gamma_j = 0$, and let $\beta = \sum_{j=1}^{q} \mu_j \beta_j$. Then

$$0 > \beta^T P_k \beta = \beta^T T_k^T X_k \left(\sum_{j=1}^{q} \mu_j X_{k+1}^{\dagger} \mathcal{B}_k T_k \beta_j \right) = \beta^T T_k^T X_k \left(\sum_{j=1}^{q} \mu_j \gamma_j \right) = 0,$$

a contradiction. Now suppose that $\gamma = \sum_{j=1}^{q} \mu_j \gamma_j = \sum_{j=1}^{p} \lambda_j \alpha_j \neq 0$, and let $\beta = \sum_{j=1}^{q} \mu_j \beta_j$ be as before. Then

$$0 > \beta^T P_k \beta = \beta^T T_k^T X_k X_{k+1}^{\dagger} \mathcal{B}_k T_k \beta = \beta^T T_k^T X_k X_{k+1}^{\dagger} X_{k+1} \gamma = 0,$$

a contradiction. □

3. Proof of Theorem 1

The proof of Theorem 1 is based on the following idea. By Proposition 1, the existence of a conjoined basis of (1) without focal points in $(0, N+1]$ implies positivity of $\mathcal{F}$. Now, by contradiction, if we assume that there exists another conjoined basis with more than n focal points in $(0, N+1]$ (counting multiplicities), we are able to construct an admissible pair (x, u) with $x \neq 0$ for which $\mathcal{F}(x, u) \leq 0$ which contradicts the positivity of $\mathcal{F}$.

Before starting the proof, we slightly specify the terminology concerning the multiplicity of a focal point. If $(k, k+1]$ contains a focal point of multiplicity $p + q$, where $p = \operatorname{rank} M_k$, $q = \operatorname{ind} P_k$, we say that p focal points are at $k+1$ and q focal points are in the open interval $(k, k+1)$.

Proof of Theorem 1. Let (X, U) be a conjoined basis of (1) and let the intervals

$$(k_i, k_i + 1] \subseteq (0, N+1], \quad i = 1, \dots, l, \ 0 \leq k_1 < k_2 < \cdots < k_l \leq N,$$

contain focal points of (X, U) of multiplicities m_i, $i = 1, \dots, l$. Let $m_i = p_i + q_i$, where $p_i = \operatorname{rank} M_{k_i}$, $q_i = \operatorname{ind} P_{k_i}$. For each interval $(k_i, k_i + 1]$ define the admissible pairs as follows. For $j = 1, \dots, p_i$ we set

$$x_k^{[i,j]} = \begin{cases} X_k \alpha_j^{[i]} & 0 \leq k \leq k_i, \\ 0 & k_i + 1 \leq k \leq N+1, \end{cases} \tag{12}$$
$$u_k^{[i,j]} = \begin{cases} U_k \alpha_j^{[i]} & 0 \leq k \leq k_i, \\ 0 & k_i + 1 \leq k \leq N+1, \end{cases}$$

where $\alpha_j^{[i]} \in \operatorname{Ker} X_{k_i+1} \setminus \operatorname{Ker} X_{k_i}$ are linearly independent n-dimensional vectors (see Lemma 4). For $j = p_i + 1, \ldots, p_i + q_i$ we define

$$x_k^{[i,j]} = \begin{cases} X_k \gamma_j^{[i]} & 0 \le k \le k_i, \\ 0 & k_i + 1 \le k \le N+1, \end{cases}$$

$$(13) \qquad u_k^{[i,j]} = \begin{cases} U_k \gamma_j^{[i]} & 0 \le k \le k_i - 1, \\ U_k \gamma_j^{[i]} - T_k \beta_j^{[i]} & k = k_i, \\ 0 & k_i + 1 \le k \le N+1, \end{cases}$$

where $\beta_j^{[i]}$, $j = 1, \ldots, q_i$, are orthogonal eigenvectors corresponding to the negative eigenvalues of the matrix P_{k_i}, and $\gamma_j^{[i]} = X_{k_i+1}^{\dagger} \mathcal{B}_{k_i} \beta_j^{[i]}$. By Proposition 2 we have for any $i \in \{1, \ldots, l\}$

$$\mathcal{F}(x^{[i,j]}, u^{[i,j]}) = (x_0^{[i,j]})^T u_0^{[i,j]}, \quad j = 1, \ldots, p_i,$$
$$\mathcal{F}(x^{[i,j]}, u^{[i,j]}) = (x_0^{[i,j]})^T u_0^{[i,j]} + (\beta_j^{[i]})^T P_{k_1} \beta_j^{[i]}, \; j = p_i + 1, \ldots, p_i + q_i.$$

To simplify some of the next computations, we relabel occasionally the quantities $x^{[i,j]}, u^{[i,j]}, \alpha_j^{[i]}, \ldots$ as follows. We introduce the index $\ell \in \{1, \ldots, \sum_{i=1}^{l} m_i\}$ by $[i,j] \longmapsto \ell = \sum_{s=0}^{i-1} m_s + j$, $m_0 := 0$.

Now suppose, by contradiction, that the number of focal points of (X, U) in $(0, N+1]$ exceeds n, i.e., $m := \sum_{i=1}^{l} m_i > n$. In order to make the idea of the proof more understandable, we will first suppose that $q_i = 0$, $i = 1, \ldots, l$, i.e., all focal points are at $k_i + 1$ (the kernel condition is violated but all $P_{k_i} \ge 0$). Since $\sum_{i=1}^{l} m_i = \sum_{i=1}^{l} p_i = m > n$, there exists a nontrivial linear combination

$$(14) \qquad \sum_{\ell=1}^{m} \mu_\ell x_0^{[\ell]} = 0,$$

i.e., the admissible pair (x, u) given by

$$(15) \qquad x_k = \sum_{\ell=1}^{m} \mu_\ell x_k^{[\ell]}, \quad u_k = \sum_{\ell=1}^{m} \mu_\ell u_k^{[\ell]}, \quad k = 1, \ldots, N+1,$$

satisfies $x_0 = 0 = x_{N+1}$. Moreover, the Nn-dimensional vector $x = \{x_k\}_{k=1}^{N}$ is *nonzero*. Indeed, consider first the largest focal point $k_l + 1$ in $(0, N+1]$. According to the construction of $x^{[i,j]}$ (we return to the original labeling at this moment), we have

$$x_{k_l}^{[i,j]} = 0, \quad i = 1, \ldots, l-1, \; j = 1, \ldots, p_i,$$

so if $x = 0$, i.e., in particular, $x_{k_l} = 0$, we have

$$\sum_{j=1}^{p_l} \mu_{l,j} x_{k_l}^{[l,j]} = \sum_{j=1}^{p_l} \mu_{l,j} X_{k_l} \alpha_j^{[l]} = X_{k_l} \left(\sum_{j=1}^{p_l} \mu_{l,j} \alpha_j^{[l]} \right) = 0. \tag{16}$$

Since the vectors $\alpha_j^{[l]}$, $j = 1, \dots, p_l$, form the basis of the space $\operatorname{Im} S_{k_l}$, where S_{k_l} is the same as S_k in the proof of Lemma 3 (here with $k = k_l$), and at the same time by (16)

$$\sum_{j=1}^{p_l} \mu_{l,j} \alpha_j^{[l]} \in \operatorname{Ker} X_{k_l},$$

we have $\sum_{j=1}^{p_l} \mu_{l,j} \alpha_j^{[l]} = 0$ because of Lemma 2, which means that that $\mu_{l,j} = 0$, $j = 1, \dots, p_l$, since the vectors $\alpha_j^{[l]}$ are linearly independent. Repeating the previous argument for $k = k_{l-1}, \dots, k = k_1$, we find that $\mu_{i,j} = 0$, $i = 1, \dots, l$, $j = 1, \dots, p_i$, which contradicts our assumption that the linear combination (14) is nontrivial. Therefore, $x \neq 0$ in the admissible pair given by (15).

Now, let $z^{[\kappa]} = (x^{[\kappa]}, u^{[\kappa]})$, $z^{[\ell]} = (x^{[\ell]}, u^{[\ell]})$, $\kappa, \ell \in \{1, \dots, m\}$, be two admissible pairs constructed by (12). Then substituting into the formula in Lemma 1 we find that

$$\mathcal{F}(z^{[\kappa]}, z^{[\ell]}) = \begin{cases} 0, & \kappa \neq \ell, \\ (x_0^{[\ell]})^T u_0^{[\ell]}, & \kappa = \ell. \end{cases} \tag{17}$$

Consequently, for $z = (x, u)$ given by (15)

$$\begin{aligned} \mathcal{F}(z) &= \mathcal{F}\big(\sum_{\ell=1}^{m} \mu_l z^{[\ell]}\big) = \sum_{\kappa,\ell=1}^{m} \mu_\kappa \mu_\ell \mathcal{F}(z^{[\kappa]}; z^{[\ell]}) \\ &= \left(\sum_{\ell=1}^{m} \mu_\ell x_0^{[\ell]} \right)^T \left(\sum_{\ell=1}^{m} \mu_\ell u_0^{[\ell]} \right) = x_0^T u_0 = 0, \end{aligned}$$

since $x_0 = 0$ by (14). This contradicts the positivity of $\mathcal{F}$.

Suppose now that at least one of the q_i, $i = i, \dots, l$, is positive. Then we have for this index

$$\mathcal{F}(x^{[i,j]}, u^{[i,j]}) = (x_0^{[i,j]})^T u_0^{[i,j]} + (\beta_j^{[i]})^T P_{k_i} \beta_j^{[i]},$$

$j = 1, \dots, p_i$, and we have admissible pairs defined both by (12) and (13). In the previous part of the proof we have already computed $\mathcal{F}(z^{[\kappa]}; z^{[\ell]})$ for admissible pairs given by (12). It remains to compute this bilinear form if one or both admissible pairs are of the form (13). We will perform the computation in the latter case. In the former case (i.e., one of the admissible pairs is given by (12) and the second one by (13)), substituting into the formula in Lemma 1 we get again (17). So, let $z^{[\kappa]}$, $z^{[\ell]}$ be two admissible pairs given (13). If they are associated with the different focal intervals (i.e., the integers k_i in (13) are different for $z^{[\kappa]}$,

$z^{[\ell]}$), using Lemma 1 we find again that (17) holds. Therefore, suppose finally that $z^{[\kappa]}$, $z^{[\ell]}$ correspond to the same focal interval (k_i, k_i+1). Then

$$\begin{aligned}\mathcal{F}(z^{[\kappa]}; z^{[\ell]}) &= (x_0^{[\kappa]})^T u_0^{[\ell]} + (x_{k_i}^{[\kappa]})^T \{\mathcal{C}_{k_i-1} x_{k_i-1}^{[\ell]} + \mathcal{D}_{k_i-1} u_{k_i-1}^{[\ell]} - u_{k_i}^{[\ell]}\} \\ &= (x_0^{[\kappa]})^T u_0^{[\ell]} + (\gamma^{[\kappa]})^T X_{k_i} T_{k_i} \beta^{[\ell]} \\ &= (x_0^{[\kappa]})^T u_0^{[\ell]} + (\beta^{[\kappa]})^T P_{k_i} \beta^{[\ell]}.\end{aligned}$$

If $\kappa \neq \ell$, the vectors $\beta^{[\kappa]}$, $\beta^{[\ell]}$ are orthogonal eigenvectors of the matrix P_{k_i} and thus $(\beta^{[\kappa]})^T P_{k_i} \beta^{[\ell]} = 0$.

Summarizing our previous computations, for $z = (x, u)$ given by (15) (i.e., $x_0 = 0$ by (14)), we have (again with the two-indices labeling)

$$\mathcal{F}(x, u) = \sum_{i=1}^{l} \sum_{j=1}^{q_i} (\beta_j^{[i]})^T P_{k_i} \beta_j^{[i]} < 0,$$

which again contradicts the positivity of $\mathcal{F}$. Note that $x = \{x_k\}_{k=1}^N$ is again nontrivial, since for each $i = 1, \ldots, l$ the vectors $\alpha_j^{[i]}, \gamma_s^{[i]}$, $j = 1, \ldots, p_i$, $s = 1, \ldots, q_i$, are linearly independent (Lemma 4) and one can repeat the same argument as used in that part of the proof where we supposed that $q_i = 0$, $i = 1, \ldots, l$. □

4. Remarks

In this last section we collect various remarks, comments and open problems related to the results presented in the previous part of the paper.

(i) In [19, Section VII.7], one can also find a more general separation theorem for focal points of conjoined bases of the linear Hamiltonian differential system (7) than that mentioned in Remark 1. Namely, under the assumption of identical normality, the numbers of focal points of two conjoined bases in any interval differ by at most n. This statement is based on the concept of *broken extremals* and its proof substantially uses the assumption of identical normality. Since we impose *no normality* assumption on the symplectic difference system (1), we were able to prove a separation theorem only in the (weaker) form presented here.

(ii) The quadratic functional $\mathcal{F}$ is a "normal" quadratic form which has its index and nullity on the finitedimensional space of admissible (x, u) (more precisely, one can speak about admissible x only since the value of the functional $\mathcal{F}$ actually does not depend on u which appears in the equation of motion (3), see e.g. [15]). A natural question is what is the relationship between this index and nullity of the quadratic form $\mathcal{F}$ and the number of focal points of a suitably chosen conjoined basis (depending on the boundary condition for admissible pairs).

(iii) In [8, Theorem 1, (iv) and (ix)], it is claimed that the principal solution of (1) at $k = 0$ has no focal point in $(0, N+1]$ if and only if the principal solution of this system at $k = N + 1$ (i.e., the solution given by $X_{N+1} = 0$, $U_{N+1} = -I$) has no focal point in $[0, N + 1)$, where the "no focal point" of the principal solution at $N + 1$ is defined by

$$\operatorname{Ker} X_k \subseteq \operatorname{Ker} X_{k+1} \quad \text{and} \quad X_{k+1} X_k^{\dagger} \mathcal{B}_k^T \geq 0 \quad \text{for } k = 0, \dots, N.$$

A natural question is whether it can be formulated some statement about *the number* of focal points of the principal solutions at $k = 0$ and $k = N + 1$. A statement of this kind would be a discrete analogue of the statement for (7) that the number of focal points in $(a, b]$ of the conjoined basis given by $X(a) = 0$, $U(a) = I$ is *the same* as the number of focal points in $[a, b)$ of the basis given by $X(b) = 0$, $U(b) = -I$.

(iv) The separation theorem for (7) mentioned in the part (i) has a nice geometrical interpretation using the concepts of trigonometric system and trigonometric transformation of Hamiltonian differential systems, see [4, 14, 11]. The discrete trigonometric symplectic systems were introduced in [3] and the discrete trigonometric transformation (i.e., a possibility to transform any symplectic difference system (1) into a trigonometric difference system by a transformation preserving focal points) was established in [9]. A subject of the present investigation is to find a geometric interpretation of the concept of the focal point of (1) in terms of the eigenvalues of certain unitary matrices, similarly as it is done for (7) in the above mentioned papers.

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