On Boundary Value Problems for Surfaces of a Prescribed Line Element with Positive Curvature

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Abstract

In this paper the Dirichlet problem for local convex surfaces with a prescribed line element is studied. Utilizing the geometric invariants of surfaces and curves, the invariants of the Dirichlet data will be transformed into invariants of the surface. There are only two other very specific results concerning boundary value problems in the context of surfaces with prescribed line element. These results go back to Heinz [6] and to Hong [8] and. As an application of the transformation process some a priori estimates for the Dirichlet problem for surfaces of prescribed line element are proved.

1 Introduction

Let $B \subset \mathbb{R}^2$ be the open unit disk. In the closure $\overline{B}$ of B, a Riemannian metric

$$g_{ij} = g_{ij}(x^1, x^2) : \overline{B} \to \mathbb{R} \in C^{4,\alpha}(\overline{B})$$

with positive Gaussian curvature

$$K = K(x^1, x^2) : \overline{B} \to (0, \infty) \in C^{2,\alpha}(\overline{B})$$

is given, where $\alpha \in (0, 1)$ is denoting a Hölder exponent. The matrix $(g_{ij})_{i,j=1,2}$ is assumed to be positive-definite: i.e. $g_{11} > 0$, $\det(g_{ij}) > 0$ in $\overline{B}$. Like all other matrices arising in this paper, it is a 2×2 matrix. Consider a solution

$$\mathbf{X} = \mathbf{X}(x^1, x^2) : \overline{B} \to \mathbb{R}^3 \in C^3(\overline{B}, \mathbb{R}^3)$$

of the system of partial differential equations

$$(\partial_i \mathbf{X}, \partial_j \mathbf{X}) = g_{ij} \quad \text{in} \quad B, \quad i,j = 1,2. \tag{1}$$

In this paper the partial derivative with respect to x^i is denoted by ∂_i and the canonical scalar product by $(\, . \, , \, . \,)$.

The solvability of the system (1) is often studied in literature published this far, see [5] as comprehensive compendium for this topic with further references.

Only two papers are covering boundary value problems for that system. The first paper [6] goes back to Heinz. There was discussed a Dirichlet Problem for a conjugate-conformal reparametrization and not any boundary values for the surface itself. The other work goes back to Hong [8], where solutions with boundary values variing on a plane: $(\mathbf{X}, \mathbf{X}_0) \equiv 0$ on ∂B are studied.

Inspired by Heinz' work the author could translate the Dirichlet boundary data $\mathbf{Y} : \partial B \to \mathbb{R}^3$ to first order boundary data of some conjugate-conformal reparametrisation and vice versa using the tool of geometric invariants.

Consider

$$\mathbf{Y} = \mathbf{Y}(x^1, x^2) : \partial B \to \mathbb{R}^3 \in C^{4,\alpha}(\partial B, \mathbb{R}^3) \tag{2}$$

a closed curve and a solution $\mathbf{X}$ of the system boundary value problem

$$\begin{aligned} &\mathbf{X} : \overline{B} \to \mathbb{R}^3 \in C^3(\overline{B}) \\ &(\partial_i \mathbf{X}, \partial_j \mathbf{X}) = g_{ij} \quad \text{in} \quad B \\ &\mathbf{X}|_{\partial B} = \mathbf{Y}. \end{aligned} \tag{3}$$

Under further assumptions it will be shown that the solvability of problem (3) is equivalent to the solvability of the following problem

$$\begin{aligned} &z = (z^1, z^2) : \overline{E} \to \overline{B} \in C^2(E) \cap C^1(\overline{E}) \\ &\Delta z = h_{ij}(z) \nabla z^i \nabla z^j \quad \text{in} \quad E \\ &\partial z_r = k_i(z) z_t^i \quad \text{on} \quad \partial E. \end{aligned}$$

Here and up to the end the Einstein summation convention is used. The functions $h_{ij} : B \to \mathbb{C}$ and $k_i : \partial B \to \mathbb{C}$ are determined by the data g_{ij} and $\mathbf{Y}$ of the problem (3) and they will be denoted explicitly. The set E can be chosen arbitrarily but for the sake of simplicity in this paper it is the open unit disk. Moreover from a-priori estimates valid for conformal mappings z a-priori estimates of the $C^{3,\alpha}(B)$-norm of a solution $\mathbf{X}$ of (3) are deduced.

In the literature the process of showing the equivalency of the solvability of a system of differenial equations as in problem (3) to the solvability of a system as in problem (4) without any boundary data is already studied, see [6] for instance. The author added in that process the covering of boundary data. In order to establish a-priori estimates the translation process even of the systems of differential equations has to be studied in a more precise way.

Therefore, from here to the end of this section the well-known process of the translation of the systems is discussed. Consider a regular C^3-surface $\mathbf{X}$, for simplicity, defined on the unit disk B with the linear independent tangentials $\partial_i \mathbf{X}$ and the unit normal

$$\mathbf{N} = \frac{\partial_1 \mathbf{X} \wedge \partial_2 \mathbf{X}}{|\partial_1 \mathbf{X} \wedge \partial_2 \mathbf{X}|}.$$

As mentioned above the consideration of the geometric invariants is succesful, so one considers its first fundamental form

$$g_{ij} = (\partial_i \mathbf{X}, \partial_j \mathbf{X})$$

and its second fundamental form

$$b_{ij} = -(\partial_i \mathbf{X}, \partial_j \mathbf{N}) = (\partial_{ij} \mathbf{X}, \mathbf{N}).$$

The vectors $\partial_1 \mathbf{X}, \partial_2 \mathbf{X}$ and $\mathbf{N}$ form a trihedron, and expressing its first derivatives as linear combination of the trihedron one arrives at the so-called Gauss-Weingarten system

$$\begin{array}{rcl} \partial_{ij} \mathbf{X} &=& \Gamma_{ij}^k \partial_k \mathbf{X} + b_{ij} \mathbf{N} \\ \partial_i \mathbf{N} &=& -b_{ij} g^{jk} \partial_k \mathbf{X} \end{array}$$

with the inverse matrix g^{jk} and the Christoffel symbols $\Gamma_{ij}^k = \frac{1}{2} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$.

By an elementary calculation one shows the validity of the neccesary conditions of integrability. These are called the Codazzi-Mainardi equations

$$\partial_k b_{ij} - \Gamma_{ik}^m b_{mj} = \partial_j b_{ik} - \Gamma_{ij}^m b_{mk}$$

and the Gaussian theorema egregium

$$\begin{array}{rcl} b_{11} b_{22} - b_{12}^2 &=& -\frac{1}{2}(\partial_{11} g_{22} - 2\partial_{12} g_{12} + \partial_{22} g_{11}) - g_{kl}(\Gamma_{11}^m \Gamma_{22}^k - \Gamma_{12}^m \Gamma_{12}^k) \\ &=:& K \det(g_{ij}). \end{array}$$

But also the converse is true.

Theorem 1.1. (Fundamental theorem of surface theory) *Let g_{ij} be a symmetric positive-definite matrix of class $C^2(\overline{B})$ and b_{ij} be a symmetric matrix of class $C^1(\overline{B})$. Let further the Gauss and Codazzi-Mainardi equations be fullfilled. Then there exists up to isometric transformations (translations and rotations) exactly one regular surface $\mathbf{X} \in C^3(\overline{B})$ with normal $\mathbf{N} \in C^2(\overline{B})$, having g_{ij} and b_{ij} as its first and second fundamental form.*

For this theorem, many references are valid, see [3, pp. 237], [14, pp. 146], [13, Vol. IV, pp. 61] or [1, pp. 138]. So the system (1) for a given metric g_{ij} is solved, iff one can state a matrix b_{ij} fulfilling the neccesary and (by the theorem above) also sufficient conditions of integrability. Thus the question of how to solve the conditions of integrability now arrises.

Inspired by the theorema egregium the Gaussian curvature K of a metric is defined by

$$K := \frac{-\frac{1}{2}(\partial_{11}g_{22} - 2\partial_{12}g_{12} + \partial_{22}g_{11}) - g_{kl}(\Gamma^m_{11}\Gamma^k_{22} - \Gamma^m_{12}\Gamma^k_{12})}{\det(g_{ij})}. \tag{4}$$

Therefore K belongs to the data of the problem and is assumed to be positive.

Again, consider a solution $\mathbf{X} \in C^3(\overline{B})$ of (1) with $K > 0$. By the Gauss equation its second fundamental form has a positive determinant. Eventually after renaming the independend variables, one can assume it to be positive-definite. Therefore globally conformal parameters with a mapping

$$z = (z^1(w), z^2(w)) : \overline{E} \to \overline{B} \in C^2(E) \cap C^1(\overline{E}) \tag{5}$$

with Jacobian $\nabla z^1 \wedge \nabla z^2 := \partial_1 z^1 \partial_2 z^2 - \partial_2 z^1 \partial_1 z^2 > 0$ are introduced. The b_{ij}'s and the mapping z are connected by the relations of conformality

$$b_{ij}(z)\partial_w z^i \partial_w z^j = 0 \quad \text{in} \quad E$$

with the Wirtinger derivative $\partial_w = \frac{1}{2}(\partial_1 + i\partial_2)$. Due to

$$\beta(z) = (\det(b_{kl}(z)))^{1/2} > 0$$

the inverse b^{jk} exists and the relations of conformality can be restated in the form

$$(\beta(z)b^{ij}(z)) = \left.\frac{\nabla z^i \cdot \nabla z^j}{\nabla z^1 \wedge \nabla z^2}\right|_{z^{-1}(z)}. \tag{6}$$

By elementary linear algebra these relations are fulfilled, iff the Cauchy-Riemann-Beltrami equations

$$\begin{aligned} \partial_1 z^1 &= \beta^{-1}(z) b_{2i}(z) \partial_2 z^i, & \partial_2 z^1 &= -\beta^{-1}(z) b_{2i}(z) \partial_1 z^i \\ \partial_1 z^2 &= -\beta^{-1}(z) b_{1i}(z) \partial_2 z^i, & \partial_2 z^2 &= \beta^{-1}(z) b_{1i}(z) \partial_1 z^i \end{aligned}$$

are fulfilled.

Differentiating this, one arrives at

$$\Delta z^i \quad = \quad \partial_{z^j} \left(\beta(z) b^{ji}(z) \right) \nabla z^1 \wedge \nabla z^2 \quad \text{in} \quad B.$$

Introducing the conditions of integrability and the equations (6) one arrives at the system

$$\mathcal{D}_{\overline{w}} \left(\mathcal{D}_w z^k \right) + \mathcal{D}_w \left(\mathcal{D}_{\overline{w}} z^k \right) + \Gamma^k_{ij} \left(\mathcal{D}_w z^i \mathcal{D}_{\overline{w}} z^j + \mathcal{D}_{\overline{w}} z^i \mathcal{D}_w z^j \right) = 0 \tag{7}$$

with the Darboux derivatives $\mathcal{D}_w = \sqrt{K(z)}\partial_w$ and $\mathcal{D}_{\overline{w}} = \sqrt{K(z)}\partial_{\overline{w}}$. Note, this is a system of the form

$$\Delta z = h_{ij}(z) \nabla z^i \nabla z^j$$

with coefficients $h_{ij}(z)$, depending only on g_{ij} and their first three derivatives. Moreover, a diffeomorphic solution of (7) leads by the definition

$$b^{ij}(z) := (K(z) \det(g_{kl}(z))^{-1/2} \left. \frac{\nabla z^i \cdot \nabla z^j}{\nabla z^1 \wedge \nabla z^2} \right|_{z^{-1}(z)}$$

to b_{jk}'s which fulfill the conditions of integrability. Remarking the fundamental theorem the system (1) is therefore solved iff the system (7) can be solved in the class of diffeomorphisms. For a reference of this statements see for instance [6] or [12].

In order to consider boundary value problems for the system (1) the author could reformulate the boundary data for the conjugate-conformal mapping in a similar manner.

There were several steps in doing so:

1. Identify the geometric invariants of normal cuts of curves and prove an analogon of the fundamental theorem for normal cuts of curves.

2. Express the invariants of normal cuts of curves in terms of invariants of the solution surface.

3. Transform these equations in conjugate-conformal parametrisation and obtain boundary values for the conjugate-conformal transformation z.

4. Check, whether the converse holds true, i.e. does the right boundary values of the solution reached if the obtained boundary values for the conjugate-conformal transformation are prescribed.

During this process it became as clear that a-priori estimates holds true. These estimates are formulated, too.

2 Invariants of curves and normal cuts.

It is convinient to fix the objects the next sections talk about. To avoid multiple defintions there is one central. Remark $\alpha \in (0, 1)$ is a Hölder exponent again, for $\alpha = 0$ the class $C^{k,0}$ denotes the class C^k.

Definition 2.1. *1. Let*

$$\mathbf{Y} = \mathbf{Y}(t) : [0, T] \to \mathbb{R}^3$$

be a differentiable mapping with $\mathbf{P}(t) = \mathbf{Y}'(t)$, $t \in [0, T]$. *If*

$$\mathbf{P}^2(t) > 0 \quad \textit{for} \quad t \in [0, T]$$

is fulfilled, then $\{\mathbf{Y}\}$ *is called a* curve. *If*

$$\mathbf{P}^2(t) = 1 \quad \textit{for} \quad t \in [0, T]$$

is fulfilled, then $\{\mathbf{Y}\}$ *is called a* curve in arc-length parametrisation.

2. Let $\{\mathbf{Y}\}$ *be a curve (in arc-length parametrisation). Let*

$$\mathbf{N} = \mathbf{N}(t) : [0, T] \to S^2$$

be a mapping under the condition

$$(\mathbf{P}(t), \mathbf{N}(t)) \equiv 0 \quad \textit{for} \quad t \in [0, T].$$

Then the tupel $\{\mathbf{Y}, \mathbf{N}\}$ *is called a* normal cut (in arc-lenght parametrisation) with normal $\mathbf{N}$ *or shortly a* normal cut.

3. *Let* $\{\mathbf{Y}\}$ *be a curve resp.* $\{\mathbf{Y},\mathbf{N}\}$ *be a normal cut. If*

$$\mathbf{P},\mathbf{N}\in C^{k,\alpha}[0,T]$$

is fulfilled, then $\{\mathbf{Y}\}$ *resp.* $\{\mathbf{Y},\mathbf{N}\}$ *is called a* curve *resp. a* normal cut of class $C^{k,\alpha}[0,T]$.

4. *Let* $\{\mathbf{Y}\}$ *be a curve resp.* $\{\mathbf{Y},\mathbf{N}\}$ *be a normal cut of the class* $C^{k,\alpha}[0,T]$. *If:*

   (a) $\mathbf{Y}(0)=\mathbf{Y}(T)$ *and*

   (b) the mapping $\mathbf{P},\mathbf{N}$ *can be extended periodically to* $\mathbf{P},\mathbf{N}\in C^{k,\alpha}(\mathbb{R})$

   is fulfilled, then $\{\mathbf{Y}\}$ *resp.* $\{\mathbf{Y},\mathbf{N}\}$ *is called a* closed curve *resp. a* closed normal cut of class $C^{k,\alpha}[0,T]$.

5. *Let* $\{\mathbf{Y},\mathbf{N}\}$ *be a normal cut in arc-length parametrisation of class* $C^1[0,T]$. *In order of the later purposes the vector* $\mathbf{Q}:=-\mathbf{P}\wedge\mathbf{N}$ *is called the* radial tangential *and the vector*

$$\mathbf{K}=(\mathbf{K}^{(1)}(t),\mathbf{K}^{(2)}(t),\mathbf{K}^{(3)}(t)):[0,T]\to\mathbb{R}^3$$

is called the curvature vector *with the components*

$$\begin{array}{rclclcl} -\mathbf{K}^{(1)} & = & -(\mathbf{Q},\mathbf{N}') & = & (\mathbf{Q}',\mathbf{N}) & = & \det(\mathbf{N},\mathbf{N}',\mathbf{P}) \\ \mathbf{K}^{(2)} & = & -(\mathbf{P},\mathbf{N}') & = & (\mathbf{P}',\mathbf{N}) & = & -\det(\mathbf{P},\mathbf{P}',\mathbf{Q}) \\ -\mathbf{K}^{(3)} & = & (\mathbf{P}',\mathbf{Q}) & = & -(\mathbf{P},\mathbf{Q}') & = & \det(\mathbf{P},\mathbf{P}',\mathbf{N}). \end{array} \tag{8}$$

which are called

$$\begin{array}{rl} -\mathbf{K}^{(1)} & \textit{(geodesic) torsion,} \\ \mathbf{K}^{(2)} & \textit{normal curvature,} \\ -\mathbf{K}^{(3)} & \textit{geodesic curvature.} \end{array}$$

Remark 2.2. *1. Arc-lenght parametrisation can be introduced. See [3, pp. 6], [14, pp. 54] or [1, pp. 15]. One has to consider the inverse* $t(s)$ *of*

$$s(t)=\int_0^t|\mathbf{P}(\tau)|\,d\tau.$$

Regularity questions are left for the reader.

2. *Let $\{\mathbf{Y}\}$ be a curve in arc-length parametrisation of class $C^{k,\alpha}[0,T]$. The implicit functions theorem states the existence of a normal $\mathbf{N}$ such that $\{\mathbf{Y},\mathbf{N}\}$ is a normal cut of class $C^{k,\alpha}[0,T]$.*

3. *Let $k \in \mathbb{N}$ and $\{\mathbf{Y}\}$ be a (closed) curve in arc-length parametrisation of class $C^{k,\alpha}[0,T]$ under the condition*

$$P'(t) \neq 0 \quad \textit{for} \quad t \in [0,T].$$

Then the mapping $\mathbf{N}(t) := \mathbf{P}(t) \wedge \frac{\mathbf{P}'(t)}{|\mathbf{P}'(t)|}$ fulfills $(\mathbf{P},\mathbf{N}) \equiv 0$ and $\{\mathbf{Y},\mathbf{N}\}$ is a (closed) normal cut in arc-length parametrisation of the regularity class $C^{k,\alpha}[0,T]$. In that case we have

$$\begin{array}{rclcl} -\mathbf{K}^{(1)} & = & -(\mathbf{Q}',\mathbf{N}) & = & \dfrac{\det(\mathbf{P},\mathbf{P}',\mathbf{P}'')}{(\mathbf{P}',\mathbf{P}')} \\ \mathbf{K}^{(2)} & = & -\det(\mathbf{P},\mathbf{P}',\mathbf{Q}) & = & 0 \\ -\mathbf{K}^{(3)} & = & (\mathbf{P}',\mathbf{Q}) & = & |\mathbf{P}'|. \end{array} \tag{9}$$

These formulae are well-known in differential geometry of curves, see again [3], [14] or [1]. By these comments a curvature vector is defined in a natural way.

4. *Let $\mathbf{X} : \overline{B} \to \mathbb{R}^3 \in C^3(\overline{B})$ be a regular, conform parametrised surface with normal $\mathbf{M}$. Then the mappings $\mathbf{Y}(t) := \mathbf{X}(\cos t, \sin t)$, $\mathbf{N}(t) := \mathbf{M}(\cos t, \sin t)$ define a closed normal cut $\{\mathbf{Y},\mathbf{N}\}$. Moreover we have $\mathbf{P} = \lambda \partial_t \mathbf{X}$ and $\mathbf{Q} = \mu \partial_r \mathbf{X}$ for the partial derivatives with $x + iy = re^{it}$ with some functions λ, μ. This explains the notion of radial tangential.*

5. *The components of the curvature vector are the geometric invariants of a normal cut. This is point 1 of the program to do.*

The next lemma also contributes to point 1. of the program.

Lemma 2.3. *Let $k \in \mathbb{N}$ and*

$$\mathbf{K} = (\mathbf{K}^{(1)}(t), \mathbf{K}^{(2)}(t), \mathbf{K}^{(3)}(t)) : [0,T] \to \mathbb{R}^3 \in C^{k-1,\alpha}[0,T]$$

be a vector-valued function. Then there exists up to isometric transformations (translations and rotations) exactly one normal cut in arc-length parametrisation of class $C^{k,\alpha}[0,T]$ having $\mathbf{K}$ as its curvature vector.

Proof. 1. The is separated in different parts. At first, the analogon of the Gauss-Weingarten equations is pro ved. Let $\{\mathbf{Y}, \mathbf{N}\}$ be a normal cut in arc-length parametrisation of class $C^{k,\alpha}[0,T]$ with curvature vector $\mathbf{K}$. Let $\mathbf{P} = \mathbf{Y}'$ and $\mathbf{Q} = -\mathbf{P} \wedge \mathbf{N}$, then the vectors $\mathbf{P}, \mathbf{Q}, \mathbf{N}$ form by their definition an orthonormal trihedron for every $t \in [0,T]$. Because of $k \in \mathbb{N}$, the first derivatives exist and can be developed then in terms of that trihedron. For $\mathbf{P}'$ the equation

$$\mathbf{P}' = \alpha\mathbf{P} + \beta\mathbf{Q} + \gamma\mathbf{N}$$

with functions α, β, γ holds true. An elementrary calculation yields $\alpha = 0$, $\beta = -\mathbf{K}^{(3)}$ and $\gamma = \mathbf{K}^{(2)}$. An analog consideration leads to the Frenet-like system

$$\begin{array}{rclcccc} \mathbf{P}' & = & & - & \mathbf{K}^{(3)}\mathbf{Q} & + & \mathbf{K}^{(2)}\mathbf{N} \\ \mathbf{Q}' & = & \mathbf{K}^{(3)}\mathbf{P} & & & - & \mathbf{K}^{(1)}\mathbf{N} \\ \mathbf{N}' & = & -\mathbf{K}^{(2)}\mathbf{P} & + & \mathbf{K}^{(1)}\mathbf{Q} & & \end{array} \tag{10}$$

Let $\mathbf{Z}_i = (\mathbf{P}_i, \mathbf{Q}_i, \mathbf{N}_i)$ be the vector of the i-th component of the trihedron for $i = 1, 2, 3$. Then the system above can be rewritten in the form

$$\mathbf{Z}_i' = \mathbf{K} \wedge \mathbf{Z}_i \quad \text{for} \quad i = 1, 2, 3. \tag{11}$$

2. Now the existence of a normal cut with prescribed curvature vector $\mathbf{K}$ is proven. Let $\mathbf{Z}_i = (\mathbf{Z}_i^{(1)}, \mathbf{Z}_i^{(2)}, \mathbf{Z}_i^{(3)})$ be the (unique) solution of the initial value problem

$$\begin{array}{rcl} \mathbf{Z}_i'(t) & = & \mathbf{K} \wedge \mathbf{Z}_i(t) \quad \text{for} \quad t \in [0,T], \\ \mathbf{Z}_i(0) & = & e_i, \end{array}$$

with the canonical unit vector e_i for $i = 1, 2, 3$. Rearranging the columns and lines by

$$\begin{array}{rcl} \mathbf{P} & = & (\mathbf{Z}_1^{(1)}, \mathbf{Z}_2^{(1)}, \mathbf{Z}_3^{(1)}), \\ \mathbf{Q} & = & (\mathbf{Z}_1^{(2)}, \mathbf{Z}_2^{(2)}, \mathbf{Z}_3^{(2)}), \\ \mathbf{N} & = & (\mathbf{Z}_1^{(3)}, \mathbf{Z}_2^{(3)}, \mathbf{Z}_3^{(3)}) \end{array}$$

the system (10) holds true. Only the trihedron property is left to check. From the system (11) it follows

$$(\mathbf{Z}_i, \mathbf{Z}_j)' = \det(\mathbf{K}, \mathbf{Z}_i, \mathbf{Z}_j) + \det(\mathbf{Z}_i, \mathbf{K}, \mathbf{Z}_j) = 0$$

and therefore

$$(\mathbf{Z}_i(t), \mathbf{Z}_j(t)) = (\mathbf{Z}_i(0), \mathbf{Z}_j(0)) = (e_i, e_j),$$

and by continuity

$$\det(\mathbf{Z}_1(t), \mathbf{Z}_2(t), \mathbf{Z}_3(t)) = \det(\mathbf{Z}_1(0), \mathbf{Z}_2(0), \mathbf{Z}_3(0)) = 1.$$

So $\mathbf{Z}_i$ for $i = 1, 2, 3$ form an orthonormal trihedron and $\mathbf{P}, \mathbf{Q}, \mathbf{N}$ do likewise. It follows $\mathbf{P}^2 \equiv 1$, and from the system (10) follows that the normal cut really has $\mathbf{K}$ as its curvature vector.

3. Finally the uniqueness up to isometric transformation will be shown. Let $\{\mathbf{Y}, \mathbf{N}\}$ and $\{\overline{\mathbf{Y}}, \overline{\mathbf{N}}\}$ be normal cuts in arc-length parametrisation with tangentials $\mathbf{P}$, $\overline{\mathbf{P}}$ and radial tangentials $\mathbf{Q}$, $\overline{\mathbf{Q}}$, which have the same curvature vector $\mathbf{K}$. Taking $\det(\overline{\mathbf{P}}(0), \overline{\mathbf{Q}}(0), \overline{\mathbf{N}}(0)) = 1 = \det(\mathbf{P}(0), \mathbf{Q}(0), \mathbf{N}(0))$ and the definitions

$$\tilde{\mathbf{P}}(t) := \mathcal{P} \circ \overline{\mathbf{P}}(t), \quad \tilde{\mathbf{Q}}(t) := \mathcal{P} \circ \overline{\mathbf{Q}}(t), \quad \tilde{\mathbf{N}}(t) := \mathcal{P} \circ \overline{\mathbf{N}}(t)$$

with the orthogonal matrix

$$\mathcal{P} = (\mathbf{P}(0), \mathbf{Q}(0), \mathbf{N}(0)) \circ (\overline{\mathbf{P}}(0), \overline{\mathbf{Q}}(0), \overline{\mathbf{N}}(0))^{-1}$$

into account $\{\tilde{\mathbf{Y}}, \tilde{\mathbf{N}}\}$ is a normal cut in arc-length parametrisation with $\tilde{\mathbf{Y}} = \int \tilde{\mathbf{P}}\, dt + \mathbf{Y}(0)$. This normal cut has the same curvature vector $\mathbf{K}$ as $\{\overline{\mathbf{Y}}, \overline{\mathbf{N}}\}$ and by assumption as $\{\mathbf{Y}, \mathbf{N}\}$. Therefore $\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}, \tilde{\mathbf{N}}$ and $\mathbf{P}, \mathbf{Q}, \mathbf{N}$ are solutions of the same system (10) with the same initial values. By the theory of ordinary differential equations they have to coincide. So $\{\mathbf{Y}, \mathbf{Q}\}$ coincides with $\{\tilde{\mathbf{Y}}, \tilde{\mathbf{Q}}\}$.

4. The regularity assertion follows directly from the regularity theory of ordinary differential equations.

$\square$

The next observation on normal cuts is crucial. It is a transformation of the invariants of a curve to the invariants of a normal cut induced by a surface from remark 4.

Lemma 2.4. *Let $k \in \mathbb{N}$ and $\{\mathbf{Y}, \mathbf{N}\}$, $\{\mathbf{Y}, \overline{\mathbf{N}}\}$ be two normal cuts in arc-length parametrisation of class $C^{k,\alpha}[0,T]$ with tangential $\mathbf{P}$ and normals $\mathbf{N}$, $\overline{\mathbf{N}}$. Then there is an angle $\phi = \phi(t)$ such that*

$$\begin{array}{rcrcr} \overline{\mathbf{Q}} & = & \cos\phi\,\mathbf{Q} & - & \sin\phi\,\mathbf{N}, \\ \overline{\mathbf{N}} & = & \sin\phi\,\mathbf{Q} & + & \cos\phi\,\mathbf{N}. \end{array}$$

Moreover, the curvature vectors $\mathbf{K}$ and $\overline{\mathbf{K}}$ transform like

$$\begin{array}{rcrrcr} \overline{\mathbf{K}}^{(1)} & = & & \mathbf{K}^{(1)} + \phi', & & \\ \overline{\mathbf{K}}^{(2)} & = & & \cos\phi\,\mathbf{K}^{(2)} & + & \sin\phi\,\mathbf{K}^{(3)}, \\ \overline{\mathbf{K}}^{(3)} & = & - & \sin\phi\,\mathbf{K}^{(2)} & + & \cos\phi\,\mathbf{K}^{(3)}. \end{array}$$

The existence of ϕ is evident. The second part can be directly calculated from the definition and is left for the reader. In virtue of this lemma and remark 2.2 part 2. and 3. it is clear that the curvature κ of a curve $\{\mathbf{Y}\}$ can be written in terms

$$\kappa = \sqrt{(\mathbf{K}^{(2)})^2 + (\mathbf{K}^{(3)})^2}$$

with any curvature vector $\mathbf{K}$ of any normal cut $\{\mathbf{Y}, \mathbf{Q}\}$.

The last two lemmas contribute to point 4. of the program in the following sense: Let there be given a curve $\mathbf{Y}$ with non-vanishing curvature, it is equivalent to a curvature vector $\mathbf{K}$ which can be transformed by lemma 2.4 to a curvature vector $\overline{\mathbf{K}}$ of a normal cut fitting to a surface. If one can realize a surface whose boundary normal cut has $\overline{\mathbf{K}}$ as its curvature vector, one has by lemma 2.4 the curvature vector $\mathbf{K}$ of the boundary curve. Lemma 2.3 gives the existence of the curve $\mathbf{Y}$. To perform that action, one must be able to calculate the angle ϕ in a unique way, which is demantated in the next section.

3 Surfaces

As indicated in the last section one can identify a curve or a normal cut by its curvature vector. To be more precise, let $\mathbf{X} : \overline{B} \to \mathbb{R}^3 \in C^3(\overline{B})$ be a regular surface with first fundamental form g_{ij}, Gaussian curvature $K > 0$ and boundary curve $\mathbf{Y} = \mathbf{Y}(t) = \mathbf{X}(x^1(t), x^2(t))$, where $(x^1(t), x^2(t))$ is a regular parametrisation of ∂B.

In order to follow point 2. of the program stated in the introduction, the second fundamental form b_{ij} will be expressed in terms of the first fundamental form and the curvature vector of a curve of $\mathbf{Y}$. Some technical difficulties arise, since $\mathbf{Y}$ is not a curve in arc-length parametrisation. So a rescaling to the formulae (8) has to be made. Taking the chain rule into account, one arrives with $|\mathbf{P}|^2 = g_{ij}\dot{x}^i\dot{x}^j$ at

$$\begin{aligned}
-\mathbf{K}^{(1)} &= |\mathbf{P}|^{-2}\det(\mathbf{N},\mathbf{N}',\mathbf{P}),\\
\mathbf{K}^{(2)} &= -|\mathbf{P}|^{-2}(\mathbf{P},\mathbf{N}'),\\
-\mathbf{K}^{(3)} &= |\mathbf{P}|^{-3}\det(\mathbf{P},\mathbf{P}',\mathbf{N})
\end{aligned}$$

for any normal cut $\{\mathbf{Y},\mathbf{N}\}$. Consider a special normal cut with $\mathbf{N}(t) = \mathbf{M}(x^1(t), x^2(t))$, where $\mathbf{M}$ is the surface normal. Express the components of the curvature vector in terms of the first and second fundamental form of the surface. Take into account

$$\begin{aligned}
\mathbf{N}' &= \partial_i\mathbf{N}\dot{x}^i = -\dot{x}^i b_{ij} g^{jk}\partial_k\mathbf{X},\\
\mathbf{P} &= \partial_i\mathbf{X}\dot{x}^i,\\
\mathbf{P}' &= (\ddot{x}^k + \dot{x}^i\dot{x}^j\Gamma^k_{ij})\partial_k\mathbf{X} + \dot{x}^i\dot{x}^j b_{ij}\dot{x}^j\mathbf{N}
\end{aligned}$$

and arrive at

$$\begin{aligned}
-\mathbf{K}^{(1)} &= -|\mathbf{P}|^{-2}\dot{x}^i b_{ij} g^{jk}\dot{x}^l\det(\mathbf{N},\partial_k\mathbf{X},\partial_l\mathbf{X})\\
&= -|\mathbf{P}|^{-2}\dot{x}^i b_{ij} g^{jk} e_{kl}\dot{x}^l\\
\mathbf{K}^{(2)} &= |\mathbf{P}|^{-2}\dot{x}^i\dot{x}^j b_{jk} g^{kl}(\partial_i\mathbf{X},\partial_l\mathbf{X})\\
&= |\mathbf{P}|^{-2}\dot{x}^i b_{ij}\dot{x}^j\\
-\mathbf{K}^{(3)} &= |\mathbf{P}|^{-3}\dot{x}^l(\ddot{x}^k + \dot{x}^i\dot{x}^j\Gamma^k_{ij})\det(\partial_l\mathbf{X},\partial_k\mathbf{X},\mathbf{N})\\
&= |\mathbf{P}|^{-3}\dot{x}^l e_{lk}(\ddot{x}^k + \dot{x}^i\dot{x}^j\Gamma^k_{ij})
\end{aligned} \tag{12}$$

with $e_{ij} = \det(\mathbf{N},\partial_i\mathbf{X},\partial_j\mathbf{X})$. Therefore the geodesic curvature $-\mathbf{K}^{(3)}$ depends only on the first fundamental form. For that reason, one can calculate the angle ϕ from lemma 2.4. Further from $K > 0$ the inequality

$$\mathbf{K}^{(2)} > 0$$

is deduced. So it is no restriction of generality to assume for the curvature of the curve $\{\mathbf{Y}\}$

$$\kappa > 0.$$

Therefore for κ one gets

$$\kappa^2 = \frac{(\mathbf{P} \wedge \mathbf{P}')^2}{|\mathbf{P}|^6}$$

after rescaling, using the chain rule as above and the expression (9).

Lemma 3.1. *Let there be given a positive-definite metric g_{ij} with $K > 0$ on the closed unit disk. Let further be given a closed curve $\mathbf{Y}$ parametrised compatible to g_{ij}, i.e. $\mathbf{P}^2 = g_{ij}\dot{x}^i\dot{x}^j$ for a parametrisation of ∂B denoted by $x(t) = (x^1(t), x^2(t))$. Then the normal $\mathbf{N}$ of a surface having g_{ij} as first fundamental form and $\mathbf{Y}$ as boundary curve is up to isometric transformations uniquely defined. This implies the uniqueness of the angle ϕ in lemma 2.4.*

Proof. Taking into account that $\mathbf{K}^{(3)}$ depends only on g_{ij} and $x(t)$, one has for the normal of the surface the following linear system

$$\begin{aligned}
|\mathbf{P}|^{-3}(\mathbf{P} \wedge \mathbf{P}') \cdot \mathbf{N} &= -\mathbf{K}^{(3)} \\
|\mathbf{P}|^{-2}\mathbf{P}' \cdot \mathbf{N} &= \mathbf{K}^{(2)} = \sqrt{\kappa^2 - (\mathbf{K}^{(3)})^2} \\
|\mathbf{P}|^{-1}\mathbf{P} \cdot \mathbf{N} &= 0
\end{aligned}$$

This is a system the right handed side of which depends only on $\mathbf{Y}$ and g_{ij} and their derivatives. The determinant of the coefficient matrix

$$|\mathbf{P}|^{-6}(\mathbf{P} \wedge \mathbf{P}')^2 = \kappa^2 > 0$$

is non-vanishing and there is a unique solution $\mathbf{N}$. □

Collecting the assertions of lemma 2.3, 2.4 and 3.1 one has

Lemma 3.2. *In the situation of 3.1 there is exactly one curvature vector $\mathbf{K}$ for the normal cut $\{\mathbf{Y}, \mathbf{N}\}$, where $\mathbf{N}$ is the surface normal restricted to the boundary of an arbitrary solution of problem* (3)*.*

Finding a surface $\mathbf{X}$ whose boundary normal cut has $\mathbf{K}$ as its curvature vector leads to a realization of $\mathbf{Y}$. But more is true. The first two equations

(12) are a linear system for the unknown $\dot{x}^i b_{ij}$ with a non-vanashing determinant. So the complete system is fulfilled iff one has, additionally to the third equation the equations

$$\dot{x}^i b_{ij} = \dot{x}^i(\mathbf{K}^{(2)} g_{ij} - \mathbf{K}^{(1)} e_{ij}).$$

One has reached at

Lemma 3.3. *In the situation of 3.1 there is for every solution of problem* (3) *for the second fundamental form the equation*

$$\dot{x}^i b_{ij} = \dot{x}^i(\mathbf{K}^{(2)} g_{ij} - \mathbf{K}^{(1)} e_{ij})$$

valid. Moreover, if the second fundamental form fulfilles the equation above, then the curvature vector of the boundary normal cut is exactly $\mathbf{K}$*.*

The next lemma will allow to express the b_{ij}'s explicitly:

Lemma 3.4. *In the situation of lemma 3.1 there is for the mean curvature of any solution of problem* (3) *the equation*

$$H = \frac{K + (\mathbf{K}^{(1)})^2 + (\mathbf{K}^{(2)})^2}{2\mathbf{K}^{(2)}} \tag{13}$$

valid.

Proof. Denoting by $p_{ij} = (\partial_i N, \partial_j N)$ the third fundamental form and obtaining from the well-known [11, Ch. XI, §2] equality

$$p_{ij} - 2H b_{ij} + K g_{ij} = 0$$

one gets by multiplication with $\dot{x}^i \dot{x}^j$ and summation

$$(\mathbf{N}', \mathbf{N}') - 2H(\mathbf{N}', \mathbf{P}) + K|\mathbf{P}|^2 = 0.$$

Equivalently one has

$$H = \frac{K + |\mathbf{P}|^{-2}(\mathbf{N}', \mathbf{N}')}{2|\mathbf{P}|^{-2}(\mathbf{N}', \mathbf{P})}.$$

Taking the chain rule into accound and the system (10) in arc-length parametrisation, one arrives at the assertion. □

Consider again the first two equations of (12) and the third equation

$$\frac{K + |\mathbf{P}|^{-2}(\mathbf{N}', \mathbf{N}')}{2|\mathbf{P}|^{-2}(\mathbf{N}', \mathbf{P})} = H = \operatorname{tr} b_{ij} g^{jk}.$$

This is again a linear system for the unknown $b_{11}, b_{12} = b_{21}, b_{22}$ with a non-vanishing determinant. So the following lemma holds true.

Lemma 3.5. *In the situation of lemma 3.1 there is at ∂B*

$$b_{ij} = \frac{\dot{x}^k \dot{x}^l}{\dot{x}^p g_{pq} \dot{x}^q \mathbf{K}^{(2)}} \left(K e_{ki} e_{lj} + \left(\mathbf{K}^{(2)} g_{ki} - \mathbf{K}^{(1)} e_{ki} \right) \left(\mathbf{K}^{(2)} g_{lj} - \mathbf{K}^{(1)} e_{lj} \right) \right),$$

which is equivalent to the validity of (12) and (13).

The theorem describes the process of translation of the boundary data together with lemma 2.3.

Theorem 3.6. *In the situation of lemma 3.1 one has for any conjugate-conformal mapping $z = z(w) = z(re^{it})$ from (5) the differential equation*

$$\mathcal{D}_{\overline{w}} \left(\mathcal{D}_w z^k \right) + \mathcal{D}_w \left(\mathcal{D}_{\overline{w}} z^k \right) + \Gamma^k_{ij} \left(\mathcal{D}_w z^i \mathcal{D}_{\overline{w}} z^j + \mathcal{D}_{\overline{w}} z^i \mathcal{D}_w z^j \right) = 0 \qquad (14)$$

in B and the boundary data

$$z^i_r = \frac{1}{\sqrt{K}} \left(-\mathbf{K}^{(1)} \delta^i_k + \mathbf{K}^{(2)} g^{ij} e_{jk} \right) z^k_t \qquad (15)$$

on ∂B. Here the denotation is: $\mathcal{D}_w = \sqrt{K \circ z}\, \partial_w$ and $\mathcal{D}_{\overline{w}} = \sqrt{K \circ z}\, \partial_{\overline{w}}$ with the Wirthinger derivatives ∂_w and $\partial_{\overline{w}}$. The converse is also true: Let z be a diffeomorphic solution of the boundary value problem above, then there is a surface $\mathbf{X}$ with first fundamental form g_{ij} whose boundary normal cut has the curvature vector $\mathbf{K} = (\mathbf{K}^{(1)}, \mathbf{K}^{(2)}, \mathbf{K}^{(3)})$.

Proof. 1. Evaluate the first two equations (12) in conjugate-conformal parametrisation. Therefore one has by the relations of conformality $b_{ij} = \sqrt{K \det(g_{ij})} \nabla z^1 \wedge \nabla z^2 \delta_{ij}$. Using further a parametrisation of ∂B (u^1, u^2) with $(\dot{u}^1)^2 + (\dot{u}^2)^2 = 1$ arrive at

$$\mathbf{K}^{(2)} = \frac{b_{ij} \dot{x}^i \dot{x}^j}{g_{kl} \dot{x}^k \dot{x}^l} = \frac{\sqrt{K \det(g_{ij})} \nabla z^1 \wedge \nabla z^2}{g_{pq} z^p_k z^q_l \dot{u}^k \dot{u}^l} = \sqrt{K} \frac{e_{kl} z^k_r z^l_t}{g_{pq} z^p_t z^q_t}.$$

Taking into account

$$\begin{aligned}
\sqrt{\det(G_{ij})}\dot{u}^i\delta_{ij}G^{jk}E_{kl}\dot{u}^l &= -\sqrt{\det(G_{ij})}\dot{u}^i\delta_{ij}E^{jk}G_{kl}\dot{u}^l \\
&= \dot{u}^i\delta_{ij}\epsilon^{jk}z_k^p g_{pg} z_l^q \dot{u}^l \\
&= g_{pq}z_r^p z_t^q
\end{aligned}$$

with the first fundamental form in conjugate-conformal parametrisation G_{jk}, its inverse G^{kl}, the antisymmetric surface element in conjugate-conformal parametrisation E_{kl}, its inverse E^{jk} and the total antisymmetric $\epsilon^{12} = 1 = -\epsilon^{21}$ one arrives at

$$-\mathbf{K}^{(1)} = \sqrt{K}\frac{g_{kl}z_r^k z_t^l}{g_{pq}z_t^p z_t^q}.$$

Alternating insertion in (15) checks the equivalency of that formulae with (15). The validity of the differential system was already proved in the introduction.

2. Checking the converse. The existence of a surface with first fundamental form g_{ij} was already pointed out in the introduction. By the first part of the proof the equation (15) is equivalent to the first two of (12). So the curvature vector is $\mathbf{K} = (\mathbf{K}^{(1)}, \mathbf{K}^{(2)}, \mathbf{K}^{(3)})$ with $\mathbf{K}^{(3)}$ given in (12), too.

□

4 A Priori Estimates

As an application of the translation process a-priori estimates for the following problem are proved.

Problem 4.1. *Let $C > 0$ be a constant and $\alpha \in (0, 1)$ a Hölder exponent. The problem described below will be denoted by $\mathcal{P}(C)$.*

1. *Let there be a symmetric, positive-definite matrix $g_{ij} \in C^{4,\alpha}(B) \cap C^{3,\alpha}(\overline{B})$ given with Gaussian curvature defined in formula (4). Assume*

$$\|g_{ij}\|_{3,\alpha}^B \leq C, \qquad \inf_B K \geq \frac{1}{C}, \qquad \inf_B \det(g_{ij}) \geq \frac{1}{C}.$$

2. *Let* $\mathbf{Y} = \mathbf{Y}(t) : [0, 2\pi] \to \mathbb{R}^3$ *be a curve with tangential* $\mathbf{P}(t) = \mathbf{Y}'(t)$ *and curvature*

$$\kappa(t) = \left(\frac{(\mathbf{P} \wedge \mathbf{P}')^2}{|\mathbf{P}|^6} \right)^{\frac{1}{2}} \Bigg|_t \quad \textit{for} \quad t \in [0, 2\pi].$$

Assume

$$\|\mathbf{Y}\|_{4,\alpha}^{[0,2\pi]} \le C, \qquad \inf_{[0,2\pi]} |\mathbf{P}| \ge \frac{1}{C}.$$

3. *By* $x(t) = (x^1(t), x^2(t)) = (\cos t, \sin t) : [0, 2\pi] \to \partial B$ *denote a parametrisation of the boundary subject to the compatibility condition*

$$|\mathbf{P}(t)|^2 = g_{ij}(x)\dot{x}^i \dot{x}^j|_t \quad \textit{for} \quad t \in [0, 2\pi]$$

with the geodesic curvature

$$\kappa_g(t) = \frac{\dot{x}^k e_{kl}(\ddot{x}^l + \Gamma^l_{ij}(x)\dot{x}^i \dot{x}^j)}{(g_{pq}(x)\dot{x}^p \dot{x}^q)^{\frac{3}{2}}} \Bigg|_t \quad \textit{for} \quad t \in [0, 2\pi].$$

Assume

$$\inf_{[0,2\pi]} \left(\kappa^2 - \kappa_g^2\right) \ge \frac{1}{C}.$$

Search a surface

$$\mathbf{X} = \mathbf{X}(x^1, x^2) : \overline{B} \to \mathbb{R}^3 \in C^4(B) \cap C^3(\overline{B})$$

subject to the condition

$$(\partial_i \mathbf{X}, \partial_j \mathbf{X}) = g_{ij} \quad \textit{in} \quad B$$

and

$$\mathbf{X}(x^1(t), x^2(t)) = \mathbf{Y}(t) \quad \textit{for} \quad t \in [0, 2\pi].$$

A theorem of F. Sauvigny [11, chapter XII, §7, Theorem 2] is used. It can be restated in the following form

Theorem 4.2. *Let* $a_{ij} \in C^0(\overline{B}) \cap C^{1,\alpha}(\partial B)$ *be a symmetric positive-definite matrix subject to the following condititions: For* $\lambda > 0$ *one has*

$$\frac{1}{\lambda}\xi^2 \le a_{ij}(x)\xi^i \xi^j \le \lambda \xi^2 \quad \textit{for} \quad x \in \overline{B} \quad \textit{and} \quad \xi = (\xi^1, \xi^2) \in \mathbb{R}^2$$

and

$$\|a_{ij}(\cos t, \sin t)\|_{1,\alpha}^{[0,2\pi]} \le \gamma$$

for a $\gamma < \infty$. *Let further* $z = z(w) = (z^1(w), z^2(w)) : \overline{E} \to \overline{B} \in C^2(E) \cap C^1(\overline{E})$ *be a conformal mapping relatively to* a_{ij} *with* $z(0) = 0$ *and a positively oriented diffeomorphic solution of the system*

$$\Delta z^k = h_{ij}^k(z)\nabla z^i \nabla z^j \quad \text{in} \quad E$$

with coefficients $h_{ij}^k : \overline{B} \to \mathbb{R} \in C^{0,\alpha}(\overline{B})$ *and*

$$\|h_{ij}^k\|_{0,\alpha}^B \le \gamma.$$

Then one has $z \in C^{2,\alpha}(\overline{E})$ *and there are constants* $\Theta = \Theta(\lambda, \gamma, \alpha) < \infty$ *and* $\Lambda = \Lambda(\lambda, \gamma, \alpha) > 0$ *such that*

$$\|z\|_{2,\alpha}^E \le \Theta \quad \text{and} \quad \inf_E (\nabla z^1 \wedge \nabla z^2) \ge \Lambda.$$

In order to establish a priori estimates introduce an isothermal parameterisation in the second fundamental form b_{ij} of the surface $\mathbf{X}$ with a mapping z from (5) and $z(0) = 0$. This mapping is a solution of the Darboux equation (7). Rewriting this equation one has a system of the form of theorem 4.2, see [6], for instance. The coefficients are

$$\begin{array}{rclcrcl} h_{11}^1 &=& -\frac{\partial_1 K}{2K} - \Gamma_{11}^1, & \qquad & h_{11}^2 &=& -\Gamma_{11}^2, \\ h_{12}^1 &=& -\frac{\partial_2 K}{2K} - \Gamma_{12}^1, & & h_{12}^2 &=& -\Gamma_{12}^2, \\ h_{21}^1 &=& -\Gamma_{12}^1, & & h_{21}^2 &=& -\frac{\partial_1 K}{2K} - \Gamma_{12}^2, \\ h_{22}^2 &=& -\Gamma_{22}^1, & & h_{22}^2 &=& -\frac{\partial_2 K}{2K} - \Gamma_{22}^2. \end{array} \tag{16}$$

One can find a constant $\gamma_1 = \gamma_1(C) < \infty$ such that $\|h_{ij}^k\|_{0,\alpha}^B \le C$ holds true. Moreover, $h_{ij}^k \in C^{1,\alpha}(B)$ is true. By remark 2.2, point 3., lemma 2.4 and lemma 3.1 one can find a constant $D_1 = D_1(C) < \infty$ such that the curvature vector $\mathbf{K}$ of the normal cut $\{\mathbf{Y}, \mathbf{N} \wedge \mathbf{P}\}$ is estimated by $\|\mathbf{K}\|_{1,\alpha}^{0,2\pi} \le D_1$ in non-arc-length parametrisation, too. By lemma 3.5 one can find a constant $\gamma_2 = \gamma_2(C) < \infty$ such that $\|b_{ij}(\cos t, \sin t)\|_{1,\alpha}^{0,2\pi} \le \gamma_2$. For the ellipticity estimate note that $L + N \le 2H(E + G)$ and therefore

$$\frac{K}{2H}\frac{EG - F^2}{E + G}\xi^2 \le b_{ij}(x)\xi^i\xi^j \le 2H(E + G)\xi^2$$

holds true. So estimates on H will give an estimate of the ellipticity. Due to the regularity assumption on $\mathbf{X}$ one has $H \in C^2(B) \cap C^1(\overline{B})$. So the continuous function H will attend a maximum either in B or on ∂B. If it attends its maximum an ∂B, estimate H with lemma 3.4. If it attends its maximum in B, say in $x_0 \in B$, apply the famous Weyl inequality

$$H(x_0)^2 \leq K - \frac{1}{4K} \mathbf{\Delta}_{g_{ij}} K \bigg|_{x_0}.$$

So a constant $D_2 = D_2(C) < \infty$ can be found such that $\sup_B H \leq D_2$ holds true. For a reference of the Weyl inequality see [9] or [2], for instance. Note, to prove the inequality one needs the second order neccesary condition for a maximum, therefore impose $H \in C^2(B)$, which leads to $b_{ij} \in C^2(B)$. But the second derivatives do not contribute to the result quantitatively. That is the reason one has to impose $g_{ij} \in C^{4,\alpha}(B)$, because then $h_{ij}^k \in C^{1,\alpha}(B)$ holds true and the conjugate conformal mapping will belong to class $C^{3,\alpha}(E)$ by standard elliptic regularity theory found in [4]. After setting (6) one has $b_{ij} \in C^{2,\alpha}(B)$.

After applying Sauvignys theorem the following theorem is proved

Theorem 4.3. *Let $\mathbf{X}$ be a solution of $\mathcal{P}(c)$. Then for all positive-oriented diffeomorphic conjugate-conformal mappings*

$$z = z(w) = (z^1(w), z^2(w)) : \overline{E} \to \overline{B} \in C^2(E) \cap C^1(\overline{E})$$

of $\mathbf{X}$ with $z(0) = 0$ there is $z \in C^{3,\alpha}(E) \cap C^{2,\alpha}(\overline{E})$. Moreover, there are constants $\Theta = \Theta(C, \alpha) < \infty$ and $\Lambda = \Lambda(C, \alpha) > 0$ such that

$$\|z\|_{2,\alpha}^E \leq \Theta \quad \textit{and} \quad \inf_E (\nabla z^1 \wedge \nabla z^2) \geq \Lambda$$

holds true.

Taking (6) and the Gauss-Weingarten equations into account, one has

Theorem 4.4. *Let $\mathbf{X}$ be a solution of $\mathcal{P}(c)$. Then $\mathbf{X} \in C^{4,\alpha}(B) \cap C^{3,\alpha}(\overline{B})$ holds true.Moreover there is a constant vector $\mathbf{X}_0$ and a constant*

$$\Theta = \Theta(C, \alpha) < \infty$$

such that

$$\|\mathbf{X} - \mathbf{X}_0\|_{3,\alpha}^B \leq \Theta$$

holds true.

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