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Abstract

Companies in the same industry sector are usually stronger correlated than firms in different sectors, as they are similarly affected by macroeconomic effects, political decisions, and consumer trends. In spite of many stock return models taking account of this fact there are only a few credit default models taking it into consideration. In this paper we present a default model based on nested Archimedean copulas which is able to capture hierarchical dependence structures among the obligors in a credit portfolio. Nested Archimedean copulas have a surprisingly simple and intuitive interpretation. The dependence among all companies in the same sector is described by an inner copula; the sectors are then coupled via an outer copula. Consequently, our model implies a larger default correlation for companies in the same industry sector compared to companies in different sectors. A calibration to CDO tranche spreads of the European iTraxx portfolio is performed to demonstrate the fitting capability of our model. This portfolio consists of CDS on 125 companies from six different industry sectors. It is therefore an excellent portfolio to compare our generalized model to a traditional copula model of the same family, which does not account for different sectors.

1 Introduction

The knowledge about the industry sector of the portfolio constituents is often neglected in the modeling of credit portfolios. Not considering this information has the unrealistic effect that default correlations do not explicitly change whether two companies are in the same industry sector or not. Incorporating sector effects to existing credit-portfolio models is straightforward in structural and factor models by introducing sector-specific risk factors. Examples are Hull, Predescu, White (2006) or Kiesel, Scherer (2007) for the former, and multi-sector generalizations of Kalemanova, Schmid, Werner (2007) for the latter. In contrast, tractable portfolio models relying on copulas are usually based on specific choices of exchangeable Archimedean copulas, as the sampling algorithm of Marshall, Olkin (1988) allows to obtain an approximate loss distribution via a conditionally independent approach. As a drawback, homogeneous pairwise correlations among obligors are inherited from the symmetry of exchangeable Archimedean copulas. In this paper, we want to introduce an intuitive generalization which allows for asymmetry.

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More generally, we introduce a hierarchical structure to the popular copula models of Schönbucher, Schubert (2001) and Li (2000). Our generalization allows to classify the firms in a credit portfolio according to some attribute, which is the industry sector in our application. Alternative classification criteria, such as geographic regions or political unions, can similarly be used. We achieve this segmentation by introducing nested Archimedean copulas with two levels. This approach allows us to first couple the firms of the same sector by some inner copula. Then, we combine these sectors by an outer copula, both of the Archimedean class.

As a main application we shall treat the pricing of CDOs, for which we show that our approach yields significant smaller pricing errors than the current standard approach using exchangeable copulas. As our investigation involves several Archimedean copulas, our results also indicate which class of Archimedean copulas might be preferable for the modeling of CDOs. As benchmark for this calibration, we also include the Gauss copula as market standard. Our simulation studies, in which we fit different families of copulas to portfolio CDS and CDO quotes, are based on fast simulation algorithms for nested Archimedean copulas. The resulting Monte Carlo engine is combined with an efficient parameter optimizing algorithm, which allows to conduct a calibration of the model in reasonable amounts of time on standard PCs.

The Archimedean copulas used in our model show different kinds of tail dependence and several of our results can be linked to this property. For instance, it is known that the limit of default correlations as maturity decreases to zero can be expressed via the parameter of upper tail dependence of the specific copula, compare Chapter 10 of Schönbucher (2003). This result can be generalized in our nested framework to the parameter of tail dependence of the respective sector or outer copula. The result is that the limit of default correlations of companies within the same industry sector is larger than of companies in different sectors.

The paper is organized as follows. In Section 2 we review the intensity-based approach to derive marginal default probabilities and the concept how defaults are made dependent via copulas. Section 3 then presents sampling algorithms for exchangeable and nested Archimedean copulas and explains the implied default correlation generated by such copulas. The payment streams of portfolio CDS and CDO contracts and the Monte-Carlo pricing approach are presented in Section 4. In Section 5 we then show how our model can be calibrated to market quotes. Finally, Section 6 concludes.

2 The model

We work on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P}$ is a pricing measure which is calibrated to market quotes of portfolio derivatives in Section 5. The modeled portfolio contains I credit-risky assets, whose payment streams dependent on the default status of one of the I firms. These companies are assigned to one of S industry sectors, or classified in one of S sectors according to some alternative attribute. The default times of these companies are denoted by τ_i. The default status of each company is modeled via a simple intensity model, the intensity of company $i \in \{1, \ldots, I\}$ is assumed to be a deterministic, nonnegative function, denoted by λ_i. The term structures of survival probabilities $p_i(t)$ and default probabilities $\bar{p}_i(t)$ of this company are then given by

$$p_i(t) = \exp(-\int_0^t \lambda_i(s)ds), \quad \bar{p}_i(t) = 1 - p_i(t), \tag{1}$$

respectively. At this point, let us remark that other models for the individual default probabilities are also possible, as our model is purely designed to explain the dependence among the firms. A classical result, compare Bielecki, Rutkowski (2002), page 183, or Schönbucher (2003), page 122, allows the following canonical construction of τ_i, which is extremely useful for simulating τ_i. Let U_i be uniformly distributed on $[0, 1]$ and independent of $\mathcal{F}$, then

$$\tau_i \stackrel{\mathrm{d}}{=} \inf\Big\{t \geq 0 : p_i(t) \leq U_i\Big\}. \tag{2}$$

Hence, having drawn a random variate U_i, we simply have to compute $\tau_i = p_i^{-1}(U_i)$.

The relevant quantity for pricing portfolio derivatives and for risk management purposes is the portfolio-loss process. Given the default time, the recovery rate, and the nominal of each firm, this stochastic process is easily derived. Undisputably, corporate defaults are not mutually independent. In the considered model, dependence among the default times is introduced by making the random variables U_i dependent. Therefore, we assume the vector of trigger variables $(U_1, \ldots, U_I)$ to be jointly distributed according to some copula C. In general, the distribution of the loss process is not analytically available. For some exchangeable Archimedean copulas it is possible to approximate the portfolio-loss distribution via a conditionally independent approach, compare Schönbucher (2003), Chapter 10.8.2. Unfortunately, this does not easily generalize to nested Archimedean copulas. However, as long as we can efficiently sample from the chosen copula C, it is possible to simulate the loss process and to price portfolio derivatives such as CDOs. Efficient sampling algorithms are known for several exchangeable Archimedean copulas and have lately been suggested for some nested Archimedean copulas. These algorithms are introduced in the following chapter. For a sampling algorithm of the Gauss copula, see Embrechts, Lindskog, McNeil (2001).

3 Exchangeable and nested Archimedean copulas

We assume the reader to be familiar with bivariate Archimedean copulas, an excellent introduction is provided by Nelson (1998). We will review basic facts about Archimedean copulas, however, we focus on our multidimensional setting. In order to distinguish between the standard symmetric Archimedean copulas and the nonsymmetric nested ones, we refer to the former as exchangeable and to the latter as nested Archimedean copulas.

Archimedean copulas are convenient to work with as they are fully specified by some generator function. Important for modeling dependent defaults is the property that Archimedean copulas are able to capture different tail dependencies. In our model, company k (resp. l) defaults up to time t if and only if its trigger variable U_k (resp. U_l) satisfies $U_k \geq p_k(t)$ (resp. $U_l \geq p_l(t)$), i.e. if and only if it is close to one. Therefore, a joint default occurs if both trigger variables are close to one simultaneously. Thus upper tail dependence is an important property for the copulas used in our model. In a multidimensional setting, the symmetry inherent in the class of Archimedean copulas is usually a drawback. However, by nesting Archimedean copulas one can bring asymmetries into play.

A multidimensional exchangeable Archimedean copula is given by

$$C(\mathbf{u}) = C(u_1, \ldots, u_I; \varphi_0) = \varphi_0^{-1}[\varphi_0(u_1) + \cdots + \varphi_0(u_I)], \ \mathbf{u} \in [0,1]^I, \tag{3}$$

where the generator $\varphi_0 : [0,1] \mapsto [0,\infty]$ is continuous and strictly decreasing, satisfying $\varphi_0(1) = 0$. The inverse φ_0^{-1} is assumed to be *completely monotonic* on $[0,\infty)$, i.e. $(-1)^k \frac{d^k}{dt^k}\varphi_0^{-1}(t) \geq 0$ for any $t \in (0,\infty)$ and $k \in \mathbb{N}_0$.

3.1 Sampling exchangeable Archimedean copulas

For the pricing of portfolio derivatives we have to rely on Monte Carlo techniques, as closed-form expressions of the portfolio-loss distribution are not available in our hierarchical framework. The difficult step in the simulation of the portfolio-loss process is the simulation of uniformly distributed random variables with some copula describing their dependence structure. For exchangeable Archimedean copulas, Marshall, Olkin (1988) present an algorithm which is especially efficient for large dimensions.

By Bernstein's Theorem, see Feller (1971), page 439, φ_0^{-1} is the Laplace-Stieltjes transform of a distribution function F_0 concentrated on $[0,\infty)$, shortly $\varphi_0^{-1} = \mathcal{LS}(F_0)$. The following algorithm of Marshall, Olkin (1988) exploits this relation for sampling $(U_1, \ldots, U_I)$ from a multidimensional exchangeable Archimedean copula, assuming $F_0 = \mathcal{LS}^{-1}(\varphi_0^{-1})$ is known.

Algorithm 1 (Marshall, Olkin)
(1) Sample $V_0 \sim F_0$.

(2) Sample i.i.d. realizations $X_i \sim U[0,1]$, $i \in \{1, \ldots, I\}$.

(3) Return $(U_1, \ldots, U_I)$, *where* $U_i = \varphi_0^{-1}(-\log(X_i)/V_0)$, $i \in \{1, \ldots, I\}$.

3.2 Sampling nested Archimedean copulas

Nested Archimedean copulas appear e.g. in Joe (1997), Whelan (2004), Savu, Trede (2006), and McNeil (2007). The structures we are interested in are *partially nested*

Archimedean copulas of the form

$$\begin{aligned}
C(\mathbf{u}) &= C(C(u_{11},\dots,u_{1d_1};\varphi_1),\dots,C(u_{S1},\dots,u_{Sd_S};\varphi_s);\varphi_0)\\
&= \varphi_0^{-1}[\varphi_0(\varphi_1^{-1}[\varphi_1(u_{11})+\dots+\varphi_1(u_{1d_1})])+\dots\\
&\quad+\varphi_0(\varphi_S^{-1}[\varphi_S(u_{S1})+\dots+\varphi_S(u_{Sd_S})])]\\
&= \varphi_0^{-1}\left[\sum_{s=1}^{S}\varphi_0\left(\varphi_s^{-1}\left[\sum_{l=1}^{d_s}\varphi_s(u_{sl})\right]\right)\right],
\end{aligned} \tag{4}$$

$u_{sl} \in [0,1]$, $s \in \{1,\dots,S\}$, $l \in \{1,\dots,d_s\}$, where S is the number of *sectors* and $I=\sum_{s=1}^{S} d_s$ is the dimension. This copula model comprises $S+1$ different bivariate margins and is hence much more flexible than (3). A sufficient condition for (4) being a proper copula is that all involved nodes of the form $\varphi_0 \circ \varphi_s^{-1}$ for any $s \in \{1,\dots,S\}$ have completely monotonic derivatives, see McNeil (2007). For the nested Archimedean copulas we address in this paper, this condition is equivalent to $\vartheta_0 \le \vartheta_s$ for any $s \in \{1,\dots,S\}$, where ϑ_k denotes the parameter corresponding to φ_k, $k \in \{0,\dots,S\}$. This is often the case when generators belonging to the same Archimedean family are nested, see Hofert (2007a).

The following algorithm is for sampling the partially nested Archimedean copula (4), see McNeil (2007). The generator inverses $\exp(-v\varphi_0 \circ \varphi_s^{-1}(t))$ are denoted by $\varphi_{0,s}^{-1}(t;v)$, $s \in \{1,\dots,S\}$. The involved *outer* and *inner* distribution functions F_V are denoted by $F_0 = \mathcal{LS}^{-1}(\varphi_0^{-1})$ and $F_{0,s} = \mathcal{LS}^{-1}(\varphi_{0,s}^{-1}(t;v))$, $s \in \{1,\dots,S\}$, respectively.

Algorithm 2 (McNeil)

(1) Sample $V_0 \sim F_0$.

(2) For $s \in \{1,\dots,S\}$, sample $(X_{s1},\dots,X_{sd_s}) \sim C(u_{s1},\dots,u_{sd_s};\varphi_{0,s}(\cdot;V_0))$ using Algorithm 1.

(3) Return the vector $(U_{11},\dots,U_{Sd_S})$, where $U_{sl}=\varphi_0^{-1}(-\log(X_{sl})/V_0)$, $s \in \{1,\dots,S\}$, $l \in \{1,\dots,d_s\}$.

Table 1 lists the Archimedean families of Ali-Mikhail-Haq (A), Clayton (C), Frank (F), Gumbel (G), Joe (J), and the outer power family based on Clayton's generator, in short outer power Clayton family (opC), together with lower and upper tail dependence parameters. The algorithms for sampling these copulas and specific information about the implementation are given in the Appendix.

3.3 Default correlations in a nested framework

As a risk measure for dependent defaults, the concept of default correlation is widely used by risk managers, rating agencies, and regulators. In model (2), the default correlation of two companies k and l up to time t is defined as

$$\rho_{k,l}(t) = \mathrm{Cor}(\mathbb{1}_{\{\tau_k \le t\}}, \mathbb{1}_{\{\tau_l \le t\}}).$$

Family	ϑ	$\varphi(t)$	$\varphi^{-1}(t)$	λ_L	λ_U
A	$[0,1)$	$\log\frac{1-\vartheta(1-t)}{t}$	$\frac{1-\vartheta}{e^t-\vartheta}$	0	0
C	$(0,\infty)$	$t^{-\vartheta}-1$	$(1+t)^{-\frac{1}{\vartheta}}$	$2^{-1/\vartheta}$	0
opC	$(0,\infty)$	$(t^{-\vartheta_c}-1)^{\vartheta}$, $\vartheta_c>0$	$(1+t^{1/\vartheta})^{-\frac{1}{\vartheta_c}}$	$2^{-1/(\vartheta\vartheta_c)}$	$2-2^{1/\vartheta}$
F	$(0,\infty)$	$-\log\frac{1-e^{-\vartheta t}}{1-e^{-\vartheta}}$	$-\frac{1}{\vartheta}\log(e^{-t}(e^{-\vartheta}-1)+1)$	0	0
G	$[1,\infty)$	$(-\log t)^{\vartheta}$	$e^{-t^{\frac{1}{\vartheta}}}$	0	$2-2^{1/\vartheta}$
J	$[1,\infty)$	$-\log(1-(1-t)^{\vartheta})$	$1-(1-e^{-t})^{\frac{1}{\vartheta}}$	0	$2-2^{1/\vartheta}$

Table 1 Parameter ranges, generators, corresponding inverses and tail dependence parameters for the used Archimedean families.

By assuming identical intensities, the following result was obtained by Schönbucher (2003), page 348.

Theorem 3.1
Let C denote the copula of the default triggers $(U_1,\ldots,U_I)$. Marginal default probabilities are specified by model (2) with identical intensities, i.e. $\lambda_i(s)=\lambda(s)$, $s\geq 0$, for every $i\in\{1,\ldots,I\}$.

(a) The default correlation $\rho(t)$ between two companies k and l up to time t in (2) is given by

$$\rho(t)=\frac{C_{k,l}(p(t),p(t))-p(t)^2}{p(t)(1-p(t))},\ t>0,$$

where $C_{k,l}$ denotes the (k,l)-th margin of C. For an Archimedean copula with generator φ, we have $C_{k,l}(p(t),p(t))=\varphi^{-1}[2\varphi(p(t))]$.

(b) For copulas with existing upper tail dependence parameter λ_U, the limit of the default correlation $\rho(t)$ as t decreases to zero equals λ_U.

We notice that the original model with exchangeable Archimedean copulas implies identical pairwise default correlations for any two firms. In our framework, the pairwise default correlation explicitly depends on whether or not two firms are in the same sector. More precisely, we find the following result.

Theorem 3.2
(a) If C is the partially nested Archimedean copula (4), we have at most $S+1$ different default correlations, given by

$$\rho_0(t)=\frac{\varphi^{-1}[2\varphi(p(t))]-p(t)^2}{p(t)(1-p(t))},\ t>0,\quad \rho_s(t)=\frac{\varphi_s^{-1}[2\varphi_s(p(t))]-p(t)^2}{p(t)(1-p(t))},\ t>0,$$

depending on whether the two companies under consideration belong to different sectors or the same sector $s\in\{1,\ldots,S\}$, respectively.

(b) For the partially nested Archimedean copula (4), we obtain

$$\lim_{t \searrow 0} \rho_0(t) = \lambda_U, \quad \lim_{t \searrow 0} \rho_s(t) = \lambda_{U,s},$$

where the first limit corresponds to companies belonging to different and the second corresponds to companies belonging to the same sector $s \in \{1, \dots, S\}$. In this formula, $\lambda_{U,s}$ denotes the upper tail dependence parameter of the Archimedean copula generated by φ_s.

Proof
Note that Theorem 3.1 of Schönbucher (2003) only depends on the bivariate marginal Archimedean copula. For nested Archimedean copulas, we have at most $S+1$ different bivariate margins, leading to the result as stated. □

Corollary 3.3
The copulas of the Archimedean families we nest are ordered in the concordance ordering. This implies, that larger values of the outer parameter ϑ_0 imply larger default correlations between companies belonging to different sectors. Similarly, larger values of the inner parameter ϑ_s lead to larger default correlations between companies belonging to the same sector s. Further note that the restriction $\vartheta_0 \leq \vartheta_s$ for any $s \in \{1, \dots, S\}$ for the nested Archimedean families we consider implies that a pair of random variables belonging to the same sector is at least as concordant as a pair of random variables belonging to different sectors. This implies $\rho_0(t) \leq \rho_s(t)$ for any $t > 0$ and $s \in \{1, \dots, S\}$.

4 Portfolio CDS and CDOs

Portfolio-credit derivatives, such as portfolio CDS and CDOs, have shown an impressive growth in terms of outstanding notional over the last decade. Especially CDOs have attracted attention among researchers and practitioners as their prices are driven, to large extend, by the dependence among the obligors in the considered portfolio. The idea of CDOs is to pool credit risky assets and to resell the portfolio in slices with different seniority, called the tranches of a CDO.

4.1 The payment streams

We assume a portfolio consisting of I obligors, each contributing $1/I$ to the unit nominal of the portfolio. The time to maturity (in years) is denoted by T. As common for spread products, we have to consider two payment streams, the premium and the default leg. Premium payments are made at certain dates, the resulting payment schedule is denoted by $\mathcal{T} = \{t_0 = 0 < t_1 < \cdots < t_n = T\}$. Note that defaults may happen anywhere in $[0, T]$, but to simplify the computation of the default leg, we defer all default payments in between two premium payment dates to the next scheduled payment date. For assessing accrued interest, i.e. the interest accumulated between a default and the last payment date, we assume that defaults happen at the midpoint of two payment dates. Therefore, accrued interest for defaulted companies is considered by taking the midpoint $(t_k + t_{k+1})/2$ as reference for a default $\tau_i \in [t_k, t_{k+1})$.

Portfolio CDS and the tranches of a CDO can both be interpreted as options on the portfolio-loss process L_t. In what follows we assume an identical deterministic recovery rate R for all companies, which simplifies the computation of L_t to

$$L_t = \frac{1-R}{I} \sum_{i=1}^{I} \mathbb{1}_{\{\tau_i \le t\}}, \ t \in [0,T].$$

Based on the overall portfolio loss, the loss affecting tranche $j \in \{1, \dots, J\}$ of a CDO contract is given by

$$L_{t,j} = \min\{\max\{0, L_t - l_j\}, u_j - l_j\}, \ t \in [0,T], \tag{5}$$

where l_j and u_j, $j \in \{1, \dots, J\}$, denote the lower and upper attachment points for the CDO tranches, respectively.

Given L_t, $t \in [0,T]$, the remaining nominal of the considered portfolio CDS is given by

$$N_t = 1 - \frac{L_t}{1-R}, \ t \in [0,T],$$

i.e. the remaining nominal is reduced by $1/I$ after each default. For tranche $j \in \{1, \dots, J\}$ of a CDO contract, the remaining nominal is determined by

$$N_{t,j} = (u_j - l_j - L_{t,j}), \ t \in [0,T]. \tag{6}$$

Given the payment schedule $\mathcal{T}$, the annualized portfolio-CDS spread s_T^{pCDS} (quoted in bp), and the discount factors d_{t_k} corresponding to the time point t_k, the expected discounted premium and default leg of the portfolio CDS are given by

$$EDPL_T = \mathbb{E}\Big[\sum_{k=1}^{n} d_{t_k} s_T^{pCDS} \Delta t_k \left(N_{t_k} + (N_{t_{k-1}} - N_{t_k})/2\right)\Big], \tag{7}$$

$$EDDL_T = \mathbb{E}\Big[\sum_{k=1}^{n} d_{t_k} (L_{t_k} - L_{t_{k-1}})\Big], \tag{8}$$

where $\Delta t_k = (t_k - t_{k-1})$ and the summands $d_{t_k} s_T^{pCDS} \Delta t_k (N_{t_{k-1}} - N_{t_k})/2$ in Equation (7) account for accrued interest.

For tranche $j \in \{1, \dots, J\}$ of a CDO contract, the corresponding legs are given by

$$EDPL_{T,j} = \mathbb{E}\Big[\sum_{k=1}^{n} d_{t_k} s_{T,j}^{CDO} \Delta t_k \left(N_{t_k,j} + (N_{t_{k-1},j} - N_{t_k,j})/2\right)\Big], \tag{9}$$

$$EDDL_{T,j} = \mathbb{E}\Big[\sum_{k=1}^{n} d_{t_k} (L_{t_k,j} - L_{t_{k-1},j})\Big], \tag{10}$$

where $s_{T,j}^{CDO}$ denotes the annualized spread of the respective tranche (quoted in bp for all but the equity tranche). The fair spreads $s_T^{pCDS,f}$ and $s_{T,j}^{CDO,f}$ of the portfolio CDS and the tranches of the CDO, respectively, are computed by equating the respective expected discounted premium and default leg and solving for the spread. It became market standard to assume a running spread of 500 bp for the most subordinate tranche, called equity tranche. Therefore, an upfront payment (quoted as a percentage of the nominal of the equity tranche) is introduced to correct for this artificial spread. This upfront payment, for simplicity denoted by $s_{T,1}^{CDO}$ in the sequel, satisfies the relation

$$s_{T,1}^{CDO}\left(u^1 - l^1\right) + \mathbb{E}\Big[\sum_{k=1}^{n} d_{t_k} 0.05 \Delta t_k (N_{t_k,1} + (N_{t_{k-1},1} - N_{t_k,1})/2)\Big] = EDDL_{T,1}. \quad (11)$$

4.2 The pricing approach

This section presents the pricing algorithms used to calibrate the model. We begin with the algorithm for pricing portfolio CDS. If deterministic discount factors are assumed, Equations (7) and (8) only require the computation of the expected portfolio loss at the premium-payment dates. Using linearity, this expectation is simply given as the mean of individual default probabilities times the loss given default, which is easily computed in our framework. The computation of the expected remaining nominal is done similarly. Hence, the expected discounted premium and default legs can be evaluated for any dependence structure between the companies. This allows us to find the fair spread for a portfolio CDS via the following algorithm.

Algorithm 3 (The fair spread of a portfolio CDS)

(1) ***Setup****. Specify the payment schedule $\mathcal{T}$, the number of companies I, the intensity function $\lambda_i(t)$ of each firm, the recovery rate R, and the discount factors d_{t_k}.*

(2) ***Expected discounted premium and default legs****. Compute the expected discounted premium leg with $s_T^{pCDS} = 1$ and the default leg from Equations (7) and (8), respectively, by the argument explained above. Assess the fair spread by*

$$s_T^{pCDS,f} = \frac{EDDL_T}{EDPL_T}.$$

Principally, the same argument applies to Equations (9) and (10) for the different CDO tranches. However, as we observe from Equation (5), the loss affecting a certain tranche is not a linear functional in the default indicators $\mathbb{1}_{\{\tau_i \leq t\}}$, $i \in \{1, \ldots, I\}$. Due to the dependence structure of the trigger variables, it is therefore not straightforward to compute the resulting expected loss of a certain tranche analytically. Therefore, we use the following algorithm for pricing the tranches of a CDO, which is based on a Monte-Carlo simulation.

Algorithm 4 (Pricing CDO tranches via Monte Carlo)

(1) ***Setup****. Specify $\mathcal{T}$, I, $\lambda_i(t)$, R, and d_{t_k} as in Algorithm 3. Moreover, specify the number of simulation runs N, the attachment points l_j and u_j of each tranche, and*

a copula C for the default triggers $(U_1, \dots, U_I)$. If C is the Gauss copula or an exchangeable Archimedean copula as listed in Table 1, only one parameter ϑ_0 has to be chosen. However, if C is a nested Archimedean copula, the parameter vector $\boldsymbol{\vartheta} = (\vartheta_0, \dots, \vartheta_S)$ has to be specified.

(2) ***Survival probabilities***. *Compute all $p_i(t_k)$ as in Equation (1).*

(3) ***Monte Carlo simulation***. *For each of the N runs, do:*

   (3.1) *Sample $(U_1, \dots, U_I) \sim C$ according to Theorem A.1.*

   (3.2) *Compute the corresponding default times via $\tau_i = p_i^{-1}(U_i)$.*

   (3.3) *Compute the loss process L_{t_k}, $t_k \in \mathcal{T}$, of the current Monte Carlo run.*

   (3.4) *For each tranche $j \in \{1, \dots, J\}$ and each $t_k \in \mathcal{T}$, compute $L_{t_k,j}$ and $N_{t_k,j}$ via Equations (5) and (6), respectively.*

(4) ***Expected discounted premium and default legs***. *Based on Step (3), compute for each tranche $j \in \{1, \dots, J\}$ the N discounted premium and default legs of all Monte Carlo runs and estimate the expectations in Equations (9) and (10) by their sample means $\overline{EDPL}_{T,j}$ and $\overline{EDDL}_{T,j}$, respectively. For assessing the fair spreads of all tranches, compute the premium legs with spreads $s_{T,j}^{CDO} = 1$, $j \in \{2, \dots, J\}$, and set*

$$\hat{s}_{T,j}^{CDO,f} = \frac{\overline{EDDL}_{T,j}}{\overline{EDPL}_{T,j}}, \; j \in \{2, \dots, J\},$$

and determine $\hat{s}_{T,1}^{CDO,f}$ via Equation (11).

Estimating CDO spreads via Monte Carlo naturally invokes the question on confidence intervals. Given the straightforward asymptotic confidence intervals for the expected discounted premium and default legs, asymptotic confidence intervals for the tranche spreads are easily found via the following lemma.

Lemma 4.1 (Asymptotic confidence intervals for CDO spreads)
Given a significance level $\alpha \in [0, 1]$ and a tranche $j \in \{1, \dots, J\}$ of a CDO contract with maturity T, asymptotic $(1-\alpha/2)$-confidence intervals for the expected discounted default and premium legs $EDDL_{T,j}$ and $EDPL_{T,j}$ are given by

$$[l_{EDDL_{T,j}}, u_{EDDL_{T,j}}] = \Big[\overline{EDDL}_{T,j} - \frac{q_{1-\alpha/2}}{\sqrt{N}} s_{DDL_{T,j}}, \overline{EDDL}_{T,j} + \frac{q_{1-\alpha/2}}{\sqrt{N}} s_{DDL_{T,j}}\Big],$$
$$[l_{EDPL_{T,j}}, u_{EDPL_{T,j}}] = \Big[\overline{EDPL}_{T,j} - \frac{q_{1-\alpha/2}}{\sqrt{N}} s_{DPL_{T,j}}, \overline{EDPL}_{T,j} + \frac{q_{1-\alpha/2}}{\sqrt{N}} s_{DPL_{T,j}}\Big],$$

respectively, where $s_{DDL_{T,j}}$ and $s_{DPL_{T,j}}$ denote the sample standard deviations for the simulated discounted default and premium legs, respectively, and $q_{1-\alpha/2}$ denotes the $1-\alpha/2$ quantile of the standard normal distribution. This implies that

$$\Big[\frac{l_{EDDL_{T,j}}}{u_{EDPL_{T,j}}}, \frac{u_{EDDL_{T,j}}}{l_{EDPL_{T,j}}}\Big]$$

is an asymptotic $(1-\alpha)$*-confidence interval for the fair spread* $s_{T,j}^{CDO,f}$ *of the CDO tranche* j*. For the upfront payment, an asymptotic* $(1-\alpha)$*-confidence interval is given by*

$$\Big[\bar{Y} - \frac{q_{1-\alpha}}{\sqrt{N}} s_Y, \bar{Y} + \frac{q_{1-\alpha}}{\sqrt{N}} s_Y\Big],$$

where

$$Y_l = \frac{1}{u^1 - l^1} \sum_{k=1}^{n} d_{t_k} (L_{t_k,1} - L_{t_{k-1},1} - 0.05 \Delta t_k (N_{t_k,1} + (N_{t_{k-1},1} - N_{t_k,1})/2)),$$

$l \in \{1, \dots, N\}$*, and* $\bar{Y}$ *and* s_Y *denote the sample mean and standard deviations of the variates* Y_l*, respectively.*

Proof
All results are straightforward applications of the central limit theorem or involve only basic calculations. □

5 Calibration of the model

One difficulty in calibrating a portfolio model to CDO quotes is that the upfront payment of the equity tranche complicates the comparison of pricing errors in this tranche to pricing errors in more senior tranches. By far the most spread is payed for the equity tranche, we therefore calibrate our model to match the upfront payment (up to bid-ask spreads reflected by ε_{CDO}) of the equity tranche and to minimize the distance of market to model spreads over the remaining tranches. Hence, we aim for

$$D_1 := |s_{T,1}^{CDO,f} - s_{T,1}^{CDO,m}| \le \varepsilon_{CDO}, \quad D_2 := \sum_{j=2}^{J} |s_{T,j}^{CDO,f} - s_{T,j}^{CDO,m}| \to \min, \tag{12}$$

where the minimization is taken over the involved copula parameters.

5.1 Data and setup

We calibrate our model to portfolio CDS and CDO market quotes of the seventh iTraxx Europe series, the former are denoted by $s_T^{pCDS,m}$, the latter by $s_{T,j}^{CDO,m}$, $j \in \{1, \dots, J\}$, for the tranches of the CDO. Portfolio CDS spreads are available for contracts maturing in three, five, seven, and ten years and spreads of the first five tranches of the CDO are liquidly traded for the maturities five, seven, and ten years. All data was retrieved from the Bloomberg database. We calibrated our model to five trading days, 2007-06-12, 2007-06-14, 2007-06-19, 2007-06-21, and 2007-06-26. The iTraxx Europe series implies a quarter-yearly payment schedule $\mathcal{T}$, i.e. $n = 4T$, where we chose $T \in \{5, 10\}$. Further, the portfolio consists of $I = 125$ companies which are mapped to one of the six business sectors: Auto (10), Consumer (30), Energy (20), Financials (25), Industrials (20), and

TMT (20). For simplicity, we assume a homogeneous portfolio with piecewise constant default intensities of the form

$$\lambda(t) = \lambda_i(t) = \lambda_5 \mathbb{1}_{[0,5]}(t) + \lambda_{10} \mathbb{1}_{(5,10]}(t),\ t \in [0,10],$$

for each $i \in \{1, \dots, I\}$, where in the sequel λ_5 and λ_{10} are used to denote the constant intensities on the intervals $[0,5]$ and $(5,10]$, respectively. The constant recovery rate is chosen as $R = 40\%$ for all firms, a commonly accepted assumption. The attachment points of the five traded tranches of the iTraxx Europe portfolio are given by [0%,3%], [3%,6%], [6%,9%], [9%,12%], and [12%,22%]. The continuously compounded interest rates are derived from par yields obtained from Bloomberg (ticker symbols C9603M, C9601Y, ..., C96010Y) by the standard bootstrap method, see Hull (2005), Chapter 4. Interest rates corresponding to noninteger maturities were linearly interpolated.

5.2 Calibration of the individual default probabilities

One major advantage of our model is that default probabilities and dependence structure are specified independent of each other. This allows to proceed in two steps, initially fitting the default probabilities to portfolio CDS quotes via adjusting the default intensities and then fitting the dependence structure to CDO quotes by appropriately setting the parameters of the respective copula. The first step is done with Algorithm 5, the second step with Algorithm 6.

Algorithm 5 (Fitting intensities via Algorithm 3)

(1) ***Setup****. In our calibration, we specify the parameters as described in Section 5.1 according to the iTraxx Europe standards. Portfolio CDS market spreads are denoted by* $s_5^{pCDS,m}$ *and* $s_{10}^{pCDS,m}$ *for maturities* $T = 5$ *and* $T = 10$*, respectively.*

(2) ***Fitting*** λ_5 ***and*** λ_{10}*. Use a numerical root-finding procedure to find* $\hat{\lambda}_5$*, such that* $s_5^{pCDS,f} = s_5^{pCDS,m}$*. For this, the model spread* $s_5^{pCDS,f}$*, which is an increasing function of* λ_5*, is computed using Algorithm 3 with* $T = 5$*. Given* $\hat{\lambda}_5$*, use the numerical root-finding procedure a second time to find* $\hat{\lambda}_{10}$ *satisfying* $s_{10}^{pCDS,f} = s_{10}^{pCDS,m}$*. For this, the model spread* $s_{10}^{pCDS,f}$ *is computed as a function of* λ_{10} *with fixed* $\lambda_5 = \hat{\lambda}_5$ *using Algorithm 3 with* $T = 10$*.*

5.3 Calibration of the dependence structure

The dependence structure of our model is specified by the copula from which the default triggers $(U_1, \dots, U_I)$ are drawn. We used all Archimedean families listed in Table 1, each in their exchangeable and nested version. For each family we further considered the corresponding survival copulas. Given a sample $(U_1, \dots, U_I)$ from a copula, a sample from the corresponding survival copula is given by $(1 - U_1, \dots, 1 - U_I)$. We hereby excluded the case of Frank, as this copula is radially symmetric. The remaining freedom of choice for the parameter ϑ_c of the outer power Clayton copula is used by setting

$\vartheta_c = 0.1$. The reason for this choice is that we preferred a rather small value for ϑ_c in order to be able to capture a large interval of possible upper tail dependence parameters for the Clayton survival copula. As a benchmark for our studies, we also included the Gauss copula. Overall, this results in a pool of eleven different copulas, being able to capture different kinds of tail dependence.

For each family, our goals are two-fold. We first test if the exchangeable copula of the family under consideration is able to produce sufficient dependence to match the upfront payment of the equity tranche. If not, we feel that this family is not suitable for modeling CDOs in the suggested framework. If so, we try to improve the fitting quality in a second step by using the nested copula of the same family. To decrease the dimension of the parameter space in the second step, we assume identical parameters $\vartheta_s = \vartheta_1$ for all sectors $s \in \{2, \ldots, S\}$.

The objecting function for assessing the fitting quality of our model is stated in Equation (12). As fair spreads are found by simulation we developed our own optimizer. This routine exploits the specific structure of the problem to achieve the required precision in relatively small amounts of time. The idea of our routine is to first choose the parameter ϑ_0 of the exchangeable Archimedean copula such that the upfront payment is matched, this position is denoted by $\hat{\vartheta}$. Then, the second step starts from position $(\hat{\vartheta}, \hat{\vartheta})$ with the nested Archimedean copula of the same class and follows the level curve satisfying $D_1 = 0$ on a fine two-dimensional grid. Important for tracking the level curve on which the model matches the quoted upfront payment is the fact that the upfront payment is decreasing in the dependence among the firms. Due to the concordance orderings of both the outer and the inner Archimedean families of the nested Archimedean families we consider, the level curve is a monotonically nonincreasing function in the $(\vartheta_0, \vartheta_1)$ plane, see Corollary 3.3 and Figure 2, which justifies and illustrates this algorithm, respectively.

Algorithm 6 (CDO calibration)

(1) ***Setup***. *In our calibration we use parameters as described in Section 5.1. In particular, CDO market spreads are denoted by* $s_5^{CDO,m}$ *and* $s_{10}^{CDO,m}$ *for maturities* $T = 5$ *and* $T = 10$*, respectively. We further have to specify a copula* C *for the default triggers* $(U_1, \ldots, U_I)$*, which we assume to be one of the copulas mentioned above. Denote by* $[\vartheta_l, \vartheta_u]$ *the parameter space for the optimization of* ϑ_0*, depending on the chosen family. If* C *is a nested Archimedean copula, choose the number of sectors* S *(e.g. 6) and the companies in each sector* $(d_1, \ldots, d_S)$ *(e.g.* $(10, 30, 20, 25, 20, 20)$*), in correspondence to the iTraxx Europe specifications. Choose* ε_{CDO} *(e.g.* 0.0004*) as pricing error for the upfront payment of the first tranche and choose the number* N *(e.g.* $500,000$*) of simulation runs. Specify the number of subdivisions of each dimension for the 2d-optimizer* m_1 *(e.g.* 200*,* 300*, or* 400*, depending on the length* $\vartheta_u - \vartheta_l$ *of the starting interval) and the number* m_2 *(e.g. 3) of subdivisions for refinement.*

(2) ***Fitting*** λ_5 ***and*** λ_{10}. *Calibrate the model to match portfolio CDS spreads. For this, use Algorithm 5 to obtain the fitted intensities* $\hat{\lambda}_5$ *and* $\hat{\lambda}_{10}$.

(3) ***1d-optimization***. *For the parameter* $\vartheta_0 \in [\vartheta_l, \vartheta_u]$ *of the chosen copula* C*, find a value* $\hat{\vartheta}$ *satisfying* $D_1 < \varepsilon_{CDO}$*, compare Equation (12), by using a bisection. If there is no such parameter, stop, and conclude that* C *is not adequate for our modeling purpose.*

(4) ***2d-optimization***. *If* C *is a nested Archimedean copula use* $\boldsymbol{\vartheta}_0 = (\vartheta_0, \vartheta_1) = (\hat{\vartheta}, \hat{\vartheta})$ *from the 1d-optimization in Step (3) as initial vector for the minimization of* D_2 *over all* $(\vartheta_0, \vartheta_1) \in [\vartheta_l, \vartheta_u]^2$ *satisfying the constraint* $D_1 < \varepsilon_{CDO}$*, compare Equation (12). Note that being the result of the 1d-optimization, the vector* $(\hat{\vartheta}, \hat{\vartheta})$ *implies* $D_1 < \varepsilon_{CDO}$*. For the minimization of* D_2*, define a fine grid on the parameter space* $(\vartheta_0, \vartheta_1) \in [\vartheta_l, \vartheta_u]^2$ *with mesh* $l = (\vartheta_u - \vartheta_l)/m_1$*. Then repeat*

  (4.1) *For each of the parameter constellations* $\boldsymbol{\vartheta}_1 = (\vartheta_0 - l, \vartheta_1)$*,* $\boldsymbol{\vartheta}_2 = (\vartheta_0 - l, \vartheta_1 + l)$*, and* $\boldsymbol{\vartheta}_3 = (\vartheta_0, \vartheta_1 + l)$ *derive CDO tranche spreads with Algorithm 4.*

  (4.2) *If the upfront payment of at least one of the parameter pairs* $\boldsymbol{\vartheta}_k$*,* $k \in \{1, 2, 3\}$*, is found to satisfy* $D_1 < \varepsilon_{CDO}$*, set* $\boldsymbol{\vartheta}_0$ *to* $\boldsymbol{\vartheta}_k$*.*

  (4.3) *If none of the three search directions fulfills* $D_1 < \varepsilon_{CDO}$*, consider the direction* $\boldsymbol{\vartheta}_k$ *minimizing* D_1 *and subdivide the segment from* $\boldsymbol{\vartheta}_0$ *to* $\boldsymbol{\vartheta}_k$ *into* m_2 *equally spaced parts. Then, apply Algorithm 4 with the* m_2 *copula parameters* $\boldsymbol{\vartheta}_l$*,* $l \in \{1, \dots, m_2\}$ *and set* $\boldsymbol{\vartheta}_0$ *to* $\operatorname{argmin}_{l \in \{0,\dots,m_2\}} D_1$*.*

until $\boldsymbol{\vartheta}_0 \notin [\vartheta_l, \vartheta_u]^2$*, i.e.* $\vartheta_0 \le \vartheta_l$ *or* $\vartheta_1 \ge \vartheta_u$*. Given all visited pairs* $(\vartheta_0, \vartheta_1)$ *satisfying* $D_1 < \varepsilon_{CDO}$*, choose* $(\hat{\vartheta}_0, \hat{\vartheta}_1)$ *to be the minimizer of* D_2*.*

5.4 Results of the calibration

First of all, let us remark that the calibration results we obtained are similar for the five considered trading days. Therefore, we only list the first trading day in detail, the results for this day are listed in Tables 2 and 3 for maturities $T = 5$ and $T = 10$, respectively. The average results for all trading days are presented in condensed form in Table 4. In all tables, we identify exchangeable and nested Archimedean copulas by using the leading characters "e" and "n", respectively. Further, the survival copulas for each family are denoted by a trailing "s". The Gauss copula, abbreviated by "Ga", is used as a benchmark. In conjunction to the notation for Archimedean copulas, we use "eGa" for the Gauss copula with homogeneous correlation and "nGa" for the Gauss copula parametrized such that firms in the same sector have correlation ϑ_1 and firms in different sectors have correlation ϑ_0.

The Gauss copula as market standard is outperformed by several exchangeable Archimedean families. Those which are able to capture upper tail dependence provided good calibration results for all analyzed trading days and maturities. Our generalization to nested Archimedean copulas reduced the pricing errors for all trading days and maturities, especially for the families which are upper tail dependent, the error was significantly reduced. We emphasize that this improvement is already obtained by introducing a single additional parameter. The outer power Clayton copula provided the most accurate fit to market quotes. For this copula, Table 5 lists computed 98% confidence intervals for the upfront payment and fair spreads computed with 500,000 runs. The concern that the generalization to nested Archimedean copulas is computationally too expensive is not justified, as we may infer from mean computational times as listed in Table 4, where $\bar{\boldsymbol{\kappa}}$ denotes mean runtimes for the optima taken over all five trading days.

2007-06-12	Dependence		CDO upfront and spreads $\hat{s}_{5,j}^{CDO,f}$					Error
Copula	$\hat{\boldsymbol{\vartheta}}$	$\boldsymbol{\rho}$ in %	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	D_2
eA	-	-	-	-	-	-	-	-
nA	- -	- -	-	-	-	-	-	-
eAs	0.69	3.55	7.42	123.06	2.13	0.00	0.00	94.95
nAs	0.12 0.94	0.23 17.88	7.42	108.12	11.38	0.82	0.02	69.93
eC	1.94	3.32	7.40	98.67	19.27	3.84	0.26	64.21
nC	1.87 2.03	3.21 3.48	7.42	97.02	18.21	3.50	0.24	61.84
eCs	0.08	3.28	7.40	91.79	21.41	5.76	0.64	58.32
nCs	0.08 0.09	3.15 3.42	7.42	88.80	19.89	5.22	0.63	53.28
eopC	1.06	8.07	7.42	37.31	17.51	11.73	6.96	24.86
nopC	1.05 1.08	6.10 10.44	7.43	44.88	19.77	11.42	5.37	17.64
eopCs	-	-	-	-	-	-	-	-
nopCs	- -	- -	-	-	-	-	-	-
eF	2.77	3.28	7.44	92.97	22.34	5.98	0.64	60.65
nF	2.72 2.83	3.22 3.36	7.42	90.74	20.92	5.26	0.55	56.38
eG	1.06	8.19	7.42	35.94	17.58	11.62	6.81	26.04
nG	1.05 1.09	5.93 11.10	7.38	45.32	20.66	11.83	5.33	18.86
eGs	1.24	3.24	7.43	109.41	13.10	0.90	0.01	71.96
nGs	1.24 1.27	3.09 3.73	7.39	105.17	12.45	0.89	0.01	67.08
eJ	1.07	8.45	7.37	32.01	17.06	11.88	7.24	30.14
nJ	1.04 1.10	5.41 11.83	7.44	45.38	21.24	11.80	5.16	19.31
eJs	-	-	-	-	-	-	-	-
nJs	- -	- -	-	-	-	-	-	-
eGa	0.20	3.31	7.43	89.85	21.68	5.94	0.76	56.71
nGa	0.16 0.33	2.49 6.99	7.37	89.44	20.51	5.52	0.70	54.77
Market			7.40	45.12	11.84	5.18	2.14	

Table 2 Simulation results for 2007-06-12 based on 500,000 runs.

Figure 1 shows the implied default correlations for all fitted copulas for the first day for maturity $T=5$. As interpretation we notice the large variety of implied term structures of default correlations. Additionally considering the calibration results allows us to

2007-06-12	Dependence		CDO upfront and spreads $\hat{s}_{10,j}^{CDO,f}$					Error
Copula	$\hat{\boldsymbol{\vartheta}}$	$\boldsymbol{\rho}$ in %	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	D_2
eA	0.71	4.79	36.86	367.06	150.76	62.84	11.77	129.48
nA	0.70 0.73	4.72 4.92	36.87	365.35	149.86	62.57	11.58	126.81
eAs	0.27	2.41	36.88	536.29	108.22	4.04	0.01	286.13
nAs	0.01 0.83	0.06 19.59	36.91	427.53	150.32	41.73	2.99	178.79
eC	0.62	4.30	36.91	388.18	150.82	55.82	8.08	147.34
nC	0.62 0.62	4.29 4.30	36.85	386.60	150.09	55.77	8.19	144.88
eCs	0.07	4.07	36.87	390.46	146.77	54.33	8.59	143.58
nCs	0.07 0.09	3.87 4.84	36.92	389.40	145.19	53.45	8.47	140.16
eopC	1.10	12.58	36.86	279.29	89.68	47.39	24.35	57.10
nopC	1.05 1.16	6.68 17.92	36.92	315.12	115.26	57.25	19.13	44.13
eopCs	-	-	-	-	-	-	-	-
nopCs	- -	- -	-	-	-	-	-	-
eF	1.56	6.06	36.91	327.08	143.55	70.88	20.08	95.40
nF	1.55 1.56	6.02 6.10	36.85	325.38	141.58	69.84	19.66	90.28
eG	1.12	14.28	36.90	248.17	85.72	50.78	27.66	98.88
nG	1.05 1.20	6.65 21.06	36.92	296.78	115.70	61.22	20.70	68.46
eGs	1.10	2.97	36.89	466.13	146.36	22.38	0.40	235.38
nGs	1.09 1.19	2.58 5.67	36.92	451.47	147.39	29.54	1.03	213.95
eJ	1.17	16.56	36.89	213.34	76.36	50.83	30.52	145.99
nJ	1.02 1.29	2.66 25.72	36.89	313.24	133.75	70.58	17.17	75.86
eJs	-	-	-	-	-	-	-	-
nJs	- -	- -	-	-	-	-	-	-
eGa	0.13	4.36	36.89	381.96	146.96	57.13	10.02	136.62
nGa	0.13 0.16	4.15 5.43	36.87	381.17	146.16	56.53	10.01	134.45
Market			36.88	316.90	93.36	42.53	13.39	

Table 3 Simulation results for 2007-06-12 based on 500,000 runs.

	$T=5$			$T=10$		
Copula	$\bar{\boldsymbol{\kappa}}$ in sec	$\bar{\boldsymbol{\rho}}$ in %	$\bar{D}_2$	$\bar{\boldsymbol{\kappa}}$ in sec	$\bar{\boldsymbol{\rho}}$ in %	$\bar{D}_2$
eA	-	-	-	28.76	4.94	138.14
nA	-	-	-	47.42	4.78 5.56	134.97
eAs	18.84	3.44	91.82	29.34	2.46	303.47
nAs	29.53	0.25 16.96	67.53	37.92	0.08 20.23	189.67
eC	30.19	3.18	59.35	40.81	4.50	158.12
nC	59.93	3.04 3.33	56.06	80.94	4.40 4.65	154.57
eCs	30.13	3.12	52.51	40.55	4.20	153.31
nCs	200.13	2.99 3.32	48.95	220.29	4.12 4.40	149.50
eopC	45.01	7.73	27.06	55.13	12.90	57.28
nopC	70.21	4.88 11.24	16.18	80.16	7.77 17.53	47.98
eF	23.86	3.14	54.71	34.23	6.30	101.86
nF	61.59	3.04 3.26	50.54	69.53	6.22 6.38	97.84
eG	34.14	7.83	29.18	44.34	14.63	99.35
nG	58.00	4.52 11.93	17.44	68.61	7.53 20.88	72.70
eGs	34.65	3.10	69.72	44.86	3.05	246.46
nGs	58.41	2.88 3.93	66.10	69.11	2.67 5.68	223.42
eJ	33.68	8.09	32.54	43.91	16.98	147.90
nJ	55.50	4.22 12.51	18.11	65.99	3.33 25.84	80.84
eGa	63.74	3.15	51.24	73.83	4.53	145.65
nGa	63.73	2.87 4.21	48.92	73.90	4.29 5.67	143.36

Table 4 Average calibration results based on 500,000 runs.

2007-06-12		Two-sided 98% confidence intervals for $s_{T,j}^{CDO,f}$				
Copula	T	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
eopC	5	[7.33,7.51]	[36.40,38.21]	[16.85,18.17]	[11.18,12.28]	[6.55,7.37]
nopC	5	[7.34,7.52]	[43.88,45.87]	[19.07,20.46]	[10.89,11.95]	[5.02,5.72]
eopC	10	[36.76,36.95]	[277.79,280.80]	[88.72,90.65]	[46.65,48.12]	[23.83,24.87]
nopC	10	[36.82,37.02]	[313.47,316.77]	[114.16,116.35]	[56.46,58.04]	[18.71,19.56]

Table 5 Confidence intervals for the CDO upfront payment and spreads for the outer power Clayton copula fitted to 2007-06-12 based on 500,000 runs.

assign the families to three classes. The class that performed best consists of the outer power Clayton copula family, the families of Gumbel and Joe, each of which is able to capture upper tail dependence, a fact which is reflected in default correlations starting above zero. The implied default correlations of these families are relatively constant over time, which is obviously desirable if CDOs with nonstandard maturities, e.g. four years, have to be priced. Also, the absolute level of implied default correlations, the difference of intra- to inter-sector correlations and the improvement in fitting quality of the nested compared to the exchangeable Archimedean families is similar for the members of this class. The second class encompasses Ali-Mikhail-Haq's family, Clayton's family and its corresponding survival copula family, the family of Frank, the survival copula family based on Gumbel's family, and the Gauss copula. Except for the Clayton survival copula, these copula families are not able to capture upper tail dependence, which implies vanishing default correlations at time zero and forces the term structure of default correlations to increase over time. Although the Clayton survival copula is theoretically able to capture upper tail dependence, the fitted parameters imply only a negligible upper tail dependence. We may also infer from Tables 2, 3, and 4 that the nested copulas of this second class perform only slightly better than their exchangeable counterparts. Also, the difference of intra- to inter-sector correlations is relatively small. The last class of copulas only consists of the Ali-Mikhail-Haq survival copula. As most of the members of the second class, this copula also do not show upper tail dependence, but the improvement in fitting quality, as well as the difference of intra- to inter-sector correlations, is large.

Figure 2 illustrates the effectiveness of Algorithm 6 using the outer power Clayton copula exemplarily. The relevant parameter space consists of all $(\vartheta_0, \vartheta_1)$ satisfying $\vartheta_0 \leq \vartheta_1$. At first, a bisection is used to find the parameter $\hat{\vartheta}$ of the exchangeable copula for which the upfront payment is matched. The points considered in this bisection are interpreted as points on the diagonal $\vartheta_0 = \vartheta_1$ in the parameter space and displayed in Figure 2 in black color. Then, starting from this optimal point on the diagonal, the two-dimensional optimizer follows the level curve on which the nested copula matches the upfront payment. For all visited points on this level curve we compute the errors D_1 and D_2, the former is illustrated by different plot symbols, the latter by different shades of gray. The optima for the exchangeable and nested Archimedean copula are also reported.

6 Conclusion

We introduced the class of nested Archimedean copulas to the copula approach of Li (2000) and Schönbucher, Schubert (2001) for the modeling of dependent defaults. This class of copulas induces a hierarchical structure on the obligors in the considered credit portfolio, which, depending on the classification criterion, allows for different economical interpretations. To demonstrate the advantage of using nested Archimedean copulas over

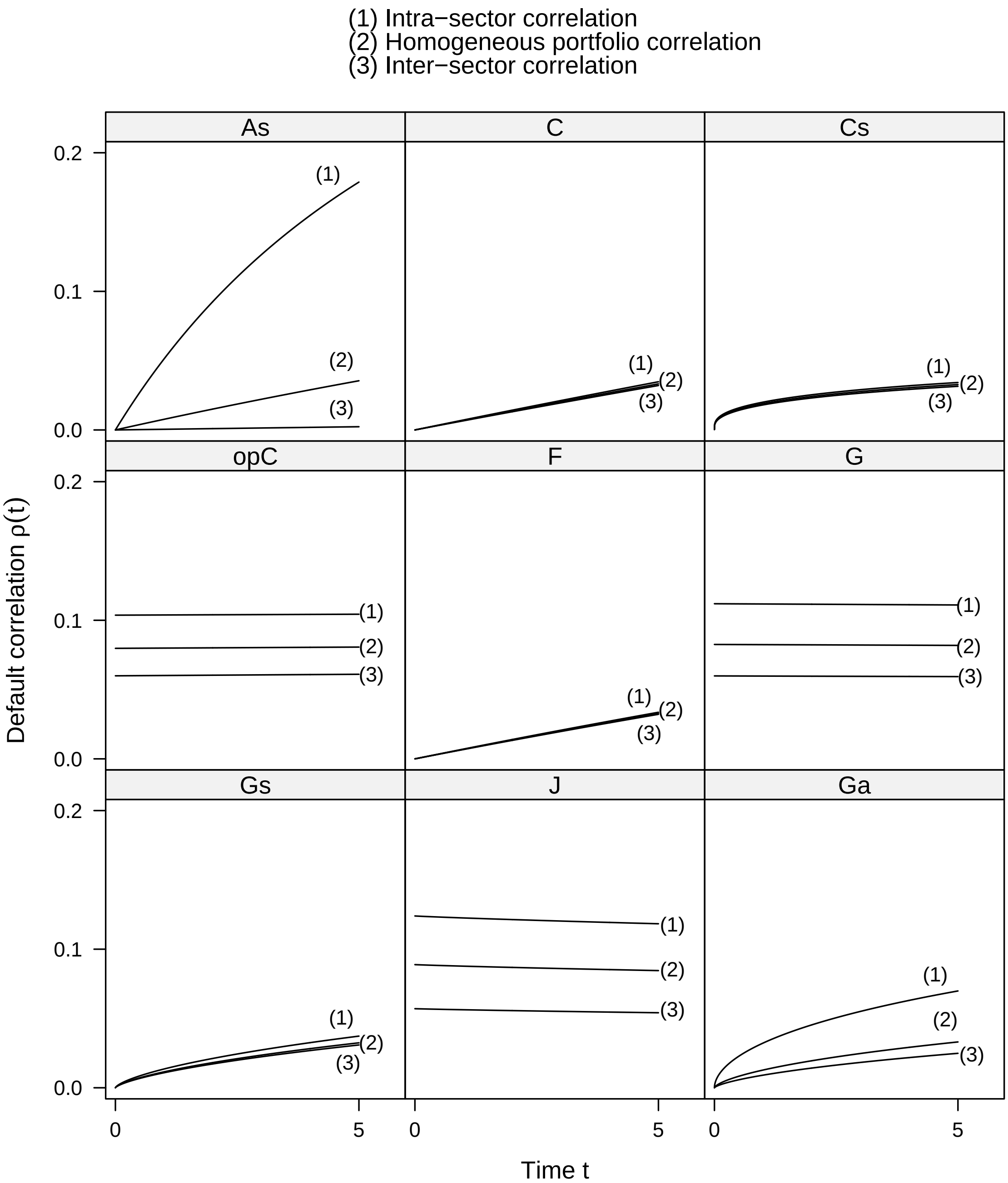

Figure 1 Default correlations for the fitted copulas for 2007-06-12 for maturity $T = 5$.

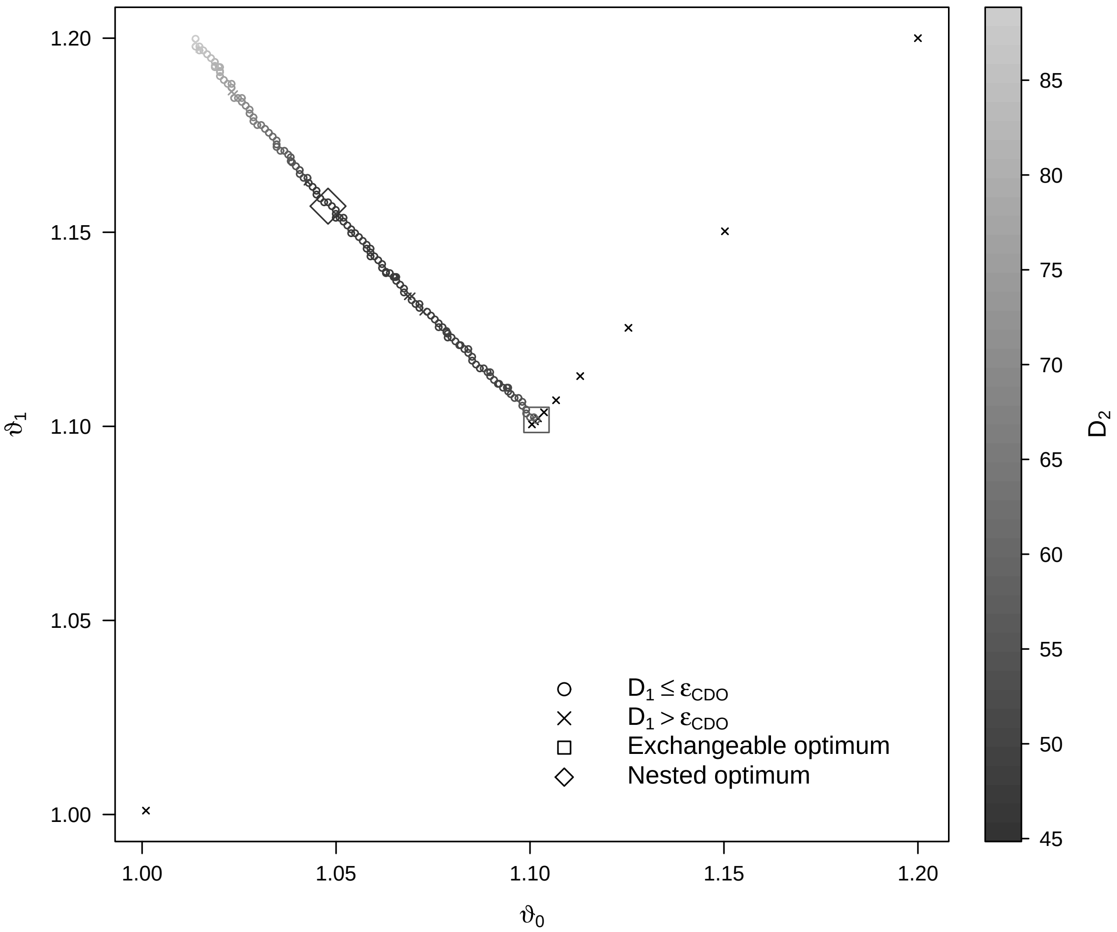

Figure 2 Optimization level curve for the outer power Clayton copula fitted to 2007-06-12 for maturity $T = 10$.

their exchangeable versions we calibrated the model to CDO tranche spreads of the European iTraxx portfolio. The hierarchical structure for this calibration was defined by the original iTraxx industry sector segmentation. Our analysis includes several exchangeable Archimedean families, some of which were newly applied within this framework, and therefore indicates which copulas might be preferable for modeling CDOs. The calibration results show considerably reduced pricing errors by using nested Archimedean families, even if we restrict our model to identical sector parameters. Moreover, our results also indicate that copulas which are able to capture upper tail dependence generally provide the best fits, e.g. the families of Gumbel, Joe, and outer power Clayton performed best. Technically, such a calibration requires fast simulation techniques and an optimizer which exploits the specific structure of the problem, both were introduced in this work. Further results address confidence intervals for CDO tranche spreads and the implied term structure of default correlations of the model. We showed that firms in the same sector have larger default correlations compared to firms in different sectors.

A Algorithms for sampling Archimedean copulas

The algorithms for sampling the exchangeable and nested Archimedean copulas listed in Table 1 are summarized in the following theorem. The values ϑ_0 and ϑ_s denote the parameters of the involved generators. For Gumbel's and partly for Clayton's family, these results can be found in McNeil (2007), the results for the other copulas were obtained by Hofert (2007a) and Hofert (2007b).

Theorem A.1 (Sampling exchangeable and nested Archimedean copulas)

(a) *For the family of Ali-Mikhail-Haq, F_0 is a $\mathrm{Geo}(1-\vartheta_0)$, i.e. a geometric, distribution. Further, $F_{0,s}$, $s \in \{1, \dots, S\}$, is also discrete and can be sampled via the following algorithm, where V_0 denotes a sample from F_0.*

   (1) Sample i.i.d. $V_{0,s,i} \sim \mathrm{Geo}(\frac{1-\vartheta_s}{1-\vartheta_0})$, $i \in \{1, \dots, V_0\}$.

   (2) Return $V_{0,s} = \sum_{i=1}^{V_0} V_{0,s,i}$.

(b) *For the family of Clayton, F_0 is a $\Gamma(1/\vartheta_0, 1)$, i.e. a Gamma, distribution with density $x^{1/\vartheta_0-1}e^{-x}/\Gamma(1/\vartheta_0)$, $x \in [0, \infty)$. Further, $F_{0,s}$, $s \in \{1, \dots, S\}$, has Laplace-Stieltjes transform $\varphi_{0,s}^{-1}(t; V_0) = \exp(-V_0((1+t)^\alpha - 1))$ with $\alpha = \vartheta_0/\vartheta_s$ and $V_0 \sim F_0$. Therefore, $F_{0,s}$ has an exponentially tilted Stable density given by $f_{0,s}(x) = e^{V_0-x}f(x)$, where f denotes the density of a $\mathrm{S}(\alpha, 1, (\cos(\frac{\pi}{2}\alpha)V_0)^{1/\alpha}, 0; 1)$ distribution, see Nolan (2007) for the Stable parametrization. If ϑ_0 is not too small, this density can be efficiently sampled via the following rejection algorithm.*

   (1) For $i \in \{1, \dots, \lceil V_0 \rceil\}$, sample $V_{0,s,i}$ from the distribution with Laplace-Stieltjes transform $\varphi_{0,s}^{-1}(t; \frac{V_0}{\lceil V_0 \rceil})$. This can be achieved with a rejection algorithm with envelope $\exp(\frac{V_0}{\lceil V_0 \rceil})f(x)$.

   (2) Return $V_{0,s} = \sum_{i=1}^{\lceil V_0 \rceil} V_{0,s,i}$.

Note that for a given V_0*, this algorithm for sampling* $F_{0,s}$ *has expected number of iterations* $\lceil V_0 \rceil \exp(\frac{V_0}{\lceil V_0 \rceil})$*. Using a rejection with envelope* $\exp(V_0)f(x)$ *right from the beginning, as proposed by McNeil (2007), will give an expected number of iterations of* $\exp(V_0)$*. For a given sample* V_0 *from* F_0 *we therefore use either of the algorithms, depending on whether* $\lceil V_0 \rceil \exp(\frac{V_0}{\lceil V_0 \rceil}) \le \exp(V_0)$ *or not.*

(c) *For a generator* $\varphi_0(t) = \varphi(t)^{\vartheta_0}$ *of an outer power family with base generator* φ*, the following algorithm samples from* $F_0 = \mathcal{LS}^{-1}(\varphi_0)$*. For Clayton's family, this involves sampling a Gamma distribution, as stated in (b).*

   (1) Sample $V \sim F = \mathcal{LS}^{-1}(\varphi^{-1})$*.*

   (2) Sample $S \sim \mathrm{S}(1/\vartheta_0, 1, (\cos(\frac{\pi}{2\vartheta_0}))^{\vartheta_0}, 0; 1)$*.*

   (3) Return SV^{ϑ_0}*.*

   Further, for step (2) of Algorithm 2 we may as well sample the copula corresponding to the generator inverse $\exp(-t^\alpha)$*,* $\alpha = \vartheta_0/\vartheta_s$*, and this generator inverse is the Laplace-Stieltjes transform of a* $\mathrm{S}(\alpha, 1, (\cos(\frac{\pi}{2}\alpha))^{1/\alpha}, 0; 1)$ *distribution.*

(d) *For the family of Frank,* F_0 *is a* $\mathrm{Log}(1 - \exp(-\vartheta_0))$*, i.e. a logarithmic, distribution. Further,* $F_{0,s}$*,* $s \in \{1, \dots, S\}$*, is also discrete and can be sampled via the following algorithm, where* $\alpha = \vartheta_0/\vartheta_s$ *and* V_0 *again denotes a sample from* F_0*.*

   (1) Sample i.i.d. $V_{0,s,i}$*,* $i \in \{1, \dots, V_0\}$*, with discrete probability density given by* $y_k = \binom{\alpha}{k}(-1)^{k-1}\frac{(1-e^{-\vartheta_s})^k}{1-e^{-\vartheta_0}}$ *at* k *for* $k \in \mathbb{N}$*.*

   (2) Return $V_{0,s} = \sum_{i=1}^{V_0} V_{0,s,i}$*.*

(e) *For the family of Gumbel,* F_0 *is a* $S \sim \mathrm{S}(1/\vartheta_0, 1, (\cos(\frac{\pi}{2\vartheta_0}))^{\vartheta_0}, 0; 1)$ *distribution. Further, for step (2) of Algorithm 2 we may as well sample the corresponding to the generator inverse* $\exp(-t^\alpha)$*,* $\alpha = \vartheta_0/\vartheta_s$*, and this generator inverse is the Laplace-Stieltjes transform of a* $\mathrm{S}(\alpha, 1, (\cos(\frac{\pi}{2}\alpha))^{1/\alpha}, 0; 1)$ *distribution.*

(f) *For the family of Joe,* F_0 *is a discrete distribution given by* $y_k = \binom{1/\vartheta_0}{k}(-1)^{k-1}$ *at* k *for* $k \in \mathbb{N}$*. Further,* $F_{0,s}$*,* $s \in \{1, \dots, S\}$*, is also discrete and can be sampled via the following algorithm, where* $\alpha = \vartheta_0/\vartheta_s$ *and* V_0 *denotes a sample from* F_0 *as before.*

   (1) Sample i.i.d. $V_{0,s,i}$*,* $i \in \{1, \dots, V_0\}$*, with discrete probability density given by* $y_k = \binom{\alpha}{k}(-1)^{k-1}$ *at* k *for* $k \in \mathbb{N}$*.*

   (2) Return $V_{0,s} = \sum_{i=1}^{V_0} V_{0,s,i}$*.*

B A word concerning the implementation

All numerical experiments were run on a node containing two AMD Opteron 252 processors (2.6 GHz) with 8 GB RAM as part of a Linux cluster. All algorithms were implemented in C/C++ and compiled using the GCC, version 3.3.3 (SuSE Linux) with option `-O2` for code optimization. The command `gettimeofday` was used to measure runtime as wall-clock time. For generating uniform random numbers we used an implementation of the Mersenne Twister by Wagner (2003).

For sampling the exchangeable and nested Archimedean copulas of Frank and Joe involved in Part (d) and (f) of Theorem A.1, we proceeded as follows. Given a set of parameters, we precomputed and stored the first couple of function values of both F_0 and $F_{0,s}$, involving one computation for exchangeable Archimedean copulas and two computations for nested Archimedean copulas by our assumption $\vartheta_s = \vartheta_1$ for all $s \in \{2, \dots, S\}$. All distribution functions were precomputed until either the corresponding values were greater than or equal to $1 - 10^{-8}$ or until 500,000 values were computed. For a uniform sample U greater than the maximal precomputed value of the distribution function, the quantile corresponding to this maximal precomputed value was returned.

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