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Abstract. In this paper, we introduce an adaptive wavelet method for operator equations on unbounded domains. We use wavelet bases on $\mathbb{R}^n$ to equivalently express the operator equation in terms of a well-conditioned discrete problem on sequence spaces. By realizing an approximate adaptive operator application also for unbounded domains, we obtain a scheme that is convergent at an asymptotically optimal rate. We show the quantitative performance of the scheme by various numerical experiments.

1. Introduction

Operator equations on unbounded domains are relevant in various fields where no boundary conditions, but only the asymptotic behavior of the solution is known. Examples include radiation or wave propagation processes as well as valuation problems in finance. In many cases, the asymptotic nature of the solution allows to truncate the computational domain to a bounded one and to perform all computations by standard methods on that bounded domain. Obviously, this requires a careful compromise of accuracy (sufficiently large truncation domain) and computational complexity (possibly small truncation domain). However, in more complex situations (like for complex structured financial products), such an a priori truncation is not straightforward.

There are several methods to treat problems on unbounded domains such as Infinite Elements, Inverted Finite Elements, FEM-BEM coupling and others. In this paper, we introduce an adaptive wavelet method for operator equations on unbounded domains. For the ease of presentation, we describe the idea and the analysis for the univariate case. Note that the approach is by no means restricted to 1d as we will later also describe in more detail. The key ingredient are Riesz wavelet bases for Sobolev spaces $H^t(\mathbb{R})$. Given a Riesz basis for $L_2(\mathbb{R})$ consisting of dilations and integer translates of a locally supported mother wavelet ψ,

$$\Psi^{\mathbb{R}} := \{\psi_{j,k} : \ j,k \in \mathbb{Z}\}, \quad \psi_{j,k}(x) := 2^{j/2}\psi(2^j x - k), \quad j,k \in \mathbb{Z}, \ x \in \mathbb{R}, \tag{1.1}$$

one can show (e.g. [5, 11, 32] and the references therein) that $\Psi_t^{\mathbb{R}} := \{\psi_{j,k} \times \|\psi_{j,k}\|^{-1}_{H^t(\mathbb{R})} : j,k \in \mathbb{Z}\}$ is a Riesz basis for $H^t(\mathbb{R})$ for $t \in (-\widetilde{\gamma}, \gamma)$ and $\gamma, \widetilde{\gamma}$ depending on the choice of ψ. Thus, as stated in [7, p.210], one can transform the original operator equation $\mathcal{A}u = f$ on $H^{-t}(\mathbb{R})$ into an equivalent well-posed problem $\mathbf{A}\mathbf{u} = \mathbf{f}$

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on sequence spaces ℓ_2 for the wavelet coefficients. This idea has been used e.g. in [2, 6, 7, 12, 15, 18, 26] (see also [32]) where the focus was *always* on operator equations on bounded domains. It results in adaptive wavelet methods that have been proven to converge at an optimal rate as compared with the best N-term approximation w.r.t. the same basis. To highlight the differences to bounded settings, let us mention that a wavelet basis on an interval $\Omega = (a, b) \subset \mathbb{R}$ typically takes the form

$$\Psi^\Omega := \{\psi^\Omega_{j,k} : j \geq j_0, k \in I_j\}, \tag{1.2}$$

where $|\text{supp } \psi^\Omega_{j,k}| \sim 2^{-j}$, but $\psi^\Omega_{j,k}$ may not result by scaling and translating mother wavelets, e.g. [4, 10, 13, 15]. Both $\Psi^\mathbb{R}$ and Ψ^Ω consist of infinitely many basis functions. Whereas $\Psi^\mathbb{R}$ consists of all dilates and translates, Ψ^Ω has a fixed minimal level j_0 (depending on Ω as well as the type of wavelets) and the location index $k \in I_j$ ranges over a finite index set I_j with cardinality $\#I_j \sim 2^j$.

If we can manage to design an adaptive wavelet method that is able to select appropriate subsets out of $\mathbb{Z} \times \mathbb{Z}$, then we can –in principle– use the same adaptive schemes as on Ω. This is precisely the path we follow in this paper. We introduce an adaptive selection procedure on unbounded domains and derive an asymptotically optimal adaptive wavelet method. This approach offers some interesting features:

- Though possible, the construction of wavelet bases on general domains Ω is technically challenging. Here, we completely circumvent the need of constructing a basis on a possibly complicated domain and use the most simple situation that is possible for wavelets, namely, the shift-invariant case.
- Adaptive methods are particularly favorable if the solution has local effects like a singularity of the derivative at a single point. Such effects can result from three different sources, namely the domain, the operator or the right-hand side. The first source does not appear for problems on $\mathbb{R}^n$. For the remaining two, certain a priori information is available. In fact, for example in the case $\mathcal{A} = -\Delta + \mathrm{I}$, the wavelet decomposition of the right-hand side f is already a good prediction for the relevant coefficients of the solution. Thus, this can be used as initial index set in order to improve the efficiency of the method.
- We do not need to truncate the domain, the scheme automatically detects the significant wavelets and determines a 'computational domain'. Thus, our method allows to solve a PDE problem on an unbounded domain by a compactly supported and locally refinable basis.
- This idea concerning the treatment of unbounded domains can be generalized to higher space dimensions [26], nonlocal operators or nonlinear problems [8].

Nevertheless, it is a priori not clear how actually the resolution of the asymptotic boundary conditions realized by adaptive wavelet schemes look like. As we have to take into account an infinite number of translation indices on each level (recall that in a bounded setting, this number is finite), the question arises how fast the asymptotic behavior of the best N-term approximation is reached by the algorithm.

The remainder of this paper is organized as follows. In Section 2, we review the main ingredients of adaptive wavelet methods. Section 3 contains the modification and extension to unbounded domains of the adaptive scheme from [18]. A second, heuristic adaptive scheme and a comparison of the two algorithms shall be described in Section 4. The extension to higher space dimensions is described in Section 5.

2. Adaptive Wavelet Methods

2.1. **Elliptic operator equations.** Let $H \subset L_2(\mathbb{R}^n)$ be a Hilbert space (e.g. $H^1(\mathbb{R}^n)$) and H' its dual w.r.t. $L_2(\mathbb{R}^n)$ (e.g. $H^{-1}(\mathbb{R}^n)$) where we denote by $\langle\cdot,\cdot\rangle$ the duality pairing in $H \times H'$. For a linear, self-adjoint operator $\mathcal{A} : H \to H'$ and a right-hand side $f \in H'$, we consider the operator equation: Find $u \in H$ such that

$$\mathcal{A}[u] = f \text{ in } H'. \tag{2.1}$$

We assume that the bilinear form $a(\cdot,\cdot) := \langle\,\cdot\,, \mathcal{A}[\,\cdot\,]\rangle : H \times H \to \mathbb{R}$ is symmetric, continuous and coercive, i.e., there exist constants $c_{\mathcal{A}}, C_{\mathcal{A}} > 0$ such that

$$c_{\mathcal{A}}\|v\|_H^2 \le a(v,v),\ \forall v \in H, \quad |a(w,v)| \le C_{\mathcal{A}}\|w\|_H\|v\|_H,\ \forall v,w \in H. \tag{2.2}$$

Throughout this section, we focus on the one-dimensional case $H = H^t(\mathbb{R})$ and $\mathcal{A}$ a differential operator of order $2t$ for $t \in \mathbb{N}_0$.

2.2. **Wavelets.** For the discretization of (2.1), we require a Riesz basis $\Psi_0 := \{\psi_\lambda : \lambda \in \mathcal{J}\}$ for $L_2(\mathbb{R})$ which, properly scaled by scaling factors $\mathbf{D}_\lambda^t$,

$$\Psi_t := \{\mathbf{D}_\lambda^t \psi_\lambda :\ \lambda \in \mathcal{J}\}, \tag{2.3}$$

is also a Riesz basis for $H^t(\mathbb{R})$, i.e., there exist constants $c_{\Psi_t}, C_{\Psi_t} > 0$ such that

$$c_{\Psi_t}\|\mathbf{d}\|_{\ell_2(\mathcal{J})} \le \|\mathbf{d}^T\Psi_t\|_{H^t(\mathbb{R})} \le C_{\Psi_t}\|\mathbf{d}\|_{\ell_2(\mathcal{J})}, \quad \forall \mathbf{d} \in \ell_2(\mathcal{J}). \tag{2.4}$$

Here, we denoted by $v := \mathbf{d}^T\Psi_t$ the expansion of a function $v \in H^t(\mathbb{R})$ in Ψ_t when viewing Ψ_t formally as an (infinite) column vector.

According to the remarks after (1.1), one can use the wavelet basis $\Psi_0^{\mathbb{R}} := \Psi^{\mathbb{R}}$ on $\mathbb{R}$ as long as $\gamma > t$ and rescale it to obtain a Riesz basis for $H^t(\mathbb{R})$,

$$\Psi_t^{\mathbb{R}} := \{\mathbf{D}_\lambda^t\, \psi_\lambda : \lambda \in \mathcal{J}^{\mathbb{R}}\}, \quad \mathbf{D}_\lambda^t := \|\psi_\lambda\|_{H^t(\mathbb{R})}^{-1}, \tag{2.5}$$

with $\lambda := (j,k)$, $\mathcal{J}^{\mathbb{R}} := \mathbb{Z} \times \mathbb{Z}$. In the sequel, we suppress the superscript $\mathbb{R}$ and set $\Psi_{t,-\infty} := \Psi_t^{\mathbb{R}}$, $\mathcal{J}_{-\infty} := \mathcal{J}^{\mathbb{R}}$ to underline that there is no lower bound for levels j.

As diam(supp ψ_λ) $\sim 2^{-|\lambda|}$, $|\lambda| := j$, grows exponentially with decreasing level, one can avoid arbitrarily coarse levels $j \to -\infty$ by fixing a minimal level $-\infty < j_0 < \infty$ and consider the collection

$$\Psi_{t,j_0} := \{\mathbf{D}_\lambda^t\psi_\lambda : \lambda \in \mathcal{J}_{j_0}\} := \Phi_{j_0} \cup \{\psi_{j,k}/\|\psi_{j,k}\|_{H^t(\mathbb{R})} :\ j \ge j_0, k \in \mathbb{Z}\}, \tag{2.6}$$

with $\mathcal{J}_{j_0} := \{j \ge j_0 - 1, k \in \mathbb{Z}\}$ and ($H^t(\mathbb{R})$-normalized) scaling functions (cf. [24])

$$\Phi_{j_0} := \{\varphi_{j_0,k}/\|\varphi_{j_0,k}\|_{H^t(\mathbb{R})} : k \in \mathbb{Z}\}, \quad \varphi_{j_0,k}(x) := 2^{j_0/2}\varphi(2^{j_0}x - k),\ x \in \mathbb{R},$$

where φ is a refineable function. To simplify notations, we set $\psi_{j_0-1,k} := \varphi_{j_0,k}$. With Φ_{j_0} being stable in $H^t(\mathbb{R})$, Ψ_{t,j_0} is also a Riesz basis for $H^t(\mathbb{R})$ (cf. e.g. [11, 23]). We shall assume that φ, ψ are locally supported as well as that Φ_{j_0} is exact of order d and that ψ has $\widetilde{d} \ge d$ vanishing moments. Both parameters depend on the particular choice of φ and ψ and also influence the values for γ, $\widetilde{\gamma}$. Particular instances of ψ are biorthogonal B-splines wavelets as constructed in [9]. Here, φ is a cardinal B-spline of order d and $\gamma = d - \frac{1}{2}$.

Remark 2.1. We can also use $\mathbf{D}_\lambda^t = \min\{1, 2^{-t|\lambda|}\} \sim \|\psi_\lambda\|_{H^t(\mathbb{R})}^{-1}$ in (2.5) (e.g. [3, Teorema 3.1]) since for $t \in \mathbb{N}_0$, $\|\psi_\lambda\|_{H^t(\mathbb{R})}^2 \sim \|\psi_\lambda\|_{L_2(\mathbb{R})}^2 + |\psi_\lambda|_{H^t(\mathbb{R})}^2$. Then, by a homogeneity argument, we have $\|\psi_\lambda\|_{L_2(\mathbb{R})} \sim \|\psi\|_{L_2(\mathbb{R})}$ as well as $|\psi_\lambda|_{H^t(\mathbb{R})} \sim 2^{t|\lambda|}$. Observe that in bounded domain settings (where $j_0 \ge 0$), the scaling factors simplify to $\mathbf{D}_\lambda^t = 2^{-t|\lambda|}$ which is often used in the literature.

2.3. **Wavelet discretization.** Now we use a wavelet basis $\Psi_t \in \{\Psi_{t,j_0} : j_0 \geq -\infty\}$ (where $j_0 = -\infty$ refers to (2.5)) to transform (2.1) into a well-conditioned discrete operator equation. By (2.4), we infer that there exists a unique $\mathbf{u} \in \ell_2(\mathcal{J})$ with $u = \mathbf{u}^T \Psi_t$ for the solution u of (2.1). This means that $\mathbf{u}$ is the (unknown) sequence of wavelet coefficients of u. Thus, (2.1) is equivalent to the infinite linear system

$$\mathbf{A}\mathbf{u} = \mathbf{f}, \quad \text{with } \mathbf{A} := \langle \Psi_t, \mathcal{A}[\Psi_t] \rangle,\ \mathbf{f} := \langle \Psi_t, f \rangle. \tag{2.7}$$

Note, that (2.7) is well-posed on $\ell_2(\mathcal{J})$: by (2.2) and (2.4), the symmetric bilinear form $\mathbf{a}(\mathbf{v}, \mathbf{v}) := \langle \mathbf{v}, \mathbf{A}\mathbf{v} \rangle_{\ell_2} = a(\mathbf{v}^T \Psi_t, \mathbf{v}^T \Psi_t)$ satisfies for $c_1 := c_{\Psi_t}^2 c_{\mathcal{A}}$, $c_2 := C_{\Psi_t}^2 C_{\mathcal{A}}$

$$c_1 \|\mathbf{v}\|_{\ell_2}^2 \leq \mathbf{a}(\mathbf{v}, \mathbf{v}) \leq c_2 \|\mathbf{v}\|_{\ell_2}^2, \quad \forall \mathbf{v} \in \ell_2(\mathcal{J}). \tag{2.8}$$

Therefore, $\mathbf{a}(\,\cdot\,, \cdot\,)$ is coercive and, by an analogous reasoning using (2.2), also continuous. For this reason, the operator $\mathbf{A} : \ell_2(\mathcal{J}) \to \ell_2(\mathcal{J})$ is symmetric, continuous and coercive. Moreover, (2.8) implies that $\mathbf{A}$ is boundedly invertible with

$$\|\mathbf{A}\| := \sup_{\mathbf{v} \in \ell_2(\mathcal{J})} \frac{\|\mathbf{A}\mathbf{v}\|_{\ell_2}}{\|\mathbf{v}\|_{\ell_2}} \leq c_2, \qquad \|\mathbf{A}^{-1}\| := \sup_{\mathbf{v} \in \ell_2(\mathcal{J})} \frac{\|\mathbf{A}^{-1}\mathbf{v}\|_{\ell_2}}{\|\mathbf{v}\|_{\ell_2}} \leq c_1^{-1}.$$

The condition number $\kappa(\mathbf{A}) = \frac{c_2}{c_1}$ of $\mathbf{A}$ is bounded which is in fact a crucial property for the numerical treatment. Setting $\|\mathbf{v}\|_{\mathbf{a}} := \mathbf{a}(\mathbf{v}, \mathbf{v})$ for $\mathbf{v} \in \ell_2(\mathcal{J})$, we see that the energy norm $\|\cdot\|_{\mathbf{a}}$ is equivalent to $\|\cdot\|_{\ell_2}$, i.e., $\|\mathbf{v}\|_{\mathbf{a}} \sim \|\mathbf{v}\|_{\ell_2}$. We can also define another equivalent norm for $\mathbf{v} \in \ell_2(\mathcal{J})$ by $\|\mathbf{v}\|_{\mathbf{A}} := \|\mathbf{A}\mathbf{v}\|_{\ell_2} \sim \|\mathbf{v}\|_{\ell_2}$. To avoid the use of various constants, we write $C \lesssim D$ if there exists a constant $c > 0$ such that $C \leq cD$. Analogously, we define $\gtrsim$. We use $C \sim D$ if $C \lesssim D$ and $C \gtrsim D$.

In the sequel, we shall need the restriction of infinite matrices and infinite vectors to finite index sets $\Lambda \subset \mathcal{J}$. To this end, we define for $\mathbf{v} \in \ell_2(\mathcal{J})$ the projection $\mathbf{P}_\Lambda \mathbf{v} := \mathbf{v}|_\Lambda$ and set $\mathbf{v}_\Lambda := \mathbf{P}_\Lambda \mathbf{v}$. By $\mathbf{I}_\Lambda : \ell_2(\Lambda) \to \ell_2(\mathcal{J})$, we denote the extension of a vector $\mathbf{v} \in \ell_2(\Lambda)$ by zeros. Thus, we obtain the finite Galerkin system

$$\mathbf{A}_\Lambda \mathbf{u}_\Lambda = \mathbf{f}_\Lambda, \quad \text{with } \mathbf{A}_\Lambda := \mathbf{P}_\Lambda \mathbf{A} \mathbf{I}_\Lambda,\ \mathbf{f}_\Lambda := \mathbf{P}_\Lambda \mathbf{f}. \tag{2.9}$$

One possible interpretation of many adaptive schemes is to find a sequence of index sets $\Lambda^{(0)}, \Lambda^{(1)}, \Lambda^{(2)}, \ldots$ so that the corresponding Galerkin solutions $\mathbf{u}_{\Lambda^{(k)}}$ of (2.9) converge possibly fast towards $\mathbf{u}$ with as few active wavelet coefficients as possible.

2.4. **Nonlinear approximation theory.** The analysis of adaptive schemes leads to nonlinear approximation theory. Since we want to approximate the unknown solution $\mathbf{u}$ with as few wavelet coefficients as possible, the optimum would be a *best N-term approximation* $\mathbf{u}_N$ of $\mathbf{u}$ where $\#\operatorname{supp} \mathbf{u}_N = N$, $\|\mathbf{u} - \mathbf{u}_N\|_{\ell_2} = \sigma_N(\mathbf{u})$ and

$$\sigma_N(\mathbf{u}) := \inf_{\mathbf{w} \in \Sigma_N} \|\mathbf{u} - \mathbf{w}\|_{\ell_2}. \tag{2.10}$$

Here, $\Sigma_N := \{\mathbf{v} \in \ell_2(\mathcal{J}) : \#\operatorname{supp} \mathbf{v} \leq N\}$ is a nonlinear manifold in $\ell_2(\mathcal{J})$.

In order to define (quasi-)optimality, we collect all sequences whose best N-term approximation converges with rate $s > 0$ in the *approximation class*

$$\mathcal{A}^s := \{\mathbf{v} \in \ell_2 : \sigma_N(\mathbf{v}) \lesssim N^{-s}\}, \quad s > 0. \tag{2.11}$$

A (quasi-)norm on $\mathcal{A}^s$ is given by $|\mathbf{v}|_{\mathcal{A}^s} := \sup_{N \geq 0} (N+1)^s \|\mathbf{v} - \mathbf{v}_N\|_{\ell_2}$, where $\mathbf{v}_N$ is a best N-term approximation of $\mathbf{v}$. In other words, given $\mathbf{u} \in \mathcal{A}^s$ for $s > 0$ and $\varepsilon > 0$, setting $N := \lceil \varepsilon^{-1/s} |\mathbf{u}|_{\mathcal{A}^s}^{1/s} \rceil$ yields $\|\mathbf{u} - \mathbf{u}_N\|_{\ell_2} \leq |\mathbf{u}|_{\mathcal{A}^s} N^{-s} \leq \varepsilon$ (cf. [30, Eq. (3)]). This shows that for any $\varepsilon > 0$, $\mathbf{u}$ can be approximated by $\mathbf{u}(\varepsilon)$ s.t.

$$\|\mathbf{u} - \mathbf{u}(\varepsilon)\|_{\ell_2} \leq \varepsilon, \quad \#\operatorname{supp} \mathbf{u}(\varepsilon) \lesssim \varepsilon^{-1/s} |\mathbf{u}|_{\mathcal{A}^s}^{1/s}. \tag{2.12}$$

This sets the benchmark and we call an adaptive wavelet method *(quasi-)optimal* if for $u = \mathbf{u}^T\Psi_t$ with $\mathbf{u} \in \mathcal{A}^s$ and $\varepsilon > 0$, the scheme produces an output $\mathbf{u}(\varepsilon)$ satisfying (2.12) with linear complexity in arithmetic operations and storage requirements.

The next question is of course, under which conditions on u or $\mathbf{u}$, one actually has $\mathbf{u} \in \mathcal{A}^s$. As far as the sequence $\mathbf{u}$ of the wavelet coefficients is concerned, it is well-known that certain decay rates are needed in order to ensure a certain rate of approximation. This decay is expressed by the so called weak $\ell_\tau(\mathcal{J})$-spaces defined as follows (cf. [14]). For each $0 < \tau < 2$ and $\mathbf{v} \in \ell_2(\mathcal{J})$, we define $|\mathbf{v}|_{\ell_\tau^{\mathrm{w}}} := \sup_{k\geq 1} k^{1/\tau} v_k^*$, where v_k^* is the k-th largest entry in modulus of $\mathbf{v}$ and $\mathbf{v}^* := (v_k^*)_{k\in\mathbb{N}}$. Then, we set $\ell_\tau^{\mathrm{w}}(\mathcal{J}) := \{\mathbf{v} \in \ell_2(\mathcal{J}) : |\mathbf{v}|_{\ell_\tau^{\mathrm{w}}} < \infty\}$ with the corresponding norm $\|\mathbf{v}\|_{\ell_\tau^{\mathrm{w}}} := |\mathbf{v}|_{\ell_\tau^{\mathrm{w}}} + \|\mathbf{v}\|_{\ell_2}$, $\mathbf{v} \in \ell_\tau^{\mathrm{w}}$. It is known that $\ell_\tau \hookrightarrow \ell_\tau^{\mathrm{w}} \hookrightarrow \ell_{\tau+\delta}$ for any $\delta \in (0, 2-\tau]$ explaining the notion 'weak'-ℓ_τ. With this notation at hand, $\sigma_N(\mathbf{u})$ decays with a fixed rate $s > 0$ if and only if $\mathbf{u} \in \ell_\tau^{\mathrm{w}}(\mathcal{J})$ for

$$\frac{1}{\tau} = s + \frac{1}{2}. \tag{2.13}$$

In particular, $\|\cdot\|_{\mathcal{A}^s} \sim \|\cdot\|_{\ell_\tau^{\mathrm{w}}(\mathcal{J})}$ and it holds for all $\mathbf{v} \in \ell_\tau^{\mathrm{w}}(\mathcal{J})$ that

$$\sigma_N(\mathbf{v}) \leq C_\tau \|\mathbf{v}\|_{\ell_\tau^{\mathrm{w}}} N^{-s}, \tag{2.14}$$

where the constant $C_\tau > 0$ depends only on τ (cf. [6, Proposition 3.2]). On the basis of (2.14), we refer to the largest value of s for which $\mathbf{u} \in \ell_\tau^{\mathrm{w}}$ (with τ defined in (2.13)) as the *best nonlinear approximation rate*. It turns out that this rate is related to the Besov regularity of the underlying function and the polynomial order d of the wavelets (cf. [5, 14]). More precisely, if $s < d - t$ and $u \in H^t(\mathbb{R}) \cap B_\tau^{t+s}(L_\tau(\mathbb{R}))$ with $\tau = (s + \frac{1}{2})^{-1}$, then $\mathbf{u} \in \mathcal{A}^s$, respectively $\mathbf{u} \in \ell_\tau^{\mathrm{w}}(\mathcal{J})$.

2.5. **Optimality and locality.** In order to obtain a best possible method, it is not enough to generate a scheme which converges with the same rate as a best N-term approximation. In fact, we also need to be able to compute such an approximation within linear complexity. One key ingredient is that wavelets allow for a compression of a large class of operators due to their locality and their vanishing moments.

Definition 2.2 ([7, Definition 5.8] & [19, Definition 1.1])**.** Let $s^* > 0$. An operator $\mathbf{A} : \ell_2(\mathcal{J}) \to \ell_2(\mathcal{J})$ is said to be in the class $\mathcal{B}_{s^*}$ (or s^*-*compressible*) if for each $0 < s < s^*$ and for some positive, summable sequence $(\alpha_j)_{j\geq 0}$, there exists for each $j \in \mathbb{N}_0$ a matrix $\mathbf{A}_j$ with at most $\mathcal{O}(2^j\alpha_j)$ nonzero entries per row and column s.t.

$$\|\mathbf{A} - \mathbf{A}_j\| \leq \alpha_j 2^{-js}. \tag{2.15}$$

Moreover, an operator $\mathbf{A} \in \mathcal{B}_{s^*}$ is called s^*-*computable* if each column in $\mathbf{A}_j$ can be computed within $\mathcal{O}(2^j)$ operations.

In the remainder of this section, we shall assume that there exists $s^* > 0$ such that $\mathbf{A}$ is s^*-*computable*. Compression estimates which fit into the setting of (2.15) have been discussed in detail for different types of operators (cf., e.g., [19, 26, 29]). We consider such estimates for operators and wavelet bases on unbounded domains later in Section 3.1. The compressibility of $\mathbf{A}$ can be used for the design of efficient algorithms as we shall review now. If we define $\mathbf{v}_{[j]}$ as a best 2^j–term approximation to $\mathbf{v} \in \ell_\tau^{\mathrm{w}}(\mathcal{J})$ (e.g. the first 2^j entries of $\mathbf{v}^*$), then it holds

$$\|\mathbf{v} - \mathbf{v}_{[j]}\|_{\ell_2} \leq 2^{-js}\, |\mathbf{v}|_{\mathcal{A}^s}, \tag{2.16}$$

if s is chosen as in (2.13). It can be shown that $\mathbf{A} \in \mathcal{B}_{s^*}$ is a bounded operator on $\ell^{\mathrm{w}}_\tau(\mathcal{J})$ with $\tau = (s+\frac{1}{2})^{-1}$ when $s < s^*$ (cf. [7, Proposition 5.9]) and also derive a method for approximating an infinite matrix-vector product $\mathbf{Av}$ (cf. [6, 7]).

2.6. **An optimal adaptive wavelet algorithm.** Now, we describe the adaptive wavelet scheme **ADWAV** from [18] which we use as a basis for our extension to unbounded domains. The core scheme is shown in Algorithm 1. We start with an initial error estimator $\nu_{-1} \sim \|\mathbf{f}\|_{\ell_2}$ and a desired tolerance $\varepsilon > 0$. Finally, we need to choose constants $\alpha, \gamma, \theta, \omega$ such that:

- $0 < \omega < \alpha < 1$ such that $\frac{\alpha+\omega}{1-\omega} < \kappa(\mathbf{A})^{-\frac{1}{2}}$,
- $0 < \gamma < \frac{1}{6}\kappa(\mathbf{A})^{-1/2}\frac{\alpha-\omega}{1+\omega}$ and $\theta > 0$.

Algorithm 1 $[\mathbf{u}(\varepsilon), \Lambda(\varepsilon)] = \mathbf{ADWAV}[\nu_{-1}, \varepsilon]$

1: $\Lambda^{(0)} = \emptyset$, $k := 0$, $\mathbf{w}^{(0)} := \mathbf{0}$
2: **while** with $[\Lambda^{(k+1)}, \nu_k] = \mathbf{GROW}[\mathbf{w}^{(k)}, \theta\nu_{k-1}, \varepsilon]$, $\nu_k > \varepsilon$ **do**
3: $\quad \mathbf{g}^{(k+1)} = \mathbf{P}_{\Lambda^{(k+1)}}(\mathbf{RHS}[\gamma\nu_k])$
4: $\quad \mathbf{w}^{(k+1)} = \mathbf{GALSOLVE}[\Lambda^{(k+1)}, \mathbf{g}^{(k+1)}, \mathbf{w}^{(k)}, (1+\gamma)\nu_k, \gamma\nu_k]$; $k = k+1$
5: **end do**
6: $\mathbf{u}(\varepsilon) = \mathbf{w}^{(k)}$, $\Lambda(\varepsilon) = \Lambda^{(k)}$

Before we detail the subroutines, let us recall the properties of **ADWAV**.

Theorem 2.3 ([18, Theorem 2.7] & [16, Theorem 5.5.1])**.** *The output* $\mathbf{w} = \mathbf{u}(\varepsilon)$ *of the routine* $\mathbf{ADWAV}[\nu_{-1}, \varepsilon]$ *satisfies* $\|\mathbf{Aw}-\mathbf{f}\|_{\ell_2} \le \varepsilon$. *If we assume that* $\mathbf{u} \in \ell^{\mathrm{w}}_\tau(\mathcal{J})$ *for some* $s < s^*$ ($\frac{1}{\tau} = s + \frac{1}{2}$), $\nu_{-1} \sim \|\mathbf{f}\|_{\ell_2} \gtrsim \varepsilon$, *and that* $\mathbf{g}_\eta$ *can be computed s.t.*

$$\|\mathbf{f} - \mathbf{g}_\eta\|_{\ell_2} \le \eta, \quad \#\mathrm{supp}\, \mathbf{g}_\eta \le C_s\, \eta^{-1/s}, \quad 0 < \eta \le \|\mathbf{f}\|_{\ell_2}, \tag{2.17}$$

within $\mathcal{O}(\eta^{-1/s})$ *operations for some constant* $C_s > 0$, *then* $\#\,\mathrm{supp}\,\mathbf{w} \lesssim \varepsilon^{-1/s}(|\mathbf{u}|^{1/s}_{\ell^{\mathrm{w}}_\tau})$ *and the number of arithmetic operations and storage locations is bounded by* $\mathcal{O}(\varepsilon^{-1/s}(|\mathbf{u}|^{1/s}_{\ell^{\mathrm{w}}_\tau} + C_s))$. □

The routine **GROW** (cf. Algorithm 2) computes $\Lambda^{(k+1)} \supset \Lambda^{(k)}$ such that

$$\|\mathbf{P}_{\Lambda^{(k+1)}}(\mathbf{Au}_{\Lambda^{(k)}} - \mathbf{f})\|_{\ell_2} \ge \beta\|\mathbf{Au}_{\Lambda^{(k)}} - \mathbf{f}\|_{\ell_2}, \tag{2.18}$$

for some $0 < \beta < 1$ (this is sometimes also called *saturation property*). Then, due to Galerkin orthogonality (cf. [6, Lemma 4.1]), one has the error reduction $\|\mathbf{u} - \mathbf{u}_{\Lambda^{(k+1)}}\|_{\mathbf{a}} \le (1 - \frac{c_1}{c_2}\beta^2)^{1/2}\, \|\mathbf{u} - \mathbf{u}_{\Lambda^{(k)}}\|_{\mathbf{a}}$ with the constants c_1, c_2 from (2.8).

Algorithm 2 $\mathbf{GROW}[\mathbf{w}, \bar{\nu}, \varepsilon] \to [\Lambda, \nu]$

1: Define $\zeta := 2\frac{\omega\bar{\nu}}{1-\omega}$.
2: **do** $\zeta := \zeta/2$, $\mathbf{r} :=\mathbf{RHS}[\zeta/2]-\mathbf{APPLY}[\mathbf{w}, \zeta/2]$.
3: **until** $\nu := \|\mathbf{r}\|_{\ell_2} + \zeta \le \varepsilon$ or $\zeta \le \omega\|\mathbf{r}\|_{\ell_2}$.
4: **if** $\nu > \varepsilon$ **then** determine a minimal set $\Lambda \supset \mathrm{supp}\, \mathbf{w}$ s.t. $\|\mathbf{P}_\Lambda \mathbf{r}\|_{\ell_2} \ge \alpha\|\mathbf{r}\|_{\ell_2}$.
5: **else** set $\Lambda := \emptyset$.
6: **end if**

Under the same assumptions as in Theorem 2.3 and if $\mathbf{w} \in \ell^{\mathrm{w}}_\tau(\mathcal{J})$, the number of operations and storage locations required by $[\Lambda, \nu] = \mathbf{GROW}[\mathbf{w}, \bar{\nu}, \varepsilon]$ is bounded by some absolute multiple of $\min\{\bar{\nu},\nu\}^{-1/s}[|\mathbf{w}|_{\ell_\tau^w}^{1/s} + |\mathbf{u}|_{\ell_\tau^w}^{1/s} + \bar{\nu}^{1/s}(\#\operatorname{supp}\mathbf{w}+1)]$. Moreover, we have $\nu \geq \|\mathbf{A}\mathbf{w}-\mathbf{f}\|_{\ell_2}$ and, if $\nu > \varepsilon$, the saturation property

$$\frac{\alpha-\omega}{1+\omega}\nu \leq \|\mathbf{P}_\Lambda(\mathbf{A}\mathbf{w}-\mathbf{f})\|_{\ell_2}, \qquad \#(\Lambda\backslash\operatorname{supp}\mathbf{w}) \lesssim \nu^{-1/s}|\mathbf{u}|_{\ell_\tau^w}^{1/s}, \tag{2.19}$$

holds with the constants α and ω described above.

The routine $\mathbf{RHS}[\eta]$ produces an approximation $\mathbf{g}_\eta$ to $\mathbf{f}$ such that $\|\mathbf{f}-\mathbf{g}_\eta\|_{\ell_2} \leq \eta$. To preserve the linear complexity of **ADWAV**, the length of $\mathbf{g}_\eta$ as well as the corresponding computational cost are assumed to be of order $\mathcal{O}(\eta^{-1/s})$. We shall detail a concrete version of **RHS** in Section 3.

For a given approximation $\mathbf{g}_\Lambda$ of $\mathbf{f}_\Lambda$, **GALSOLVE** computes $\widetilde{\mathbf{w}}_\Lambda$ with $\|\mathbf{A}_\Lambda\widetilde{\mathbf{w}}_\Lambda - \mathbf{f}_\Lambda\| \leq \eta$ starting with an initial guess $\mathbf{w}_\Lambda$ satisfying $\|\mathbf{A}_\Lambda\mathbf{w}_\Lambda - \mathbf{f}_\Lambda\| \leq \delta$.

Algorithm 3 $\mathbf{GALSOLVE}[\Lambda,\mathbf{g}_\Lambda,\mathbf{w}_\Lambda,\delta,\eta] \to [\widetilde{\mathbf{w}}_\Lambda]$

Determine $\mathbf{A}_J$ in the sense of (2.15) with minimal $J = J(\eta)$ s.t. $\|\mathbf{A}-\mathbf{A}_J\| \leq \frac{\eta}{3}$.
Assemble $\mathbf{B} := \mathbf{P}_\Lambda[\frac{1}{2}(\mathbf{A}_J + \mathbf{A}_J^*)]\mathbf{I}_\Lambda$ with $\mathbf{A}_J^*$ being the adjoint of $\mathbf{A}_J$.
Compute $\mathbf{r}_0 := \mathbf{g}_\Lambda - \mathbf{P}_\Lambda(\mathbf{APPLY}[\mathbf{w}_\Lambda, \frac{\eta}{3}])$.
Find $\mathbf{x}$ with $\|\mathbf{B}\mathbf{x}-\mathbf{r}_0\|_{\ell_2} \leq \frac{\eta}{3}$ by, e.g., conjugate gradients and set $\widetilde{\mathbf{w}}_\Lambda = \mathbf{w}_\Lambda + \mathbf{x}$.

One key ingredient both in **GROW** and **GALSOLVE** is the routine **APPLY** shown in Algorithm 4 which is an adaptive approximate application of the biinfinite operator $\mathbf{A}$ to a given compactly supported input $\mathbf{v}$ with the following properties. The output $\mathbf{w} = \mathbf{APPLY}[\mathbf{v},\eta]$ satisfies $\|\mathbf{A}\mathbf{v}-\mathbf{w}\|_{\ell_2} \leq \eta$ as well as $\operatorname{supp}\mathbf{w} \lesssim \|\mathbf{v}\|_{\ell_\tau^w}^{1/s}\eta^{-\frac{1}{s}}$ (cf. [6, Properties 6.4]). We remark that necessary sorting operations in **GROW** and **APPLY** which are not of linear complexity can be replaced by approximative sorting procedures introduced in [1, 15].

Algorithm 4 $\mathbf{APPLY}[\mathbf{v},\eta] \to \mathbf{w}$

Set $N := \#\operatorname{supp}\mathbf{v}$ and $k(\eta)$ as the smallest integer such that $2^{k(\eta)} \geq \eta^{-\frac{1}{s}}\|\mathbf{v}\|_{\ell_\tau^w}^{\frac{1}{s}}$.
Compute $\mathbf{v}_{[0]}, \mathbf{v}_{[i]} - \mathbf{v}_{[i-1]}$ for $i = 1,\ldots,\lfloor\log N\rfloor$ and set $\mathbf{v}_{[i]} := \mathbf{v}$ for $i > \log N$.
for $k = 1$ **to** $k(\eta)$ **do**
  $R_k := c_2\{\|\mathbf{v}-\mathbf{v}_{[k]}\|_{\ell_2} + \alpha_k 2^{-ks}\|\mathbf{v}_{[0]}\|_{\ell_2} + \sum_{\ell=0}^{k-1}\alpha_\ell 2^{-\ell s}\|\mathbf{v}_{[k-\ell]} - \mathbf{v}_{[k-\ell-1]}\|_{\ell_2}\}$
  if $R_k \leq \eta$ **exit**
end for
Compute $\mathbf{w} := \mathbf{w}_k := \mathbf{A}_k\mathbf{v}_{[0]} + \mathbf{A}_{k-1}(\mathbf{v}_{[1]} - \mathbf{v}_{[0]}) + \cdots + \mathbf{A}_0(\mathbf{v}_{[k]} - \mathbf{v}_{[k-1]})$.

Remark 2.4. Note that the actual rate s of the best nonlinear approximation is often not known in a given situation. However, it is bounded by $d-t$, see Section 2.4. This means for applying **ADWAV**, we have to require $s^* > d-t$ in order to ensure optimality.

Remark 2.5. The *symmetry* of $\mathbf{A}$ (which is induced by the fact that $\mathcal{A}$ is self-adjoint) is not a necessary condition for **ADWAV** (cf. [18, p.617]). In fact, if $\mathbf{A}$ is not symmetric, one can apply **ADWAV** to the *normal equations* $\mathbf{A}^T\mathbf{A}\mathbf{u} = \mathbf{A}^T\mathbf{f}$. Here, for simplicity, we only consider the symmetric case and refer to [7, 18] for more details on the non-symmetric case.

3. An optimal adaptive wavelet algorithm on unbounded domains

Having reviewed all ingredients of the scheme **ADWAV**, we can now identify the modifications that are necessary in order to treat problems on unbounded domains. In particular, we have to verify that $\mathbf{A} \in \mathcal{B}_{s^*}$ for $s^* > d - t$ and that a realization of **RHS** for both basis types $\Psi_{t,-\infty}$ and Ψ_{t,j_0} ($j_0 > -\infty$) is available. Note that **GROW**, **GALSOLVE** and **APPLY** do not have to be modified. Optimality and convergence of the scheme on unbounded domains follow directly from Theorem 2.3. For the ease of presentation, we collect all proofs of this section in Appendix A.

3.1. **Compressibility of differential operators on unbounded domains.** We assume that $\mathcal{A} : H^t(\mathbb{R}) \to H^{-t}(\mathbb{R})$ for $t \in \mathbb{N}_0$ can be written in the form

$$\langle v, \mathcal{A}[w] \rangle = \textstyle\sum_{\alpha \le t} \int_{\mathbb{R}} g_\alpha(x)\, \partial^\alpha v(x)\, \partial^\alpha w(x)\, \mathrm{d}x, \quad v, w \in H^t(\mathbb{R}),$$

with sufficiently smooth coefficients $g_\alpha \in L_\infty(\mathbb{R})$ for $0 \le \alpha \le t$ and $1/g_\alpha > 0$ for $\alpha \in \{0, t\}$ (which implies that $|\langle v, \mathcal{A}[v] \rangle| \sim \|v\|^2_{H^t(\mathbb{R})}$ for all $v \in H^t(\mathbb{R})$). The wavelet representation of $\mathcal{A}$ is then given by $\mathbf{A} := (a_{\lambda,\lambda'})_{\lambda,\lambda' \in \mathcal{J}}$ (cf. (2.7)) with

$$a_{\lambda,\lambda'} := \textstyle\sum_{\alpha \le t} a^{(\alpha)}_{\lambda,\lambda'}, \quad a^{(\alpha)}_{\lambda,\lambda'} := \mathbf{D}^t_\lambda\, \mathbf{D}^t_{\lambda'} \int_{\mathbb{R}} g_\alpha(x)\, \partial^\alpha \psi_\lambda(x)\, \partial^\alpha \psi_{\lambda'}(x)\, \mathrm{d}x. \tag{3.1}$$

To show s^*-compressibility of $\mathbf{A}$ for $s^* > d - t$ (cf. Remark 2.4), we use results from [15, 26] that were proven for bounded domains. To this end, we review the differences between bounded and unbounded domain settings that we have to take into account.

First, we note that infinitely many translation indices per level do not cause problems. Since $\mathcal{A}$ is a local operator and ψ_λ has local support, $a_{\lambda,\lambda'} = 0$ when $|\mathrm{supp}\, \psi_\lambda \cap \mathrm{supp}\, \psi_{\lambda'}| = 0$. Hence, in analogy to bounded domains, we have for the number of nonzeros $\# N(\lambda, \ell')$ of $\mathbf{A}$ for row $\lambda \in \mathcal{J}$ and for a column level ℓ' that

$$\# N(\lambda, \ell') \lesssim 2^{\max\{0, \ell' - |\lambda|\}}, \quad N(\lambda, \ell') := \{\lambda' \in \mathcal{J} : |\lambda'| = \ell',\, a_{\lambda,\lambda'} \ne 0\}.$$

Secondly, for $j_0 < 0$, we also have to treat negative levels. In particular, we can split $\mathbf{A}$ into four blocks w.r.t to the sign of the levels of index pairs (λ, λ')

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{+-} & \mathbf{A}_{++} \\ \mathbf{A}_{--} & \mathbf{A}_{-+} \end{pmatrix}, \tag{3.2}$$

where, e.g., $\mathbf{A}_{+-} := (a_{\lambda,\lambda'})_{|\lambda| \ge 0, |\lambda'| < 0}$. We shall see below that required estimates of the matrix entries $a^{(\alpha)}_{\lambda,\lambda'}$ may depend on the block.

To derive such estimates, as it was pointed out in [26], one replaces the scaling factors $\mathbf{D}^t_\lambda$ by $2^{-\alpha|\lambda|}$ and analyzes the compressibility of $\mathbf{B}^{(\alpha)} := (b^{(\alpha)}_{\lambda,\lambda'})_{\lambda,\lambda' \in \mathcal{J}}$ by deriving estimates for the matrix entries

$$b^{(\alpha)}_{\lambda,\lambda'} := 2^{-\alpha|\lambda|}\, 2^{-\alpha|\lambda'|} \int_{\mathbb{R}} g_\alpha(x)\, \partial^\alpha \psi_\lambda(x)\, \partial^\alpha \psi_{\lambda'}(x)\, \mathrm{d}x, \quad \forall \lambda, \lambda' \in \mathcal{J}, \tag{3.3}$$

for $0 \le \alpha \le t$. Since $|\mathbf{D}^t_\lambda| \lesssim 2^{-\alpha|\lambda|}$ for all $\lambda \in \mathcal{J}$ (cf. Remark 2.1), one can then deduce the compressibility of $\mathbf{A}$ from the compressibility of $\mathbf{B}^{(\alpha)}$. Here, we will have to distinguish between constant coefficients, i.e., $g_\alpha \equiv c_\alpha \in \mathbb{R}$ ($0 \le \alpha \le t$) and non-constant coefficients.

Before we go further, we need to review some basic assumptions on the wavelets ψ_λ where we follow the lines of [26, Section 3].

3.1.1. *Wavelet assumptions.* As already stated above, a key requirement is the local support of ψ_λ, $\operatorname{diam}(\Box_\lambda) \sim 2^{-|\lambda|}$, $\Box_\lambda := \operatorname{supp}\psi_\lambda$. Moreover, we assume that wavelets are piecewise polynomials of order d, i.e., there exist $\kappa \in \mathbb{N}$ disjoint, open subdomains $\Xi_{\lambda,1}, \ldots, \Xi_{\lambda,\kappa}$ with $\operatorname{diam}(\Xi_{\lambda,i}) \sim 2^{-|\lambda|}$ such that

$$\operatorname{supp}\psi_\lambda = \cup_{i=1}^{\kappa} \overline{\Xi}_{\lambda,i} \text{ and } \psi_\lambda|_{\Xi_{\lambda,i}} \in \mathcal{P}_{d-1}, \quad \forall \lambda \in \mathcal{J}, \tag{3.4}$$

$\mathcal{P}_p$ denoting the *polynomials of degree up to* $p \in \mathbb{N}_0$. Moreover, we suppose that

$$\psi_\lambda \in C^r(\mathbb{R}), \quad r := d-2. \tag{3.5}$$

The singular support of ψ_λ is then given by $\operatorname{singsupp}\psi_\lambda = \cup_{i=1}^{\kappa} \partial\,\Xi_{\lambda,i}$. Additionally, we assume *vanishing moments* of order $\widetilde{d} \geq d$, i.e., $(\psi_\lambda, p)_{L_2(\mathbb{R})} = 0$ for all $p \in \mathcal{P}_{\widetilde{d}-1}$. Since $|\langle \psi_\lambda, f\rangle| \leq \|\psi_\lambda\|_{L_1} \inf_{p\in\mathcal{P}_{\widetilde{d}-1}} \|f-p\|_{L_\infty(\Box_\lambda)}$ and $\|\psi_\lambda\|_{L_1} \leq 2^{-\frac{1}{2}|\lambda|}\|\psi\|_{L_1}$, this property yields by a Whitney type estimate ([5, Theorem 25.2]) for $f \in W^{\tau,\infty}(\mathbb{R})$:

$$|\langle \psi_\lambda, f\rangle| \leq C_{\psi,f}\, 2^{-(\tau+\frac{1}{2})|\lambda|}, \quad \forall \lambda \in \mathcal{J}, \quad \tau \in [0, \widetilde{d}], \tag{3.6}$$

where $C_{\psi,f} := C_W |\operatorname{supp}\psi|^{\tau+\frac{1}{2}} \|\psi\|_{L_1} |f|_{W^{\tau,\infty}(\mathbb{R})}$ and $C_W > 0$ is a constant arising from the Whitney estimate. Finally, we require that (cf. [26, Eqs. (3.2) & (3.3)])

$$|\psi_\lambda|_{W^{\tau,\infty}(\mathbb{R})} \lesssim 2^{(\frac{1}{2}+\tau)|\lambda|}, \qquad \forall \lambda \in \mathcal{J},\ \tau \in [0, d-1], \tag{3.7}$$

$$|\psi_\lambda|_{W^{\tau,\infty}(\Xi_{\lambda,i})} \lesssim 2^{(\frac{1}{2}+\tau)|\lambda|}, \qquad \forall \lambda \in \mathcal{J},\ \tau \geq 0,\ i \in \{1, \ldots, \kappa\}, \tag{3.8}$$

as well as that $\|\psi_\lambda\|_{L_2(\mathbb{R})} = 1$ for all $\lambda \in \mathcal{J}$. Note that all requirements are met e.g. by the wavelets constructed in [9] and are by no means a restriction.

3.1.2. *Compressibility.* In order to exploit the fact that wavelets ψ_λ are piecewise polynomials on subdomains $\Xi_{\lambda,i}$ and have vanishing moments of order $\widetilde{d}$, we define for the case $\Box_{\lambda,\lambda'} := \operatorname{supp}\psi_\lambda \cap \operatorname{supp}\psi_{\lambda'} \neq \emptyset$,

$$i(\lambda,\lambda') := 0 \text{ when } \begin{cases} \operatorname{dist}(\operatorname{singsupp}\psi_\lambda, \Box_{\lambda'}) > 0, \text{ or} \\ \operatorname{dist}(\operatorname{singsupp}\psi_{\lambda'}, \Box_\lambda) > 0, \end{cases}$$

and $i(\lambda,\lambda') := 1$ otherwise (cf. [26, p. 82]).

Constant coefficients. Let now $g_\alpha \equiv c_\alpha \in \mathbb{R}$ for all $\alpha \leq t$ and $\Psi_t \in \{\Psi_{t,j_0} : j_0 \geq -\infty\}$. In this case, the term $b^{(\alpha)}_{\lambda,\lambda'}$ in (3.3) simplifies to

$$b^{(\alpha)}_{\lambda,\lambda'} = 2^{-\alpha|\lambda|}\, 2^{-\alpha|\lambda'|} \int_{\mathbb{R}} c_\alpha\, \partial^\alpha\psi_\lambda(y)\, \partial^\alpha\psi_{\lambda'}(y)\,\mathrm{d}x, \quad \lambda, \lambda' \in \mathcal{J}.$$

Due to the fact that g_α is *constant*, we can proceed as in [15, Section 3] using (3.5) and (3.7). Since $\mathbf{D}^t_\lambda \lesssim 2^{-\alpha|\lambda|}$ for all $0 \leq \alpha \leq t$ and all $\lambda \in \mathcal{J}$, we infer that $|a^{(\alpha)}_{\lambda,\lambda'}| \lesssim |b^{(\alpha)}_{\lambda,\lambda'}| \lesssim 2^{-(d-\frac{1}{2}-\alpha)||\lambda|-|\lambda'||}$ and in particular, for all $\lambda, \lambda' \in \mathcal{J}$,

$$|a_{\lambda,\lambda'}| \lesssim 2^{-(d-\frac{1}{2}-t)\delta(\lambda,\lambda')}, \quad \delta(\lambda,\lambda') := \big||\lambda| - |\lambda'|\big|.$$

Note that this estimate holds for both negative and non-negative levels $|\lambda|, |\lambda'|$. It depends *only* on the level difference and is *independent* of the block in (3.2). Moreover, due to vanishing moments in conjunction with (3.4) and $\operatorname{diam}(\Box_\lambda) \lesssim 2^{-|\lambda|}$, $a_{\lambda,\lambda'} = 0$ whenever $i(\lambda,\lambda') = 0$ (cf. [15, Proposition 5.3.3]), so that we have

$$\#N(\lambda,\ell') = \#S(\lambda,\ell') \lesssim 1, \quad S(\lambda,\ell') := \{\lambda' \in \mathcal{J} : |\lambda'| = \ell',\, i(\lambda,\lambda') = 1\}, \tag{3.9}$$

uniformly in $\lambda \in \mathcal{J}$ and $\ell' \in \mathbb{Z}$. Finally, as ψ_λ is assumed to be piecewise polynomial (cf. (3.4)) and the coefficients g_α are constant, any entry $a_{\lambda,\lambda'}$ can be computed *exactly* in $\mathcal{O}(1)$. Now, one can deduce the following result from [15, Section 5.3]:

Theorem 3.1. *Let $\Psi_t \in \{\Psi_{t,j_0} : j_0 \geq -\infty\}$ and define $\mathbf{A}_j$ by dropping all entries in $\mathbf{A}$ when $\delta(\lambda,\lambda') > j$ and $i(\lambda,\lambda') = 1$. Then, the number of nonzeros in each row and column of $\mathbf{A}_j$ is of order $\mathcal{O}(j)$ and*

$$\|\mathbf{A} - \mathbf{A}_j\| \lesssim 2^{-(d-\frac{1}{2}-t)j}, \quad \forall j \in \mathbb{N}. \tag{3.10}$$

In particular, $\mathbf{A}$ is s^*-compressible and s^*-computable with $s^* = \infty$.

Proof. See Section A. □

Non-constant coefficients. Let us now consider the general case where g_α is not necessarily constant. Here, obviously, $a_{\lambda,\lambda'}$ *does not* automatically vanish when $i(\lambda,\lambda') = 0$. But, taking into account that as for bounded domains, we have

$$\#U(\lambda,\ell') \lesssim 2^{\max\{0,\ell'-|\lambda|\}}, \quad U(\lambda,\ell') := \{\lambda' \in \mathcal{J} : |\lambda'| = \ell',\, i(\lambda,\lambda') = 0\}, \tag{3.11}$$

the following result from [26] also holds in an unbounded setting:

Theorem 3.2 ([26, Theorem 4.1])**.** *Let $\Psi_t \in \{\Psi_{t,j_0} : j_0 \geq -\infty\}$ and assume that the entries $a_{\lambda,\lambda'}$ in $\mathbf{A}$ satisfy an estimate of the following type for $\alpha \leq t$:*

$$|a^{(\alpha)}_{\lambda,\lambda'}| \lesssim \begin{cases} 2^{-(\frac{3}{2}+r-\alpha)\delta(\lambda,\lambda')}\|g_\alpha\|_{W^{r+1-\alpha,\infty}(\mathbb{R})}, & i(\lambda,\lambda') = 1, \\ 2^{-(\frac{1}{2}+\widetilde{d}+\alpha)\delta(\lambda,\lambda')}\|g_\alpha\|_{W^{\widetilde{d}+\alpha,\infty}(\mathbb{R})}, & i(\lambda,\lambda') = 0. \end{cases} \tag{3.12}$$

Then, the matrix $\mathbf{A}_j$ which is obtained by dropping nonzero entries from $\mathbf{A}$ when

$$\delta(\lambda,\lambda') \cdot z^{(i(\lambda,\lambda'))} > j, \tag{3.13}$$

where $z^{(0)} := \widetilde{d} + t$ and $z^{(1)} := \frac{3}{2} + r - t$, satisfies $\|\mathbf{A} - \mathbf{A}_j\| \lesssim 2^{-j}$, $s^* := t + \widetilde{d}$ and r defined in (3.5)*. Moreover, the number of non-zeros per row and column in $\mathbf{A}_j$ is of order $\mathcal{O}(2^{j/s^*})$ showing that $\mathbf{A}$ is s^*-compressible.*

Proof. See Section A. □

Let us now comment on the *validity* of estimates of type (3.12). To this end, for possibly negative levels $j, j' \in \mathbb{Z}$ and $\lambda = (j,k)$, $\lambda' = (j',k')$, by applying the transformation of variables $y := 2^{-\ell}x$ and by introducing the notation $\lambda + \ell := (j+\ell,k)$, $\lambda' + \ell := (j'+\ell,k')$, we have that

$$b^{(\alpha)}_{\lambda,\lambda'} = 2^{-\alpha(|\lambda|+\ell)}\, 2^{-\alpha(|\lambda'|+\ell)} \int_{\mathbb{R}} g_\alpha(2^\ell y)\, \partial^\alpha \psi_{\lambda+\ell}(y)\, \partial^\alpha \psi_{\lambda'+\ell}(y)\, \mathrm{d}y. \tag{3.14}$$

Here, we choose $\ell = \max\{0, -\min\{|\lambda|,|\lambda'|\}\}$ such that, on one hand, only wavelets on non-negative scales appear under the integral, and, on the other hand, a transformation of variables is *only* applied if at least one of the levels $|\lambda|, |\lambda|$ is negative. Now, we can apply [26, Lemma 3.1] to (3.14) which yields:

$$|b^{(\alpha)}_{\lambda,\lambda'}| \lesssim \begin{cases} 2^{-(\frac{3}{2}+r-\alpha)\delta(\lambda,\lambda')}\, 2^{(r+1-\alpha)\ell}\, \|g_\alpha\|_{W^{r+1-\alpha,\infty}(\mathbb{R})}, & i(\lambda,\lambda') = 1, \\ 2^{-(\frac{1}{2}+\widetilde{d}+\alpha)\delta(\lambda,\lambda')}\, 2^{(\widetilde{d}+\alpha)\ell}\, \|g_\alpha\|_{W^{\widetilde{d}+\alpha,\infty}(\mathbb{R})}, & i(\lambda,\lambda') = 0, \end{cases} \tag{3.15}$$

where we used that $\|g_\alpha(2^\ell\,\cdot)\|_{W^{p,\infty}(\square_{\lambda+\ell,\lambda'+\ell})} \lesssim 2^{p\ell}\|g_\alpha\|_{W^{p,\infty}(\mathbb{R})}$ for $p \in \mathbb{N}$. In particular, since $|a^{(\alpha)}_{\lambda,\lambda'}| \lesssim |b^{(\alpha)}_{\lambda,\lambda'}|$ for all $\lambda,\lambda' \in \mathcal{J}$, (3.15) yields a *uniform* estimate for matrix entries in $\mathbf{A}_{++}$, i.e., for $|\lambda|,|\lambda'| \geq 0$. However, for the other blocks, we require an *upper bound* on ℓ in order to obtain a uniform estimate in the sense of (3.12). For $j_0 > -\infty$ the following result can be deduced from (3.15):

Proposition 3.3. *Let $\Psi = \Psi_{t,j_0}$ for $j_0 \in \mathbb{Z}$. If $g_\alpha \in W^{\alpha+\widetilde{d},\infty}(\mathbb{R})$ for $\alpha \le t$, then*

$$|a^{(\alpha)}_{\lambda,\lambda'}| \le C_{j_0} \begin{cases} 2^{-(\frac{3}{2}+r-\alpha)\delta(\lambda,\lambda')}\|g_\alpha\|_{W^{r+1-\alpha,\infty}(\mathbb{R})}, & i(\lambda,\lambda')=1, \\ 2^{-(\frac{1}{2}+\widetilde{d}+\alpha)\delta(\lambda,\lambda')}\|g_\alpha\|_{W^{\widetilde{d}+\alpha,\infty}(\mathbb{R})}, & i(\lambda,\lambda')=0, \end{cases} \tag{3.16}$$

for a constant $C_{j_0} \sim \max\{1, 2^{-(t+\widetilde{d})j_0}\}$ and r defined in (3.5).

Proof. See Section A. □

Remark 3.4. Proposition 3.3 in conjunction with Theorem 3.2 shows that when $\Psi_t = \Psi_{t,j_0}$ for $j_0 \in \mathbb{Z}$, $\mathbf{A}$ is s^*-compressible for $s^* = t + \widetilde{d}$. Moreover, we remark that s^*-computability of $\mathbf{A}$ can then be proven as in [26, Theorem 6.2].

Concerning $\Psi_t = \Psi_{t,-\infty}$ where we permit arbitrary coarse scales $|\lambda|, |\lambda'| < 0$ in (3.15), here, we do not have an *upper bound* on ℓ. So, we do not obtain a *uniform* estimate in the sense of (3.12) which only depends on the level difference $\delta(\lambda,\lambda')$ and the compression results from [26] *do not* permit to show $\mathbf{A} \in \mathcal{B}_{s^*}$ for $s^* > d-t$ when $j_0 = -\infty$ and g_α non-constant for $\alpha \le t$. Unfortunately, the same holds true for other results in this field, e.g., [29]. As, moreover, the numerical experiments at the end of this section indicate that the use of a minimal level is advantageous, we did not further investigate s^*-compressibility in this case.

Remark 3.5. In this section, for the ease of presentation, we only considered self-adjoint differential operators. Note that in view of Remark 2.5, following the lines of [26], the results above can be extended to operators of the form $\langle v, \mathcal{A}[w]\rangle = \sum_{\alpha,\beta\le t}\int_{\mathbb{R}} g_{\alpha,\beta}(x)\,\partial^\alpha v(x)\,\partial^\beta w(x)\,\mathrm{d}x$ for $v, w \in H^t(\mathbb{R})$.

3.2. **RHS on unbounded domains.** The main idea of **RHS** can be described as follows. For a given tolerance η, one has to construct an index set ∇_η such that $\|\mathbf{f} - \mathbf{f}|_{\nabla_\eta}\|_{\ell_2} \le \eta$ and $\#\nabla_\eta = \mathcal{O}(\eta^{-1/\bar{s}})$ where, for the same reasons given in Remark 2.4, $\bar{s} \ge d-t$. On a bounded domain, it suffices to control the maximal level in such a set ∇_η. In our case, nevertheless, we also need to bound the translation indices and, for $\Psi_t = \Psi_{t,-\infty}$ also the minimal level in ∇_η. In order to reach this goal, we need some assumptions of f, which, however, are not too restrictive.

Assumption 3.6. *We assume that $f = f_1 + f_2$ can be split into a smooth part f_1 and a singular part f_2. For some $\sigma \in (0, \widetilde{d}\,]$, we suppose that $f_1 \in W^{\sigma,\infty}(\mathbb{R}) \cap L_2(\mathbb{R}) \cap L_1(\mathbb{R})$ and satisfies*

$$\|f_1 - f_{1,R}\|_{L_2(\mathbb{R})} \le C_{f_1} R^{-\beta}, \quad \forall R > R_0, \quad f_{1,R} := f_1|_{[-R,R]}, \tag{3.17}$$

for constants $\beta > 0$, $R_0 > 0$, $C_{f_1} > 0$. Moreover, f_2 is assumed to be a finite sum of delta distributions $f_2 = \sum_{i=1}^m c_i\,\delta_{x_i}$ for $c_1,\dots,c_m \in \mathbb{R}$ and $x_1,\dots,x_m \in \mathbb{R}$.

Let us assume now that $t \in \mathbb{N}$ and Assumption 3.6 holds. Then, we can consider the *smooth* part $\mathbf{f}_1 := \langle f_1, \Psi_t\rangle$ and the *singular* part $\mathbf{f}_2 := \langle f_2, \Psi_t\rangle$ separately. Since $\mathbf{f} = \mathbf{f}_1 + \mathbf{f}_2$, it then suffices to set up index sets $\nabla_{i,\eta}$ such that $\|\mathbf{f}_i - \mathbf{f}_i|_{\nabla_{i,\eta}}\|_{\ell_2} \le \eta$ ($i \in \{1,2\}$). We point out that it is also sufficient to construct $\mathbf{f}_{i,\eta} \in \ell_2(\mathcal{J})$ such that $\|\mathbf{f}_i - \mathbf{f}_{i,\eta}\|_{\ell_2} \le \eta$ and $\#\mathrm{supp}\,\mathbf{f}_{i,\eta} \lesssim \eta^{-1/\bar{s}}$ where $\mathbf{f}_{i,\eta}$ is *not* necessarily the restriction of $\mathbf{f}_i$ to some finite index set $\nabla_{i,\eta}$ ($i \in \{1,2\}$).

3.2.1. *The case $j_0 = -\infty$.* We start by considering $\Psi_t = \Psi_{t,-\infty}$ where we assume that Ψ_t satisfies the requirements from Section 3.1.1 and show how to adapt these results to the case $j_0 > -\infty$ later.

Singular part. The approximation of the singular part $\mathbf{f}_2$ bases upon two observations. Firstly, by Remark 2.1 in conjunction with $\|\psi_\lambda\|_{L_2(\mathbb{R})} = 1$ for all $\lambda \in \mathcal{J}$, it holds for $i = 1, \dots, m$ with $d_t := \max\{1, |\psi|_{H^t}^{-1}\}$ $(t \in \mathbb{N})$

(3.18)
$$|\mathbf{D}_\lambda^t \psi_\lambda(x_i)| \leq \|\mathbf{D}_\lambda^t \psi_\lambda\|_{L_\infty} \leq \frac{2^{|\lambda|/2}\|\psi\|_{L_\infty}}{\sqrt{1 + 2^{2t|\lambda|}|\psi|_{H^t}^2}} \leq d_t \begin{cases} 2^{(\frac{1}{2}-t)|\lambda|}\,\|\psi\|_{L_\infty}, |\lambda| \geq 0, \\ 2^{|\lambda|/2}\,\|\psi\|_{L_\infty}, |\lambda| < 0. \end{cases}$$

Secondly, since diam(supp ψ_λ) $= 2^{-|\lambda|}$diam(supp ψ), the number of ψ_λ with $|\lambda| = j$ whose supports contain a given $y \in \mathbb{R}$ can be bounded independent of j and y, i.e.,

$$\#\{\lambda \in \mathcal{J} : |\lambda| = j,\, y \in \text{supp}\, \psi_\lambda\} \leq M \in \mathbb{N}, \quad \forall j \in \mathbb{Z},\, \forall y \in \mathbb{R}. \tag{3.19}$$

Now, for a tolerance $\eta > 0$, we define maximum levels $J_2^{(i,+)}(\eta)$ and minimum levels $-J_2^{(i,-)}(\eta)$ associated to $c_i\delta_{x_i}$ $(i \in \{1, \dots, m\})$ by

$$J_2^{(i,+)}(\eta) := (|\log_2(2m^2c_i^2d_t^2\|\psi\|_{L_\infty}^2 M(1 - 2^{(1-2t)})^{-1})| + 2|\log_2 \eta|)/(2t-1), \tag{3.20}$$

$$J_2^{(i,-)}(\eta) := \;|\log_2(4m^2c_i^2d_t^2\|\psi\|_{L_\infty}^2 M)| + 2|\log_2 \eta|, \tag{3.21}$$

as upper and lower level bounds on $\lambda \in \mathcal{J}$ with $x_i \in$ supp ψ_λ.

Lemma 3.7. *Let $\eta > 0$. For $i = 1, \dots, m$, we set $\mathbf{f}_2^i := (\mathbf{D}_\lambda^t\, c_i\psi_\lambda(x_i))_{\lambda\in\mathcal{J}}$ as well as $\mathbf{f}_{2,\eta} := \mathbf{f}_2^1|_{\nabla_{2,\eta}^1} + \dots + \mathbf{f}_2^m|_{\nabla_{2,\eta}^m}$, $\nabla_{2,\eta} := \cup_{i=1}^m \nabla_{2,\eta}^i$ and*

$$\nabla_{2,\eta}^i := \{\lambda \in \mathcal{J} : x_i \in \text{supp}\, \psi_\lambda, -J_2^{(i,-)}(\eta) \leq |\lambda| \leq J_2^{(i,+)}(\eta)\}. \tag{3.22}$$

Then, we have $\|\mathbf{f}_2 - \mathbf{f}_{2,\eta}\|_{\ell_2} \leq \eta$ and #supp $\mathbf{f}_{2,\eta} = \#\nabla_{2,\eta} \lesssim 1 + |\log_2 \eta|$.

Proof. See Section A. $\square$

To compute $\nabla_{2,\eta}$, we only have to take into account wavelets whose supports contain at least one of the points $x_1, \dots, x_m$. Thus, no further bound on the translation indices is required.

Smooth part. We now consider the smooth part f_1 of f. In order to compute an approximation $\mathbf{f}_{1,\eta}$ to $\mathbf{f}_1$, we need to define some quantities:

$$D_1 := (C_{f_1} c_{\Psi_t}^{-1} C_{\Psi_t}^2)^{1/\beta}, \quad D_2 := 4\, C_{\psi,f_1}^2\, d_t^2\, (1 - 2^{-2(\sigma+t)})^{-1}, \tag{3.23}$$

with the involved quantities as defined in (2.4), (3.6), (3.17) and Assumption 3.6. Next, we set for M defined in (3.19)

$$R_\eta := D_1\eta^{-1/\beta}, \quad D_3 := 2\,(2M+1)\|f_1\|_{L_1(\mathbb{R})}^2\|\psi\|_{L_\infty(\mathbb{R})}^2, \tag{3.24}$$

$$J_1^+(\eta) := \max\{[(2 + \tfrac{1}{\beta})|\log_2 \eta| + \log_2(D_1D_2)]/(2(\sigma+t)), \log_2(MR_\eta^{-1}), 0\}, \tag{3.25}$$

$$J_1^-(\eta) := \max\{2|\log_2 \eta| + |\log_2 D_3|, 1 + |\log_2(2D_1)| + |\beta^{-1}\log_2 \eta|\}. \tag{3.26}$$

These quantities permit us to set up an approximation to $\mathbf{f}_1$ by, firstly, considering only indices $\lambda \in \mathcal{J}$ with $|\text{supp}\, \psi_\lambda \cap [-R_\eta, R_\eta]| > 0$ and, secondly, using $J_1^+(\eta)$ and $J_1^-(\eta)$ as upper and lower levels bounds for these indices:

Lemma 3.8. *Let $\eta > 0$ and let Assumption 3.6 hold. Then, by setting*

$$\nabla_{1,\eta}^+ := \{\lambda \in \mathcal{J} : 0 \leq |\lambda| \leq J_1^+(\eta), |\square_\lambda \cap I_\eta| > 0\}, \tag{3.27}$$

$$\nabla_{1,\eta}^- := \{\lambda \in \mathcal{J} : -J_1^-(\eta) \leq |\lambda| < 0, |\square_\lambda \cap I_\eta| > 0\}, \tag{3.28}$$

where $I_\eta := [-R_\eta, R_\eta]$, *the vector* $\mathbf{f}_{1,\eta} := \mathbf{f}^+_{1,\eta} + \mathbf{f}^-_{1,\eta}$ *with*

$$\mathbf{f}^+_{1,\eta} := \mathbf{f}_1|_{\nabla^+_{1,\eta}}, \quad \mathbf{f}^-_{1,\eta} := \mathbf{f}_{1,R_\eta}|_{\nabla^-_{1,\eta}}, \quad \mathbf{f}_{1,R} := \langle \Psi_t, f_{1,R}\rangle, \tag{3.29}$$

satisfies $\|\mathbf{f}_1 - \mathbf{f}_{1,\eta}\|_{\ell_2} \le \eta$. *With* $\widetilde{s} := \frac{\beta(t+\sigma)}{\sigma+\beta+t+\frac{1}{2}}$ *and* $\bar{s} := \min\{\beta, \widetilde{s}\}$, *it holds that* $\#\nabla^+_{1,\eta} \lesssim \eta^{-1/\widetilde{s}}$, $\#\nabla^-_{1,\eta} \lesssim \eta^{-1/\beta}$ *and* $\#\nabla_{1,\eta} \lesssim \eta^{-1/\bar{s}}$ *for* $\nabla_{1,\eta} := \nabla^+_{1,\eta} \cup \nabla^-_{1,\eta}$.

Proof. See Section A. □

We emphasize that the truncation of f_1 to f_{1,R_η} in (3.29) is due to the fact that for $|\lambda| \to -\infty$, diam(supp ψ_λ) $\sim 2^{-|\lambda|} \to +\infty$. The diameter of I_η is *independent* of $|\lambda|$, diam$(I_\eta) \sim \eta^{-1/\beta}$. So, this approach is of advantage when numerical quadrature is required to approximate entries in $\mathbf{f}_{1,\eta}$. We comment this in Section 3.2.4.

Now, we can collect the results from Lemmata 3.7 and 3.8 to obtain:

Theorem 3.9. *Let* $\eta > 0$ *and let Assumption 3.6 hold. Then,* $\mathbf{g}_\eta := \mathbf{f}_{1,\eta/2} + \mathbf{f}_{2,\eta/2}$ *with* $\mathbf{f}_{1,\eta/2}$, $\mathbf{f}_{2,\eta/2}$ *from Lemmata 3.7, 3.8 satisfies for* $\bar{s}$ *defined in Lemma 3.8*

$$\|\mathbf{f} - \mathbf{g}_\eta\|_{\ell_2} \le \eta, \quad \#\mathrm{supp}\, \mathbf{g}_\eta \lesssim \eta^{-1/\bar{s}}. \tag{3.30}$$ □

For optimality of **ADWAV**, as stated above, we require that $\bar{s} \ge d - t$. In particular, (3.30) shows that the parameters β and σ have to be sufficiently large. Assuming exemplarily that f_1 decays exponentially and $\sigma = \widetilde{d}$, we infer from Theorem 3.9 that for any $\bar{s} < t + \widetilde{d}$, there exist constants $\beta > 0$, C_{f_1}, R_0 such that f_1 satisfies (3.17) and $\mathbf{f}$ satisfies (3.30). The limit case $\bar{s} = t + \widetilde{d}$ (which can be attained on bounded domains, see [18]) is excluded which is due to the additional bound on the translation indices which depends on η.

3.2.2. *The case* $j_0 > -\infty$. Let now $\Psi_t = \Psi_{t,j_0}$ for $j_0 < 0$ (the case $j_0 \ge 0$ is treated analogously). As $j_0 - 1$ is a natural lower bound for levels, we only have to replace $J_2^{(i,-)}(\eta)$ $(i \in \{1, \dots, m\})$ in (3.21) and $J_1^-(\eta)$ in (3.26) by $j_0 - 1$ where we recall that $\psi_{j_0-1,k} := \varphi_{j_0,k}$ $(k \in \mathbb{Z})$. Going through the corresponding proofs, this can be seen by taking into account that the basic estimates (A.8) (for Lemma 3.8) and (3.18) (for Lemma 3.7) also hold in an analogous form for scaling functions.

3.2.3. *Realization.* Let us now discuss a possible numerical realization of **RHS**. Based on Theorem 3.9, we use approximations $\mathbf{g}_{\eta_k}$ to $\mathbf{f}$ with

$$\|\mathbf{f} - \mathbf{g}_{\eta_k}\|_{\ell_2} \le \eta_k, \quad \#\mathrm{supp}\, \mathbf{g}_{\eta_k} \lesssim \eta_k^{-1/\bar{s}}, \quad \eta_k := 2^{-k}, \quad k \in \mathbb{N}, \tag{3.31}$$

which can be computed within $\mathcal{O}(\eta_k^{-1/\bar{s}})$ operations (see also Section 3.2.4). These discrete tolerances η_k are used since in praxis, it is in general not possible to set up $\mathbf{g}_\eta$ for *any* η even though the minimal tolerance for which **RHS** is called in **GROW**, can be bounded by $\eta > \varepsilon \min\left\{\frac{1}{2}\omega(1+\omega)^{-1}, \gamma\right\}$ (see [18, Proof of Theorem 2.4]).

The idea of **RHS** in Algortihm 5 is as follows. Let $\bar{\eta}$ be a current target tolerance and $\mathbf{g}_{\bar{\eta}}$ be a corresponding approximation satisfying $\|\mathbf{f} - \mathbf{g}_{\bar{\eta}}\|_{\ell_2} \le \bar{\eta}$. If **RHS** is called with a tolerance $\eta < \bar{\eta}$, we compute the largest $\eta_k \le \eta$ and $\mathbf{g}_{\bar{\eta}} := \mathbf{g}_{\eta_k}$ such that $\|\mathbf{f} - \mathbf{g}_{\bar{\eta}}\|_{\ell_2} \le \eta_k \le \eta$ within $\mathcal{O}(\eta^{-1/\bar{s}})$ operations. If **RHS** is called with $\eta \ge \bar{\eta}$, then we compute $\mathbf{g}_\eta$ as a threshold of $\mathbf{g}_{\bar{\eta}}$ satisfying $\|\mathbf{g}_{\bar{\eta}} - \mathbf{g}_\eta\|_{\ell_2} \le \eta - \bar{\eta}$ which yields $\|\mathbf{f} - \mathbf{g}_\eta\|_{\ell_2} \le \|\mathbf{f} - \mathbf{g}_{\bar{\eta}}\|_{\ell_2} + \|\mathbf{g}_{\bar{\eta}} - \mathbf{g}_\eta\|_{\ell_2} \le \eta$. Even though thresholding here requires $\mathcal{O}(\bar{\eta}^{-1/s})$ instead of $\mathcal{O}(\eta^{-1/s})$ operations which is, theoretically, not optimal, we observed that this strategy is faster than *recomputing* $\mathbf{g}_{\eta_k}$ and requires less storage than *storing* all $\mathbf{g}_{\eta_k}$ for $\eta_k \ge \bar{\eta}$.

Algorithm 5 $\mathbf{RHS}[\eta] \to \mathbf{g}_\eta$

% *Before the first call, fix some* $\bar{\eta} > \eta_0$ *and* $\mathbf{g}_{\bar{\eta}} := 0$.
1: **if** $\bar{\eta} > \eta$ **then** determine $k \in \mathbb{N}$ minimal s.t. $\eta_k \le \bar{\eta}$.
2: Compute $\mathbf{g}_{\bar{\eta}} := \mathbf{g}_{\eta_k}$, set $\bar{\eta} = \eta_k$ and return $\mathbf{g}_{\bar{\eta}}$.
3: **else** Compute $\mathbf{g}_\eta$ s.t. supp $\mathbf{g}_\eta \subseteq$ supp $\mathbf{g}_{\bar{\eta}}$ and $\|\mathbf{g}_{\bar{\eta}} - \mathbf{g}_\eta\|_{\ell_2} \le \eta - \bar{\eta}$.
4: **end if**

3.2.4. *Computability.* So far, we have neglected the issue of *computing* entries in $\mathbf{g}_\eta$. As the computation of the singular part is trivial, we focus on the smooth part of f by setting $f_2 \equiv 0$ and consider a composite quadrature rule of order p for an interval $\Omega = \cup_{i=1}^N \overline{\Omega}_i$ with N equally spaced subintervals Ω_i, i.e., $\int_\Omega g \, \mathrm{d}x \approx Q_N^\Omega(g) := \sum_{i=1}^N \sum_j \omega_j^{\Omega_i} g(x_j^{\Omega_i})$ for finite sequences of weights $(\omega_j^{\Omega_i})$ and abscissae $(x_j^{\Omega_i})$ on Ω_i. Using the fact that ψ_λ is piecewise polynomial, we define for κ and $\Xi_{\lambda,i}$, $i = 1, \dots, \kappa$ from (3.4) approximations to the entries $\mathbf{g}_{\eta,\lambda}$ in $\mathbf{g}_\eta = (\mathbf{g}_{\eta,\lambda})_{\lambda \in \nabla_{1,\eta}}$,

$$\begin{aligned}
\widetilde{\mathbf{g}}_{\eta,\lambda} &:= \mathbf{D}_\lambda^t \textstyle\sum_{i=1}^\kappa Q_N^{\Xi_i}(\psi_\lambda f_1) &&\approx \mathbf{g}_{\eta,\lambda} = \textstyle\int_{\text{supp }\psi_\lambda} \mathbf{D}_\lambda^t \psi_\lambda f_1 \, \mathrm{d}x, && \lambda \in \nabla_{1,\eta}^+,\\
\widetilde{\mathbf{g}}_{\eta,\lambda} &:= \mathbf{D}_\lambda^t \textstyle\sum_{i=1}^\kappa Q_N^{\Xi_i \cap I_\eta}(\psi_\lambda f_1) &&\approx \mathbf{g}_{\eta,\lambda} = \textstyle\int_{I_\eta} \mathbf{D}_\lambda^t \psi_\lambda f_1 \, \mathrm{d}x, && \lambda \in \nabla_{1,\eta}^-.
\end{aligned}$$

Assuming that $f_1 \in W^{p,\infty}(\mathbb{R})$ for $p \in \mathbb{N}$, the following error estimate

$$|\mathbf{g}_{\eta,\lambda} - \widetilde{\mathbf{g}}_{\eta,\lambda}| \lesssim \begin{cases} N^{-p}\, 2^{-(\frac{1}{2}+p-d^*)|\lambda|}, & |\lambda| \ge 0, \\ (2R_\eta)^{1+p}\, N^{-p}, & |\lambda| < 0, \end{cases} \qquad d^* := d - 1 - t, \tag{3.32}$$

can be derived from [19, Propositions 2.5 & 4.3]. By choosing N as well as p in dependence of $|\lambda|$, these estimates permit to set up a *computable* approximation $\widetilde{\mathbf{g}}_\eta := (\widetilde{\mathbf{g}}_{\eta,\lambda})_{\lambda \in \nabla_{1,\eta}}$ of $\mathbf{g}_\eta$ in the sense of (2.17) using techniques from [15, 19].

For convenience, we detail this exemplarily for $j_0 = -\infty$ in the following result:

Proposition 3.10. *Assume that* $\mathbf{g}_\eta$ *from Theorem 3.9 satisfies* (3.30) *with* $\bar{s} \ge d - t$ *and* β *from Assumption 3.6 satisfies* $\beta > 2\bar{s}$. *Moreover, let* $f_1 \in W^{p,\infty}(\mathbb{R})$ *for a sufficiently large* $p > \widetilde{d}$. *Then,* $\widetilde{\mathbf{g}}_\eta$ *can be computed within* $\mathcal{O}(\eta^{-1/\bar{s}})$ *operations such that* $\|\mathbf{g}_\eta - \widetilde{\mathbf{g}}_\eta\|_{\ell_2} \lesssim \eta$.

Proof. See Section A. □

3.3. **Numerical examples.** We give some examples in 1D, namely instances of the following reaction-diffusion problem in weak formulation: Find $u \in H^1(\mathbb{R})$ with

$$(\partial v, \partial u)_{L_2(\mathbb{R})} + (v, u)_{L_2(\mathbb{R})} = \langle v, f \rangle, \quad \forall v \in H^1(\mathbb{R}), \tag{3.33}$$

for $f \in H^{-1}(\mathbb{R})$. We use *biorthogonal B-splines* wavelet bases described in [9]. All examples (also those presented in Sections 4 and 5 below) are realized in C++ using the software libraries FLENS and LAWA, [22, 31]. We consider three different choices for the right-hand side f which permit a continuous reference solution with unbounded support in closed form (see Figure 3.1):

(P1) Smooth solution, exponential decay.
(P2) One peak, large significant domain, exponential decay.
(P3) Two peaks, decay: polynomial $(x \to -\infty)$, exponential $(x \to \infty)$.

3.3.1. *Parameters.* For the realization of **ADWAV**, good estimates of the constants c_1 and c_2 in the norm equivalence (2.8) are necessary for different polynomial orders d, vanishing moments $\widetilde{d}$ and different minimal levels. The values we used can be found in Table 3.1. In order to ensure a good performance of **ADWAV**, we used $\omega = 0.01$ and chose the parameter $\alpha < (1-\omega)\kappa(\mathbf{A})^{-\frac{1}{2}} - \omega$ as large as possible. Moreover, we used $\gamma = \frac{1}{12}\kappa(\mathbf{A})^{-\frac{1}{2}}\frac{\alpha-\omega}{1+\omega}$ and $\theta = \frac{2}{7}$. Note that these parameters satisfy the optimality and convergence condition stated in [18].

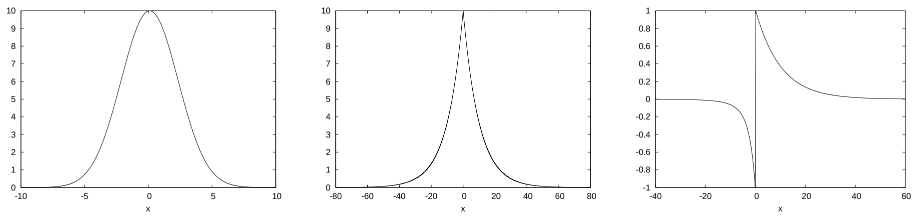


FIGURE 3.1. Solutions u_i for (P1)–(P3) from left to right.

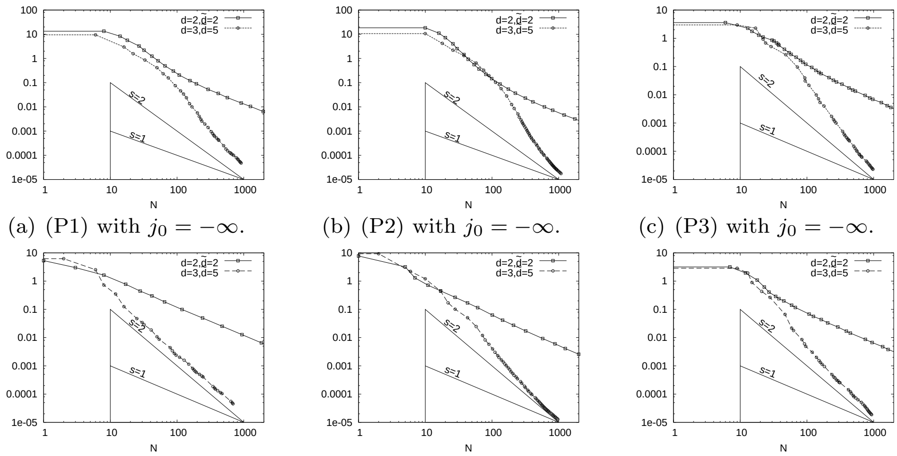
j_0		0	-1	-2	-4	-6	-20	$-\infty$
$d=2, \widetilde{d}=2$	c_1	0.37	0.56	0.56	0.44	0.34	0.21	0.19
	c_2	2.10	2.10	2.10	2.10	2.10	2.10	2.10
$d=3, \widetilde{d}=3$	c_1	0.43	0.39	0.29	0.15	0.11	0.04	0.03
	c_2	1.94	2.03	2.24	2.55	2.61	2.65	2.70
$d=3, \widetilde{d}=5$	c_1	0.45	0.41	0.32	0.18	0.16	0.15	0.15
	c_2	1.96	2.10	2.38	2.80	2.85	2.85	2.85

TABLE 3.1. Estimated bounds for c_1, c_2 from (2.8) for **ADWAV**.

Remark 3.11. Concerning the values in Table 3.1, to our knowledge, there is no method to compute these values analytically. Nevertheless, bounds for c_1, c_2 can be computed numerically. We describe the case $j_0 = -\infty$ since $j_0 > -\infty$ can be treated analogously. It suffices to consider the finite collections $\Psi_{R,J^-,J^+} := \{\mathbf{D}_\lambda^t \psi_\lambda : \operatorname{supp}\psi_\lambda \cap [-R,R] \neq \emptyset, -J^- \leq |\lambda| \leq J^+\}$ for $R > 0$, $J^-, J^+ \geq 0$ and to compute c_1, c_2 in terms of eigenvalues of the finite matrices $\mathbf{A}_{R,J^-,J^+}$ obtained by replacing Ψ_t by Ψ_{R,J^-,J^+} in (2.7). For $R \to \infty$, $J^+, J^- \to +\infty$, one can observe that the computed eigenvalues converge.

3.3.2. *Choice of a minimal level.* The bounds for c_1 and c_2 given in Table 3.1 already indicate that the choice of a minimal level j_0 is not trivial as we might have two conflicting goals: On one hand, we want to be free in the choice of a minimal level j_0 to represent both small and large supports of the numerical solution with only few degrees of freedom. On the other hand, the condition number $\kappa(\mathbf{A})$ depends strongly on j_0 where small $\kappa(\mathbf{A})$ is favorable. We choose $j_0 = |\lambda|$ where $\widehat{\mathbf{f}}_\lambda$ is an *estimate* of the largest coefficient in modulus of $\mathbf{f}$ that can be derived analytically using (3.6) and (3.18). We will further discuss this issue below.

3.3.3. *Convergence rates.* The results of our experiments concerning the convergence are shown in Figure 3.2 both for $j_0 = \infty$ and $j_0 > -\infty$ as well as wavelet bases with $d = 2$, $\widetilde{d} = 2$ and $d = 3$, $\widetilde{d} = 5$. The latter choice is due to the much better condition number $\kappa(\mathbf{A})$ if $j_0 = -\infty$ (cf. Table 3.1). We measure the error in $H^1(\mathbb{R})$ and do not show the error estimator ν_k since they basically coincide with the true error. Recall that the best nonlinear approximation s for which $\mathbf{u} \in \mathcal{A}^s$ is bounded by $d - 1$ (cf. Section 2.4). Observe that this rate is asymptotically attained. For $d = 3$, we even observe superconvergence for moderate values of N.



(a) (P1) with $j_0 = -\infty$. (b) (P2) with $j_0 = -\infty$. (c) (P3) with $j_0 = -\infty$.

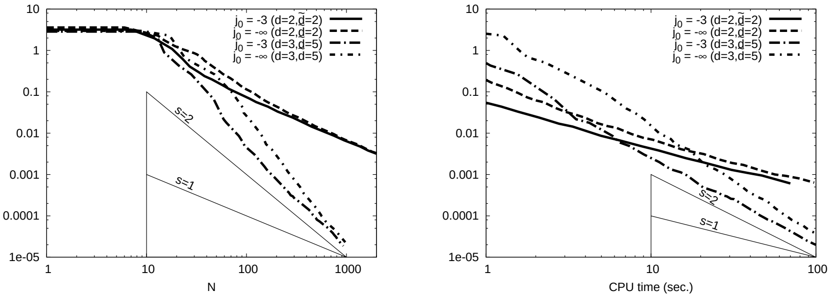
(d) (P1) with $j_0 = -2$. (e) (P2) with $j_0 = -4$. (f) (P3) with $j_0 = -3$.

FIGURE 3.2. $H^1(\mathbb{R})$-error vs. the degrees of freedom N.

3.3.4. *Discussion of the numerical results.* Despite the same asymptotic convergence rates for $j_0 = -\infty$ and $j_0 > -\infty$, there are some important quantitative differences between the two approaches that we illustrate in Figure 3.3. As an example, we consider (P3). We observe that the use of scaling functions on a minimal level j_0 significantly reduces the number of degrees of freedom for a given target accuracy. This is due to the fact that few scaling functions suffice to approximate the polynomial part of the solution, whereas, in the case $j_0 = -\infty$, we also need wavelets on very low levels which results in a higher number of degrees of freedom. Moreover, although we have a very simple structure in the basis for $j_0 = -\infty$ (we do not have to distinguish between wavelet and scaling functions), this advantage does not pay off as we can see from the computation times in Figure 3.3 b).

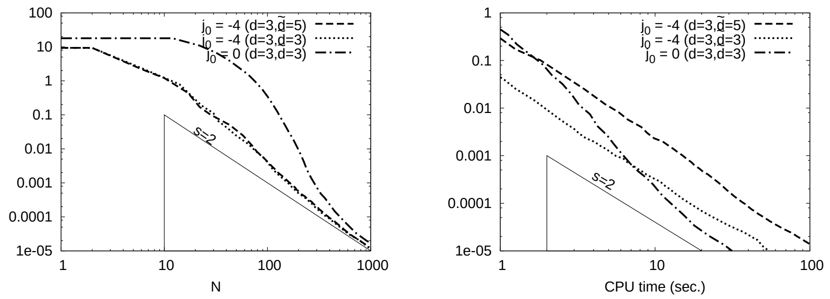
Next, we compare the influence of j_0 and the number of vanishing moments for (P2) in Figure 3.4. We observe in Figure 3.4 a) that the slope does not depend on $\widetilde{d}$ and that the minimal level $j_0 = 0$ results in worse results compared to $j_0 = -4$. However, if we take the required computation time into account (cf. Figure 3.4 b)), we observe that, due to a better condition number, the scheme converges asymptotically faster for $j_0 = 0$. Moreover, due to the fact that wavelets with $\widetilde{d} = 3$ have shorter support and fewer singular points than the one with $\widetilde{d} = 5$, the computation times are faster for $\widetilde{d} = 3$.

Our numerical results indicate that the use of $j_0 > -\infty$ is favorable. We emphasize that a better performance of **ADWAV** can be attained by increasing the



(a) N vs. error reduction for (P3). (b) Time vs. error reduction for (P3).

FIGURE 3.3. $H^1(\mathbb{R})$-error for $j_0 = \infty$ and $j_0 > -\infty$.



(a) N vs. error reduction. (b) Time vs. error reduction.

FIGURE 3.4. Influence of the minimal level j_0 and comparison between $d = 3$, $\widetilde{d} = 3$ and $d = 3$, $\widetilde{d} = 5$ for (P2).

value of α (cf. also [15, Section 5.6]). Doing so, however, we loose the guaranteed convergence. As moreover, the **APPLY**-routine is quantitatively demanding and the set up of **RHS** may be difficult, we present a heuristic algorithm in the next section which does not require these two routines.

4. A SIMPLIFIED ADAPTIVE WAVELET ALGORITHM

The simplified adaptive wavelet algorithm we present in this paragraph is a modification of the algorithm proposed in [2, 32]. To our knowledge, there is no proof of convergence or optimality. Nevertheless, numerical experiments have shown that this algorithm performs very well in practice. The simplified algorithm passes on the usage of the routines **RHS** as well as **APPLY** and provides an alternative routine **LINSOLVE** to **GALSOLVE**. Instead of **GROW**, a heuristic approach is used to determine $\Lambda^{(k+1)}$ from $\Lambda^{(k)}$ explicitly. Moreover, we do not need to consider the normal equations $\mathbf{A}^T\mathbf{A}\mathbf{u} = \mathbf{A}^T\mathbf{f}$ if $\mathcal{A}$ is not self-adjoint (cf. Remark 2.5). Motivated by the numerical results from Section 3, we consider from now on exclusively the case $\Psi_t := \Psi_{t,j_0}$ (cf. (2.6)) for $j_0 \in \mathbb{Z}$.

4.1. **Algorithm.** We start by describing the main components of the algorithm.

4.1.1. *Numerical solution of the Galerkin system.* As for **ADWAV**, we have to solve the Galerkin system (2.9) for an index set $\Lambda \subset \mathcal{J}_{j_0}$ in each iteration. But, as in the case of **GALSOLVE**, it is sufficient to solve a perturbed linear system

$$\widetilde{\mathbf{A}}_\Lambda \widetilde{\mathbf{u}}_\Lambda = \widetilde{\mathbf{f}}_\Lambda, \quad \|\mathbf{f}_\Lambda - \widetilde{\mathbf{f}}_\Lambda\|_{\ell_2} \le \eta_{\mathbf{f}} \|\mathbf{f}_\Lambda\|_{\ell_2}, \quad \|\mathbf{A}_\Lambda - \widetilde{\mathbf{A}}_\Lambda\| \le \widetilde{\eta}_{\mathbf{A}} \|\mathbf{A}\| =: \eta_{\mathbf{A}}, \tag{4.1}$$

with $0 < \eta_{\mathbf{f}}, \widetilde{\eta}_{\mathbf{A}} < 1$ given tolerances. Here, perturbations may arise from matrix compression or from the approximation of the right-hand side by numerical quadrature (cf. Section 3.2.4 or, alternatively, [2, Section 3.4]).

Within **ADWAV**, we used *absolute* tolerances in order to obtain an residual estimator ν_k within some *relative* tolerance (see lines 2 and 3 in **GROW**). For the heuristic scheme, in analogy to [2], we shall work (mostly) with relative errors w.r.t. $\|\mathbf{f}_\Lambda\|_{\ell_2}$ and explain why we use in some places also absolute tolerances.

Proposition 4.1. *Let $\Lambda \subset \mathcal{J}_{j_0}$ be finite, $\|\mathbf{A}_\Lambda - \widetilde{\mathbf{A}}_\Lambda\| \le \eta_{\mathbf{A}}$, $\eta_{\mathbf{A}} \le \frac{1}{2}c_1$ and $\|\mathbf{f}_\Lambda\|_{\ell_2} \gtrsim 1$, $\|\mathbf{f}_\Lambda - \widetilde{\mathbf{f}}_\Lambda\|_{\ell_2} \le \eta_{\mathbf{f}}\|\mathbf{f}_\Lambda\|_{\ell_2}$ for $\eta_{\mathbf{f}} < 1$. Then, $\widetilde{\mathbf{A}}_\Lambda$, $\widetilde{\mathbf{A}}_\Lambda^{-1}$ are uniformly bounded and*

$$\|\mathbf{u}_\Lambda - \widetilde{\mathbf{u}}_\Lambda\|_{\ell_2} \lesssim \eta_{\mathbf{A}} \|\mathbf{f}_\Lambda\|_{\ell_2} + \eta_{\mathbf{f}} \|\mathbf{f}_\Lambda\|_{\ell_2}, \tag{4.2}$$

where $\mathbf{u}_\Lambda$ is the solution of (2.9) *and $\widetilde{\mathbf{u}}_\Lambda$ is the solution of* (4.1).

Proof. From the assumptions, we infer from (2.8) that $|\langle \mathbf{w}_\Lambda, \widetilde{\mathbf{A}}_\Lambda \mathbf{v}_\Lambda\rangle_{\ell_2}| \le (\frac{1}{2}c_1 + c_2)\|\mathbf{w}_\Lambda\|_{\ell_2}\|\mathbf{v}_\Lambda\|_{\ell_2}$ for all $\mathbf{w}_\Lambda, \mathbf{v}_\Lambda$ with c_1, c_2 from (2.8). Moreover, we have

$$\tfrac{1}{2}c_1 \|\mathbf{v}_\Lambda\|_{\ell_2}^2 \le (c_1 - \eta_{\mathbf{A}})\|\mathbf{v}_\Lambda\|_{\ell_2}^2 \le |\langle \mathbf{v}_\Lambda, \widetilde{\mathbf{A}}_\Lambda \mathbf{v}_\Lambda\rangle_{\ell_2}|. \tag{4.3}$$

with c_1 from (2.8). Now, by (2.8), the following estimate is straightforward:

$$\begin{aligned}\|\mathbf{u}_\Lambda - \widetilde{\mathbf{u}}_\Lambda\|_{\ell_2}^2 &\le 2c_1^{-1} |\langle \mathbf{u}_\Lambda - \widetilde{\mathbf{u}}_\Lambda, \mathbf{A}_\Lambda \mathbf{u}_\Lambda - \widetilde{\mathbf{A}}_\Lambda \widetilde{\mathbf{u}}_\Lambda + \widetilde{\mathbf{A}}_\Lambda \widetilde{\mathbf{u}}_\Lambda - \mathbf{A}_\Lambda \widetilde{\mathbf{u}}_\Lambda\rangle_{\ell_2}| \\ &\le 2c_1^{-1}\big(\|(\mathbf{A}_\Lambda - \widetilde{\mathbf{A}}_\Lambda)\widetilde{\mathbf{u}}_\Lambda\|_{\ell_2}\|\mathbf{u}_\Lambda - \widetilde{\mathbf{u}}_\Lambda\|_{\ell_2} + |\langle \mathbf{u}_\Lambda - \widetilde{\mathbf{u}}_\Lambda, \mathbf{f}_\Lambda - \widetilde{\mathbf{f}}_\Lambda\rangle_{\ell_2}|\big),\end{aligned}$$

which yields $\|\mathbf{u}_\Lambda - \widetilde{\mathbf{u}}_\Lambda\|_{\ell_2} \le 2c_1^{-1}\left(\eta_{\mathbf{A}}\|\widetilde{\mathbf{u}}_\Lambda\|_{\ell_2} + \eta_{\mathbf{f}}\|\mathbf{f}_\Lambda\|_{\ell_2}\right)$ Then, we have

$$\|\widetilde{\mathbf{u}}_\Lambda\|_{\ell_2} \le 2c_1^{-1}\|\widetilde{\mathbf{f}}_\Lambda\|_{\ell_2} \le 2c_1^{-1}(1+\eta_{\mathbf{f}})\|\mathbf{f}_\Lambda\|_{\ell_2} \le 4c_1^{-1}\|\mathbf{f}_\Lambda\|_{\ell_2}, \tag{4.4}$$

where we used (4.3) and $\eta_{\mathbf{f}} < 1$. This yields the claim. ☐

Remark 4.2. We remark that ellipticity is not a necessary condition for estimate (4.2). The proof of Proposition 4.1 remains essentially the same if we only require that the norms of $\mathbf{A}$, $\mathbf{A}^{-1}$ and their perturbations are uniformly bounded.

From Proposition 4.1, we infer that there is no gain if one of the tolerances $\eta_{\mathbf{f}}$ or $\eta_{\mathbf{A}}$ is much smaller than the other. So, for a given tolerance $\mathrm{tol}_{\mathrm{iter}}$, we set

$$\eta_{\mathbf{f}} := \mathrm{tol}_{\mathrm{iter}} \text{ and } \eta_{\mathbf{A}} := \min\{\tfrac{1}{2}c_1, \mathrm{tol}_{\mathrm{iter}}\},$$

so that (4.2) can be replaced by $\|\mathbf{u}_\Lambda - \widetilde{\mathbf{u}}_\Lambda\|_{\ell_2} \lesssim \mathrm{tol}_{\mathrm{iter}}\|\mathbf{f}_\Lambda\|_{\ell_2}$ when $\|\mathbf{f}_\Lambda\|_{\ell_2} \gtrsim 1$. In particular, we have in mind to use a compressed matrix $\mathbf{A}_{J,\Lambda} = \mathbf{P}_\Lambda \mathbf{A}_J \mathbf{I}_\Lambda$ whenever $\mathbf{A} \in \mathcal{B}_{s^*}$ for some $s^* > 0$ using the compression results from Section 3. Then, we can use the routine **LINSOLVE** (Algorithm 6) to solve (2.9) approximately.

Algorithm 6 LINSOLVE$[\Lambda, \mathbf{w}_\Lambda, \mathrm{tol}_{\mathrm{iter}}] \to \widetilde{\mathbf{u}}_\Lambda$

1: Estimate $J = J(\mathrm{tol}_{\mathrm{iter}}) \in \mathbb{N}$ such that $\|\mathbf{A}_\Lambda - \mathbf{A}_{J,\Lambda}\|_{\ell_2} \le \min\{\frac{1}{2}c_1, \mathrm{tol}_{\mathrm{iter}}\}$.
2: Compute $\widetilde{\mathbf{f}}_\Lambda$ such that $\|\mathbf{f}_\Lambda - \widetilde{\mathbf{f}}_\Lambda\|_{\ell_2} \le \mathrm{tol}_{\mathrm{iter}}\|\mathbf{f}_\Lambda\|_{\ell_2}$.
3: Use a linear system solver like CG or GMRES with initial guess $\mathbf{w}_\Lambda$ to compute $\widetilde{\mathbf{u}}_\Lambda$ such that $\|\mathbf{A}_{J,\Lambda}\widetilde{\mathbf{u}}_\Lambda - \widetilde{\mathbf{f}}_\Lambda\|_{\ell_2} \le \mathrm{tol}_{\mathrm{iter}}\|\mathbf{A}_{J,\Lambda}\mathbf{w}_\Lambda - \widetilde{\mathbf{f}}_\Lambda\|_{\ell_2}$.

4.1.2. *Residual computation.* In **GROW**, we computed the residual estimator ν (cf. line 4 of Algorithm 2) by using **APPLY** and **RHS**. Instead, we now follow a heuristic strategy by constructing a so called *security zone* $\widehat{\Lambda} \supset \Lambda$ using Algorithm 7 (cf. [32, p.235]). For a constant $c > 0$, let $\square_\lambda = \operatorname{supp} \psi_\lambda =: [a_\lambda, b_\lambda]$ and

$$\mathcal{C}(\lambda, c) := \{\mu \in \mathcal{J}_{j_0} : \ |\square_\mu \cap c \cdot \square_\lambda| > 0, \ |\lambda| \leq |\mu| \leq |\lambda| + 1\}, \tag{4.5}$$

where $c \cdot \square_\lambda := [c\, a_\lambda + (1-c)z_\lambda, c\, b_\lambda + (1-c)z_\lambda]$, $z_\lambda := \frac{1}{2}(a_\lambda + b_\lambda)$. Observe that not only wavelets on higher levels are inserted into the security zone, but also further scaling function indices for $|\lambda| = j_0 - 1$. Moreover, no indices λ with levels $|\lambda| < j_0 - 1$ are inserted. Since both scaling functions and wavelets on level $j_0 - 1$ are linear combinations of scaling functions on level j_0, this would result in an over-determined system. Due to the locality of ψ_λ, $\#\mathcal{C}(\lambda, c) \lesssim 1$. Therefore, the cardinality of the output of $\mathbf{C}[\Lambda, c]$ as well as its complexity are both of order $\mathcal{O}(\Lambda)$.

Algorithm 7 $\mathbf{C}[\Lambda, c] \to \widehat{\Lambda}$

1: $\widehat{\Lambda} := \emptyset$.
2: **for** $\lambda \in \Lambda$ **do** $\widehat{\Lambda} := \widehat{\Lambda} \cup \mathcal{C}(\lambda, c)$.
3: **end for**

To estimate the residual $\mathbf{A}\mathbf{u}_\Lambda - \mathbf{f}$, we can now consider the expressions

$$\mathbf{r}_{\widehat{\Lambda}} := \mathbf{P}_{\widehat{\Lambda}}(\mathbf{A}_\Lambda \mathbf{u}_\Lambda - \mathbf{f}), \quad \widetilde{\mathbf{r}}_{\widehat{\Lambda}} := \mathbf{P}_{\widehat{\Lambda}}(\mathbf{A}_{J,\Lambda} \widetilde{\mathbf{u}}_\Lambda - \widetilde{\mathbf{f}}), \tag{4.6}$$

where $\widehat{\Lambda}$ is the output of $\mathbf{C}[\Lambda, c]$. Here, using the compressed matrix $\mathbf{A}_{J,\Lambda}$ and an approximate solution $\widetilde{\mathbf{u}}_\Lambda$ from (4.1), $\widetilde{\mathbf{r}}_{\widehat{\Lambda}}$ becomes an approximation to $\mathbf{r}_{\widehat{\Lambda}}$.

Proposition 4.3. *Let $\Lambda, \widehat{\Lambda}$ be finite subsets of $\mathcal{J}_{j_0}$ with $\widehat{\Lambda} \supset \Lambda$, $\|\mathbf{f}_\Lambda\|_{\ell_2} \gtrsim 1$ and suppose that the assumptions from Proposition 4.1 hold for both $\Lambda, \widehat{\Lambda}$. Then, by setting $\widetilde{\mathbf{A}}_{\widehat{\Lambda}} := \mathbf{A}_{J,\widehat{\Lambda}}$ for sufficiently large J, we have $\|\widetilde{\mathbf{r}}_{\widehat{\Lambda}} - \mathbf{r}_{\widehat{\Lambda}}\|_{\ell_2} \lesssim \mathrm{tol}_{\,\mathrm{iter}} \|\mathbf{f}_{\widehat{\Lambda}}\|_{\ell_2}$.*

Proof. We shall need the following notation: For a vector $\mathbf{v}_\Lambda$ with support Λ, we denote by $\mathbf{v}_{\widehat{\Lambda}}$ its extension by zeros to $\widehat{\Lambda}$. Thus, we have $\|\mathbf{v}_{\widehat{\Lambda}}\|_{\ell_2} = \|\mathbf{v}_\Lambda\|_{\ell_2}$ and, moreover, $\mathbf{A}_{\widehat{\Lambda}}\mathbf{v}_{\widehat{\Lambda}} = \mathbf{P}_{\widehat{\Lambda}}\mathbf{A}\mathbf{v}_\Lambda$. This yields the following estimate:

$$\begin{aligned}\|(\mathbf{A}_{\widehat{\Lambda}}\mathbf{u}_{\widehat{\Lambda}} - \mathbf{f}_{\widehat{\Lambda}}) - (\mathbf{A}_{J,\widehat{\Lambda}}\widetilde{\mathbf{u}}_{\widehat{\Lambda}} - \widetilde{\mathbf{f}}_{\widehat{\Lambda}})\|_{\ell_2} \leq\ & \|\mathbf{u}_\Lambda\|_{\ell_2}\|\mathbf{A}_{\widehat{\Lambda}} - \mathbf{A}_{J,\widehat{\Lambda}}\| \\ &+ \|\mathbf{A}_{J,\widehat{\Lambda}}\|\|\widetilde{\mathbf{u}}_\Lambda - \mathbf{u}_\Lambda\|_{\ell_2} + \|\mathbf{f}_{\widehat{\Lambda}} - \widetilde{\mathbf{f}}_{\widehat{\Lambda}}\|_{\ell_2}.\end{aligned}$$

From Proposition 4.1 we get that $\|\widetilde{\mathbf{u}}_\Lambda - \mathbf{u}_\Lambda\|_{\ell_2} \lesssim \mathrm{tol}_{\,\mathrm{iter}} \|\mathbf{f}_\Lambda\|_{\ell_2}$. Moreover, as in (4.4), we see that $\|\mathbf{u}_\Lambda\|_{\ell_2}\|\mathbf{A}_{\widehat{\Lambda}} - \mathbf{A}_{J,\widehat{\Lambda}}\| \lesssim \mathrm{tol}_{\,\mathrm{iter}} \|\mathbf{f}_\Lambda\|_{\ell_2}$ which yields the claim. $\square$

In view of (2.18), **RESIDUAL** is a heuristic approach as there is no proof of the existence of $0 < \beta < 1$ *independent* of Λ such that $\|\mathbf{P}_{\widehat{\Lambda}}(\mathbf{A}\mathbf{u}_\Lambda - \mathbf{f})\|_{\ell_2} \geq \beta\|\mathbf{A}\mathbf{u}_\Lambda - \mathbf{f}\|_{\ell_2}$. A fixed error reduction as for **ADWAV** can therefore not be guaranteed.

Algorithm 8 RESIDUAL$[\widehat{\Lambda}, \widetilde{\mathbf{u}}_\Lambda, \mathrm{tol}_{\,\mathrm{iter}}] \to \widetilde{\mathbf{r}}_{\widehat{\Lambda}}$

1: Estimate $J = J(\mathrm{tol}_{\,\mathrm{iter}}) \in \mathbb{N}$ such that $\|\mathbf{A}_{\widehat{\Lambda}} - \mathbf{A}_{J,\widehat{\Lambda}}\|_{\ell_2} \leq \min\{\frac{1}{2}c_1, \mathrm{tol}_{\,\mathrm{iter}}\}$.
2: Compute $\widetilde{\mathbf{f}}_{\widehat{\Lambda}}$ s.t. $\|\mathbf{f}_{\widehat{\Lambda}} - \widetilde{\mathbf{f}}_{\widehat{\Lambda}}\|_{\ell_2} \leq \mathrm{tol}_{\,\mathrm{iter}}\|\mathbf{f}_{\widehat{\Lambda}}\|_{\ell_2}$ and return $\widetilde{\mathbf{r}}_{\widehat{\Lambda}}$ according to (4.6).

Remark 4.4. The complexity of the routines **LINSOLVE** and **RESIDUAL** is in general dominated by the computation of matrix vector products of type $\mathbf{A}_{J,\Lambda}\mathbf{v}_\Lambda$ where $J = J(\mathrm{tol}_{\,\mathrm{iter}})$ depends on $\mathrm{tol}_{\,\mathrm{iter}}$. For simplicity, we consider the case where $\mathcal{A}$ is a constant coefficient differential operator and $\mathbf{A} \in \mathcal{B}_{s^*}$ with $s^* = \infty$. Thus, we can apply the compression scheme from Theorem 3.1 and infer that the matrix vector product can be computed within $\mathcal{O}(J(\mathrm{tol}_{\,\mathrm{iter}}) \cdot \#\Lambda)$ operations. To provide linear complexity for *any* tolerance $\mathrm{tol}_{\,\mathrm{iter}} > 0$ and any finite index set Λ, one has to prove the existence of a constant $C > 0$ independent of Λ such that $J(\mathrm{tol}_{\,\mathrm{iter}}) \leq C$ for all $\mathrm{tol}_{\,\mathrm{iter}} > 0$. But, as opposed to **GALSOLVE**, J *cannot* be chosen *independent* of the current target tolerance $\mathrm{tol}_{\,\mathrm{iter}}$ (even when $s^* = \infty$). Note that decreasing $\mathrm{tol}_{\,\mathrm{iter}}$ successively is necessary for the convergence of **S-ADWAV** as we shall explain in Section 4.1.4 below. So, the compression level J is a decreasing function of $\mathrm{tol}_{\,\mathrm{iter}}$, i.e., $J(\mathrm{tol}_{\,\mathrm{iter}}) \to \infty$ when $\mathrm{tol}_{\,\mathrm{iter}} \to 0$ (cf., e.g., line 1 in Algorithm 6). For these reasons, a scheme using **LINSOLVE/RESIDUAL** can in general not expected to be *asymptotically* of linear complexity.

4.1.3. *Coefficient thresholding.* Obviously, if we call iteratively $\Lambda^{(k+1)} = \mathbf{C}(\Lambda^{(k)}, c)$ starting with some initial set $\Lambda^{(0)}$, the sizes of the index sets $(\Lambda^{(k)})_{k\in\mathbb{N}}$ may grow exponentially fast. For this reason, we have to keep the index sets for which we call $\mathbf{C}$ small. This is realized by the routine **THRESH** (cf. Algorithm 9) which thresholds the wavelet coefficients in $\widetilde{\mathbf{u}}_{\Lambda^{(k)}}$ and in the estimated residual $\widetilde{\mathbf{r}}_{\Lambda^{(k)}}$. More precisely, given $\delta > 0$ and a finitely supported vector $\mathbf{v}$, **THRESH** returns a vector $\bar{\mathbf{v}}$ such that $\|\mathbf{v} - \bar{\mathbf{v}}\|_{\ell_2} \leq \delta$. Here, also approximate sorting procedures from [1, 15] can be used, so that **THRESH** can be realized in linear complexity. Note that at least one scaling function index remains in the thresholded output of $\mathbf{THRESH}[\mathbf{v}, \delta]$ (which is therefore never empty). This is important since if Λ does not contain scaling function indices, also $\mathbf{C}[\Lambda, c]$ does not contain any.

Algorithm 9 $\mathbf{THRESH}[\mathbf{v}, \delta] \to \bar{\mathbf{v}}$

1: Sort $\mathbf{v} = (\mathbf{v}_\lambda)_{\lambda\in\mathrm{supp}\,\mathbf{v}}$ by decreasing order to obtain $\mathbf{v}^* = (\mathbf{v}^*_{(i,\lambda_i)})_{i=1,\dots,N}$ where $N := \#\mathrm{supp}\,\mathbf{v}$ and (i, λ_i) for $i = 1, \dots, N$ indicate the ordering in $\mathbf{v}^*$ and the corresponding index in $\mathrm{supp}\,\mathbf{v}$. Within this sorting, find the index λ^*_S corresponding to the largest scaling function coefficient in modulus in $\mathbf{v}$.
2: Compute $\|\mathbf{v}\|_{\ell_2}$ and set $\overline{\mathbf{v}} := (\mathbf{v}_{\lambda_i})_{i=1,\dots,K} \cup \{\mathbf{v}_{\lambda^*_S}\}$ where K is the smallest integer such that $\sum_{i=1}^K |\mathbf{v}^*_{(i,\lambda_i)}|^2 \geq \|\mathbf{v}\|^2_{\ell_2} - \delta^2$.

Now, we investigate the effect of **THRESH**. Under the assumptions of Proposition 4.1, let $\mathbf{u}_\Lambda$ be the solution of (2.9) and $\widetilde{\mathbf{u}}_\Lambda := \mathbf{LINSOLVE}[\Lambda, \mathbf{w}_\Lambda, \mathrm{tol}_{\,\mathrm{iter}}]$. Then, as $\|\mathbf{u}_\Lambda - \widetilde{\mathbf{u}}_\Lambda\|_{\ell_2} \lesssim \mathrm{tol}_{\,\mathrm{iter}}\|\mathbf{f}_\Lambda\|_{\ell_2}$, it holds for $\bar{\mathbf{u}} = \mathbf{THRESH}[\widetilde{\mathbf{u}}_\Lambda, \mathrm{tol}_{\,\mathrm{iter}}]$ that $\|\mathbf{u}_\Lambda - \bar{\mathbf{u}}\|_{\ell_2} \lesssim \mathrm{tol}_{\,\mathrm{iter}}(1 + \|\mathbf{f}_\Lambda\|_{\ell_2})$, where $\mathrm{supp}\,\bar{\mathbf{u}} \subseteq \Lambda$. Hence, we get an approximation of order $\mathrm{tol}_{\,\mathrm{iter}}$ to $\mathbf{u}_\Lambda$ with, in general, a smaller support. This observation can also be made for the residual computation. Let $\mathbf{r}_{\widehat{\Lambda}}$ be the residual defined in (4.6) and $\widetilde{\mathbf{r}}_{\widehat{\Lambda}} = \mathbf{RESIDUAL}[\widehat{\Lambda}, \widetilde{\mathbf{u}}_\Lambda, \mathrm{tol}_{\,\mathrm{iter}}]$. Then, by Proposition 4.3, $\|\mathbf{r}_{\widehat{\Lambda}} - \widetilde{\mathbf{r}}_{\widehat{\Lambda}}\|_{\ell_2} \lesssim \mathrm{tol}_{\,\mathrm{iter}}\|\mathbf{f}_{\widehat{\Lambda}}\|_{\ell_2}$. Thus, for $\bar{\mathbf{r}} = \mathbf{THRESH}[\widetilde{\mathbf{r}}_{\widehat{\Lambda}}, \mathrm{tol}_{\,\mathrm{iter}}]$, it holds $\|\mathbf{r}_{\widehat{\Lambda}} - \bar{\mathbf{r}}\|_{\ell_2} \lesssim \mathrm{tol}_{\,\mathrm{iter}}(1 + \|\mathbf{f}_{\widehat{\Lambda}}\|_{\ell_2})$ and, as above, $\mathrm{supp}\,\bar{\mathbf{r}} \subseteq \mathrm{supp}\,\widetilde{\mathbf{r}}_{\widehat{\Lambda}}$.

4.1.4. *The simplified algorithm* **S-ADWAV**. With all necessary routines at hand, we can describe the complete algorithm. In each iteration of **S-ADWAV** (cf.

Algorithm 10), we compute an approximate solution $\widetilde{\mathbf{u}}^{(k)}$ to $\mathbf{A}_{\Lambda_k^{\text{cand.}}}\mathbf{u}_{\Lambda_k^{\text{cand.}}} = \mathbf{f}_{\Lambda_k^{\text{cand.}}}$ where $\Lambda_k^{\text{cand.}}$ is referred to as the set of *candidate* indices, i.e., indices that *can be activated* in the current iteration. The target precision for solving this Galerkin system is tol_{iter}. Next, we threshold the vector $\widetilde{\mathbf{u}}^{(k)}$ to obtain the *active* wavelet coefficients $\mathbf{u}^{(k)}$ that satisfy $\|\mathbf{u}^{(k)} - \widetilde{\mathbf{u}}^{(k)}\|_{\ell_2} \leq \text{tol}_{\text{iter}}$. We refer to $\Lambda^{(k)} := \text{supp } \mathbf{u}^{(k)}$ as the set of *active* indices. Around $\Lambda^{(k)}$, the security zone $\widehat{\Lambda}_k$ is constructed using the routine **C** and the residual $\mathbf{r}^{(k)}$ is computed by **RESIDUAL**. If $\|\mathbf{r}^{(k)}\|_{\ell_2} \leq \varepsilon\|\widetilde{\mathbf{f}}_{\widehat{\Lambda}_k}\|_{\ell_2}$, we accept $\mathbf{u}^{(k)}$ as solution. Otherwise, a new candidate set of activable indices $\Lambda_{k+1}^{\text{cand.}}$ is constructed by thresholding $\mathbf{r}_{\widehat{\Lambda}}$.

As we always use the same tolerance tol_{iter} for thresholding and the numerical solution of the Galerkin system, the approximation errors we generate are all of order tol_{iter} (see Propositions 4.1 and 4.3 as well as Paragraph 4.1.3). But if we fix this tolerance, it may happen that the algorithm stagnates before the target accuracy ε is reached. Namely, by thresholding the approximate Galerkin solution $\widetilde{\mathbf{u}}^{(k)}$, it may occur that no higher levels or translation indices on the coarsest level are added in the course of the algorithm and we end up with $\Lambda^{(k)} = \Lambda^{(k+1)}$. Therefore, in addition to the algorithm described in [2, 32], we decrease the threshold tolerance tol_{iter} by the factor $\frac{1}{2}$ if the difference of the residuals of two iterations is too close to zero (see line 11). In particular, when $\mathbf{r}^{(k-1)} = \mathbf{r}^{(k)}$, the threshold tolerance is decreased and we obtain in the next iteration $\widetilde{\mathbf{u}}^{(k)} = \widetilde{\mathbf{u}}^{(k+1)}$ but $\text{supp } \mathbf{u}^{(k+1)} \supseteq \text{supp } \mathbf{u}^{(k)}$. Thus, also finer information on high levels or further translations on the coarsest level remain in the set of active indices. Moreover, to prevent cycles of type $\Lambda_k = \Lambda_{k+m}$ for some $m \geq 2$, we add an inner loop with a maximal number M of iterations which also ensures that the tolerance tol_{iter} decreases. As it was pointed out in [2, p.2118], tol_{iter} should, in order to attain an approximation of order $\mathcal{O}(\varepsilon\|\mathbf{f}\|_{\ell_2})$, be *much smaller* than ε which is incorporated in line 7.

The adaptive truncation of a computational domain, i.e., $\text{supp } (\mathbf{u}^{(m,k)})^T \Psi_{t,j_0}$, is done implicitly. Every time $\mathbf{C}[\Lambda^{(k,m)}, c]$ is called, additional scaling function indices on the coarsest level are added to the security zone $\widehat{\Lambda}^{(k,m)}$. If these indices are significant, their corresponding value in $\mathbf{r}^{(k,m)}$ is relatively large and they will be added to the new candidate set $\Lambda_{k+1,m}^{\text{cand.}}$ after the call of $\mathbf{THRESH}[\mathbf{r}^{(k,m)}, \text{tol}_{\text{iter}}]$. So, in each iteration, the computational domain can be extended, but also truncated as we have another call of **THRESH** after solving the Galerkin system.

4.1.5. *Choice of a minimal level and an initial index set.* As for **ADWAV**, the choice of a minimal level is crucial. Here, we proceed as in Section 3.3.2. Moreover, we define $\Lambda_{1,1}^{\text{cand.}}$ as the scaling function index with the same level and translation index as the largest (estimated) wavelet coefficient.

4.1.6. *Convergence and complexity.* As already mentioned, there is no proof for the convergence of **S-ADWAV**. In view of Remark 4.4, this algorithm is in general *asymptotically* not of linear complexity. Nevertheless, we observed in our numerical experiments that for *moderate* target tolerances ε, **S-ADWAV** is still an efficient numerical algorithm. This is due to the facts that $J(\text{tol}_{\text{iter}})$ (cf. Remark 4.4) does not grow fast and that we use solutions from former iterations as initial guesses in **LINSOLVE** to keep the number of iterations of the linear solver small.

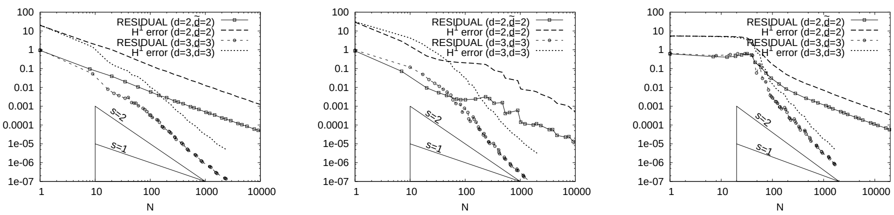
Algorithm 10 $[\mathbf{u}(\varepsilon), \Lambda(\varepsilon)] =$ **S-ADWAV**$[\varepsilon]$

Let M a fixed number of inner loops, $c > 0$ and $h > 0$, $\rho > 0$ a tuning parameters, $0 < \mathrm{tol}_{\,\mathrm{iter}} < 1$ an initial tolerance and $\Lambda_{1,1}^{\mathrm{cand.}}$ an initial index set.

for $k = 1, 2, 3, \ldots$ **do** $\mathrm{tol}_{\,\mathrm{iter}} = \frac{1}{2}\mathrm{tol}_{\,\mathrm{iter}}$
  for $m = 1, 2, \ldots, M$ **do**
    $\widetilde{\mathbf{u}}^{(k,m)} = \mathbf{LINSOLVE}[\Lambda_{k,m}^{\mathrm{cand.}}, \mathbf{u}^{(k,m-1)}, \rho \cdot \mathrm{tol}_{\,\mathrm{iter}}]$
    $\mathbf{u}^{(k,m)} = \mathbf{THRESH}[\widetilde{\mathbf{u}}^{(k,m)}, \mathrm{tol}_{\,\mathrm{iter}}]$
    $\Lambda^{(k,m)} = \mathrm{supp}\,\mathbf{u}^{(k,m)}$; $\widehat{\Lambda}_{k,m} = \mathbf{C}[\Lambda^{(k,m)}, c]$
    $\mathbf{r}^{(k,m)} = \mathbf{RESIDUAL}[\widehat{\Lambda}_{k,m}, \mathbf{u}^{(k,m)}, \mathrm{tol}_{\,\mathrm{iter}}]$
    if $\|\mathbf{r}^{(k,m)}\|_{\ell_2} \leq \varepsilon \|\widetilde{\mathbf{f}}_{\widehat{\Lambda}_{k,m}}\|_{\ell_2}$ **and** $\mathrm{tol}_{\,\mathrm{iter}} \leq \rho \cdot \varepsilon$ **then**
      return $\mathbf{u}(\varepsilon) := \mathbf{u}^{(k,m)}$, $\Lambda(\varepsilon) := \Lambda^{(k,m)}$;
    endif
    $\bar{\mathbf{r}}^{(k,m)} = \mathbf{THRESH}[\mathbf{r}^{(k,m)}, \mathrm{tol}_{\,\mathrm{iter}}]$;
    if $\|\mathbf{r}^{(k,m)} - \mathbf{r}^{(k,m-1)}\|_{\ell_2} < h \times \mathrm{tol}_{\,\mathrm{iter}} \times \|\mathbf{r}^{(k,m-1)}\|_{\ell_2}$
      $\Lambda_{k+1,1}^{\mathrm{cand.}} = \Lambda^{(k,m)} \cup \mathrm{supp}\,\bar{\mathbf{r}}^{(k,m)}$; $\mathbf{u}^{(k+1,0)} := \mathbf{u}^{(k,m)}$; $\mathbf{r}^{(k+1,0)} := \mathbf{r}^{(k,m)}$;
      break;
    else
      $\Lambda_{k,m+1}^{\mathrm{cand.}} = \mathrm{supp}\,\mathbf{u}^{(k,m)} \cup \mathrm{supp}\,\bar{\mathbf{r}}^{(k,m)}$
    end if
  end for
end for

4.2. **Numerical experiments.** In this section, we present numerical results obtained with **S-ADWAV**. We focus on the reaction-diffusion problems from Section 3.3 and compare the results with those obtained by **ADWAV**.

4.2.1. *Convergence rates.* Within **S-ADWAV**, we used the parameters $c = 0.125$, $h = 0.0001$, $M = 2$, $\rho = 0.1$, $\mathrm{tol}_{\,\mathrm{iter}} = 0.01$, $d = \widetilde{d} = 2$, $d = \widetilde{d} = 3$ and the wavelet basis from [9]. We observe in Figure 4.5 that both the output $\|\mathbf{r}^{(k,m)}\|_{\ell_2}$ of **RESIDUAL** as well as the corresponding approximation error measured in $H^1(\mathbb{R})$ converge asymptotically with the same rate as **ADWAV** (cf. Section 3.3.3).



(a) (P1) with $j_0 = -2$. (b) (P2) with $j_0 = -4$. (c) (P3) with $j_0 = -1$.

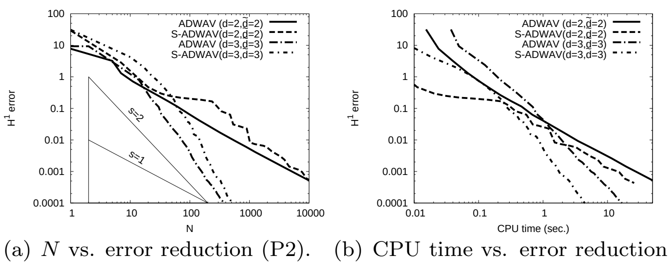
FIGURE 4.5. $H^1(\mathbb{R})$-error and $\|\mathbf{r}^{(k,m)}\|_{\ell_2}$ vs. degrees of freedom N.

4.2.2. *Thresholding and cg iterations.* Within **THRESH**, we could replace the *absolute* threshold by a *relative* threshold, i.e., replace δ by $\delta\|\mathbf{v}\|_{\ell_2}$ in Algorithm 9. In this case, the output $\bar{\mathbf{v}} =$**THRESH**$[\mathbf{v}, \delta\|\mathbf{v}\|_{\ell_2}]$ satisfies $\|\mathbf{v} - \bar{\mathbf{v}}\|_{\ell_2} \leq \delta\|\mathbf{v}\|_{\ell_2}$. A relative threshold (as it was used in [2]) seems convenient as we are using, as far as possible, only *relative* tolerances in **S-ADWAV**. Nevertheless, our experiments show that this approach may have an important drawback. We observe in Table 4.2 for (P3) that for relative thresholding, the quotient $C := \frac{\#\text{supp } \mathbf{u}^{(k,m)}}{\#\text{supp } \tilde{\mathbf{u}}^{(k,m)}}$ is much smaller than the corresponding one for absolute thresholding. Since we have to solve a linear system with $\#\text{supp } \tilde{\mathbf{u}}^{(k,m)}$ degrees of freedom, a *large* value of C is favorable to get an efficient algorithm even though **S-ADWAV** with relative threshold might produce better approximations *at early stages*. We also observe that few cg-iterations in **LINSOLVE** are sufficient.

Iteration		5	10	15	20	25	30	35
rel. thresh	N	25	59	98	131	163	233	349
	cg-its	8	7	5	7	7	7	8
	Err.	3.8e-0	1.9.0e-1	4.1e-2	1.3e-2	6.2e-3	2.7e-3	7.0e-4
	C	86%	40%	46%	46%	39%	43%	44%
abs. thresh	N	29	74	115	150	219	329	456
	cg-its	7	3	3	4	3	4	4
	Err.	3.7e-0	8.3e-2	9.0e-2	5.4e-2	1.4e-3	4.8e-4	1.8e-4
	C	93%	96%	99%	99%	97%	95%	98%

Table 4.2. Aabsolute and relative threshold for (P3) and $d = 3$ where $N := \#\text{supp } \mathbf{u}^{(k,m)}$ and Err. is the $H^1(\mathbb{R})$-error.

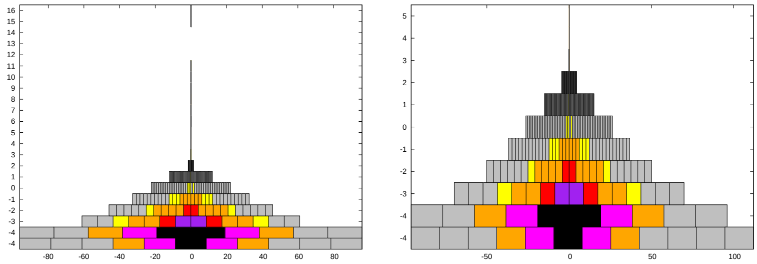
4.2.3. *Comparison of* **ADWAV** *and* **S-ADWAV**. As an example, we consider (P2) to compare the two adaptive schemes (cf. Figure 4.6). We observe that **ADWAV** needs less degrees of freedom compared to **S-ADWAV**. This is due to the fact that within **GROW** higher levels for the resolution of a singularity can be added within one iteration whereas the routine **C** can add at most wavelets on the next higher level compared to the levels $|\lambda|$, $\lambda \in \Lambda^{(k,m)}$ in the current approximation $\Lambda^{(k,m)}$. Nevertheless, the computation times state that this effect is compensated in **S-ADWAV** where we do not need the **APPLY** routine.



(a) N vs. error reduction (P2). (b) CPU time vs. error reduction (P2).

Figure 4.6. Comparison between **ADWAV** and **S-ADWAV**.

In Figure 4.7, we show examples of the structure of the index sets produced by **ADWAV** and **S-ADWAV**. We see that for a comparable size of index sets, **ADWAV** uses the information provided by **RHS** to add higher levels already at early stages of the algorithm. As already said above, this is not the case for **S-ADWAV**. Nevertheless, both algorithms detect the singularity.



(a) **ADWAV**, $\#\mathrm{supp}\,\mathbf{w}^{(k)} = 210$. (b) **S-ADWAV**, $\#\mathrm{supp}\,\mathbf{u}^{(k,m)} = 250$.

FIGURE 4.7. Coefficients $\mathbf{w}^{(k)}$, $\mathbf{u}^{(k,m)}$ for (P2) for $d = \widetilde{d} = 2$.

4.2.4. *A convection-diffusion problem.* For the reaction-diffusion examples one might argue that it would also be possible to a determine a computational domain a priori and then to use standard methods for PDEs on bounded domains. In order to treat a problem where this is not that obvious, we consider a convection diffusion problem in weak form: Find $u \in H^1(\mathbb{R})$ with

$$(\partial v, \partial u)_{L_2(\mathbb{R})} + \beta(v, \partial u)_{L_2(\mathbb{R})} + (v, u)_{L_2(\mathbb{R})} = \langle v, f\rangle, \quad \forall v \in H^1(\mathbb{R}), \tag{4.7}$$

using the right-hand side from (P1) which also fulfills all required assumptions. For increasing values of β, the solution exhibits a strong layer at $x = 0$, see the left part of Figure 4.8. On the right, we see the adaptive truncation of the computational domain. In particular, the layer is automatically detected.

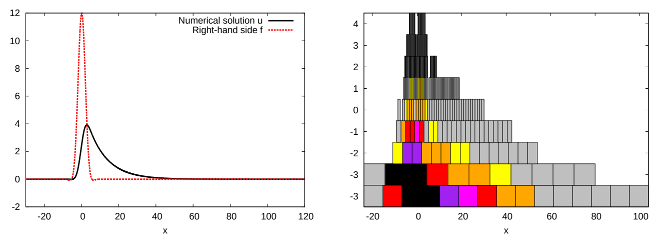


FIGURE 4.8. Solution u, right-hand side f (left) and estimated index set (right) for (4.7) with $d = \widetilde{d} = 2$, $j_0 = -2$ and $\beta = 10$.

5. EXTENSION TO HIGHER SPACE DIMENSIONS

So far, we considered the well-posed operator equation $\mathcal{A}[u] = f$ in H' in one space dimension. To use the same approach to solve (2.1) also in higher space dimensions, we briefly describe how to revise the ingredients of **ADWAV** where we focus on $H = H^t(\mathbb{R}^n)$, $t \in \mathbb{N}$. The complete analysis of the multivariate case goes beyond the scope of the present paper and can e.g. be found in [21].

5.1. **Tensor wavelet bases.** We start with the construction of a wavelet basis $\mathbf{\Psi}_{t,\mathbf{j}_0}$ for $H^t(\mathbb{R}^n)$, $t \in \mathbb{N}$. To this end, let $\Psi_{0,j_0^{(1)}}, \dots, \Psi_{0,j_0^{(n)}}$ be Riesz wavelet bases for $L_2(\mathbb{R})$ of order d with $j_0^{(i)} \in \mathbb{Z} \cup \{-\infty\}$ for $i = 1, \dots, n$ that are, properly scaled, also Riesz bases for $H^t(\mathbb{R})$ (cf. Section 2.2). Then, since we can identify $H^t(\mathbb{R}^n)$ with $\mathcal{H}^t(\mathbb{R}^n) := \cap_{k=1}^n \otimes_{m=1}^n H^{0+t\cdot\delta_{m,k}}(\mathbb{R})$, it can be proven (cf. [20]) that

$$\mathbf{\Psi}_{t,\mathbf{j}_0} := \left\{\mathbf{D}^t_{\boldsymbol{\lambda}}\boldsymbol{\psi}_{\boldsymbol{\lambda}} : \boldsymbol{\lambda} := (\lambda_1,\ldots,\lambda_n) \in \mathbf{J}_{\mathbf{j}_0} := \textstyle\prod_{i=1}^n \mathcal{J}_{j_0^{(i)}}\right\},\ \mathbf{j}_0 := (j_0^{(1)},\ldots,j_0^{(n)})$$

is a Riesz basis for $H^t(\mathbb{R}^n)$ where $\mathbf{D}^t_{\boldsymbol{\lambda}} := \|\boldsymbol{\psi}_{\boldsymbol{\lambda}}\|^{-1}_{H^t(\mathbb{R}^n)}$, $\boldsymbol{\psi}_{\boldsymbol{\lambda}} := \otimes_{i=1}^n \psi_{\lambda_i}$ for $\psi_{\lambda_i} \in \Psi_{0,j_0^{(i)}}$ $(i = 1,\ldots,n)$. Thus, there exist Riesz constants $c_{\mathbf{\Psi}_{t,\mathbf{j}_0}}, C_{\mathbf{\Psi}_{t,\mathbf{j}_0}} > 0$ such that

$$c_{\mathbf{\Psi}_{t,\mathbf{j}_0}}\|\mathbf{v}\|_{\ell_2(\mathbf{J})} \le \|\mathbf{v}^T\mathbf{\Psi}_{t,\mathbf{j}_0}\|_{H^t(\mathbb{R}^n)} \le C_{\mathbf{\Psi}_{t,\mathbf{j}_0}}\|\mathbf{v}\|_{\ell_2(\mathbf{J})}, \quad \forall \mathbf{v} \in \ell_2(\mathbf{J}). \tag{5.1}$$

We refer to $\boldsymbol{\psi}_{\boldsymbol{\lambda}}$ as an *(anisotropic) tensor* wavelet as it may have *different* levels $|\lambda_1|,\ldots,|\lambda_n|$ in different coordinate directions. For $|\boldsymbol{\lambda}| := (|\lambda_1|,\ldots,|\lambda_n|)$, we define $\mathrm{sum}(|\boldsymbol{\lambda}|) := \sum_{i=1}^n |\lambda_i|$ and $\max(|\boldsymbol{\lambda}|) := \max_{i=1,\ldots,n} |\lambda_i|$. Remind that $|\lambda_i|$ $(i = 1,\ldots,n)$ can assume negative values and therefore, $\mathrm{sum}(|\boldsymbol{\lambda}|)$, $\max(|\boldsymbol{\lambda}|)$ can get negative. In view of (2.7), we now define $\mathbf{A} := \langle \mathbf{\Psi}_{t,\mathbf{j}_0}, \mathcal{A}[\mathbf{\Psi}_{t,\mathbf{j}_0}]\rangle$ and $\mathbf{f} := \langle \mathbf{\Psi}_{t,\mathbf{j}_0}, f\rangle$.

For the construction of $\mathbf{\Psi}_{t,\mathbf{j}_0}$, the choice of the univariate Riesz bases is delicate. One can prove the following estimates for $c_{\mathbf{\Psi}_{t,\mathbf{j}_0}}$, $C_{\mathbf{\Psi}_{t,\mathbf{j}_0}}$ analogously to [15, p.80]:

$$c_{\mathbf{\Psi}_{t,\mathbf{j}_0}} = \min_i \left\{ \min\{c_{\Psi_{0,j_0^{(i)}}}, c_{\Psi_{t,j_0^{(i)}}}\} \textstyle\prod_{k\neq i} c_{\Psi_{0,j_0^{(k)}}} \right\}, \tag{5.2}$$

$$C_{\mathbf{\Psi}_{t,\mathbf{j}_0}} = \max_i \left\{ \max\{C_{\Psi_{0,j_0^{(i)}}}, C_{\Psi_{t,j_0^{(i)}}}\} \textstyle\prod_{k\neq i} C_{\Psi_{0,j_0^{(k)}}} \right\}, \tag{5.3}$$

where $c_{\Psi_{\nu,j_0^{(i)}}}, C_{\Psi_{\nu,j_0^{(i)}}}$ denote the Riesz constants of the Riesz bases $\Psi_{\nu,j_0^{(i)}}$ for $\nu \in \{0,t\}$ and $i = 1,\ldots,n$ (cf. (2.3)). Thus, in general, the Riesz constants grow *exponentially* in the dimension n. However, as a small condition number $\kappa(\mathbf{A}) \sim C^2_{\mathbf{\Psi}_{t,\mathbf{j}_0}} \cdot c^{-2}_{\mathbf{\Psi}_{t,\mathbf{j}_0}}$ is favorable (cf. Section 3.3), one can use univariate piecewise polynomial, *orthonormal multiwavelet* constructions where $c_{\Psi_{0,j_0^{(i)}}} = C_{\Psi_{0,j_0^{(i)}}} = 1$. The construction principle from Section 2.2 is still the same, the *only* difference is that we have multiple mother wavelets $\psi^1,\ldots,\psi^{m_\psi}$ and scaling functions $\varphi^1,\ldots,\varphi^{m_\varphi}$ for $m_\psi, m_\varphi \in \mathbb{N}$ (cf. (2.5) and (2.6)). As an example, an L_2-orthonormal multiwavelet basis for $L_2(\mathbb{R})$ with $j_0 = -\infty$ takes the form $\Psi_{0,-\infty} = \{\psi^\mu_{j,k} := 2^{j/2}\psi^\mu(2^j\cdot -k) : j,k \in \mathbb{Z}, \mu \in \{1,\ldots,m_\psi\}\}$. The results from Sections 3.1 and 3.2 can then be extended to multiwavelet settings if the assumptions made in Section 3.1.1 are satisfied for *each* mother wavelet (and scaling function). In the sequel (except for Section 5.2), we focus on constructions (e.g. [17]) that satisfy these assumptions.

5.2. **Nonlinear approximation for tensor wavelet bases.** Let $u = \mathbf{u}^T\mathbf{\Psi}_{t,\mathbf{j}_0}$ be the expansion of the solution of (2.1) in $\mathbf{\Psi}_{t,\mathbf{j}_0}$. It can be proven that if for $0 < s < d - t$ and $\tau^{-1} = s + \frac{1}{2}$, $u \in \bigcap_{k=1}^n \bigotimes_{\tau}{}_{i=1}^{n} B^{s+t\cdot\delta_{i,k}}_\tau(L_\tau(\mathbb{R}))$, then $\mathbf{u} \in \mathcal{A}^s$ (cf. [25, 28]). Here, $\otimes_\tau$ denotes a tensor product for *τ-placid* quasi-Banach spaces introduced in [25]. Note that for $0 < \tau < 1$, the Besov space $B^s_\tau(L_\tau(\mathbb{R}))$ is a (τ-placid) quasi-Banach space (cf. [28, p.784]). We refer to [25, 28] for the details.

We observe that a best nonlinear approximation rate *independent* of the dimension n can be attained asymptotically. This is why we choose anisotropic wavelet constructions over *isotropic* basis constructions. Given a sufficiently smooth, univariate scaling function φ of order d and a corresponding wavelet ψ, an isotropic Riesz wavelet basis for $L_2(\mathbb{R}^n)$ is exemplarily defined by, denoting by E the nontrivial vertices of $[0,1]^n$,

$$\mathbf{\Psi}^{iso}_0 := \{\boldsymbol{\psi}^{iso,e}_{j,\mathbf{k}} := 2^{jd/2}\boldsymbol{\psi}^{iso,e}(2^j\cdot -\mathbf{k}) : j \in \mathbb{Z},\ \mathbf{k} \in \mathbb{Z}^n,\ e \in E\},\ \boldsymbol{\psi}^{iso,e} := \otimes_{i=1}^n \psi^{e_i},$$

where $\psi^0 := \varphi$, $\psi^1 := \psi$. Normalizing $\boldsymbol{\psi}^{iso,e}_{j,\mathbf{k}}$ in $H^t(\mathbb{R}^n)$ yields an isotropic Riesz basis $\boldsymbol{\Psi}^{iso}_t$ for $H^t(\mathbb{R}^n)$. Here, the level j does *not* depend on the coordinate direction (explaining the notion *isotropic*). For $u = (\mathbf{u}^{iso})^T \boldsymbol{\Psi}^{iso}_t$, it can be shown (e.g. [5, Theorem 38.2]) that if for $0 < s < \frac{d-t}{n}$ and $\tau^{-1} = s + \frac{1}{2}$, $u \in B^{sn+t}_\tau(L_\tau(\mathbb{R}^n))$, then $\mathbf{u}^{iso} \in \mathcal{A}^s$. Thus, even when $u \in C^\infty$, s depends on the dimension n.

5.3. **Compressibility and computability of A.** We recall that for optimality of **ADWAV**, we require that $\mathbf{A} \in \mathcal{B}_{s^*}$ with $s^* > d-t$ (cf. Remark 2.4). In view of the numerical examples below, we state a compression scheme from [15]. Even though the original result is stated for bounded domains, it also applies to the unbounded setting as it relies on tensor product arguments applied to compression results from univariate settings where we can use the results from Section 3.1.2.

Theorem 5.1 ([15, Theorem 5.3.5])**.** *Let $\mathcal{A}[w] := -\Delta w + c \cdot w$ for $c > 0$ and $\boldsymbol{\Psi}_{t,\mathbf{j}_0}$ an orthonormal tensor wavelet basis of order $d \geq 2$ with $\mathbf{j}_0 \in (\mathbb{Z} \cup \{-\infty\})^n$. Then,*

$$\mathbf{A} = \mathbf{D}\,[\mathbf{A}^{(1)} \otimes \mathbf{I}^{(2)} \otimes \cdots \otimes \mathbf{I}^{(n)} + \cdots + \mathbf{I}^{(1)} \otimes \cdots \otimes \mathbf{I}^{(n-1)} \otimes \mathbf{A}^{(n)} + c \cdot \mathbf{I}]\,\mathbf{D},$$

where $\mathbf{A}^{(i)} := ((\partial\psi_{\lambda_i}, \partial\psi_{\lambda'_i})_{L_2})_{\lambda,\lambda'_i \in \mathcal{J}_{j_0^{(i)}}}$, $\mathbf{I}^{(i)} := (\delta_{\lambda_i,\lambda'_i})_{\lambda,\lambda'_i \in \mathcal{J}_{j_0^{(i)}}}$, $\mathbf{I} := \mathbf{I}^{(1)} \otimes \cdots \otimes \mathbf{I}^{(n)}$ and $\mathbf{D} := (\|\boldsymbol{\psi}_{\boldsymbol{\lambda}}\|^{-1}_{H^1(\mathbb{R}^n)} \cdot \delta_{\lambda_1,\lambda'_1} \cdots \delta_{\lambda_n,\lambda'_n})_{\boldsymbol{\lambda},\boldsymbol{\lambda}' \in \mathbf{J}_{\mathbf{j}_0}}$. Defining $\mathbf{A}^{(i)}_j$ by dropping all entries from $\mathbf{A}^{(i)}$ when $\delta(\lambda_i, \lambda'_i) > j$ for $i = 1, \ldots, n$ yields that $\|\mathbf{A} - \mathbf{A}_j\| \lesssim 2^{-(d-\frac{1}{2}-t)j}$ with a constant independent of n where

$$\mathbf{A}_j := \mathbf{D}\,[\mathbf{A}^{(1)}_j \otimes \mathbf{I}^{(2)} \otimes \cdots \otimes \mathbf{I}^{(n)} + \cdots + \mathbf{I}^{(1)} \otimes \cdots \otimes \mathbf{I}^{(n-1)} \otimes \mathbf{A}^{(n)}_j + c \cdot \mathbf{I}]\,\mathbf{D}.$$

In particular, the number of non-zeros in each row and column of $\mathbf{A}_j$ is of order $\mathcal{O}(nj)$ and $\mathbf{A}$ is s^*-computable with $s^* = \infty$. □

Note that $\mathbf{A}$ in Theorem 5.1 is *close* to a sparse matrix which is due to L_2-orthonormality of the univariate multiwavelets. This simple structure of $\mathbf{A}_j$ allows for a more efficient numerical implementation of **APPLY** than with other basis types where, however, similar compressibility results can be shown ([27, Proposition 8.1]).

The s^*-computability of $\mathbf{A} = (\mathbf{a}_{\boldsymbol{\lambda},\boldsymbol{\lambda}'})_{\boldsymbol{\lambda},\boldsymbol{\lambda}' \in \mathbf{J}_{\mathbf{j}_0}}$ can also be proven for non-constant, but sufficiently smooth coefficients $g_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, i.e., for $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}_0^n$ multi-indices, when

$$\mathbf{a}_{\boldsymbol{\lambda},\boldsymbol{\lambda}'} = \|\boldsymbol{\psi}_{\boldsymbol{\lambda}}\|^{-1}_{H^t(\mathbb{R}^n)} \|\boldsymbol{\psi}_{\boldsymbol{\lambda}'}\|^{-1}_{H^t(\mathbb{R}^n)} \sum_{|\boldsymbol{\alpha}|_{\ell_1}, |\boldsymbol{\beta}|_{\ell_1} \leq t} \int_{\mathbb{R}^n} g_{\boldsymbol{\alpha},\boldsymbol{\beta}}\, \partial^{\boldsymbol{\alpha}} \boldsymbol{\psi}_{\boldsymbol{\lambda}}\, \partial^{\boldsymbol{\beta}} \boldsymbol{\psi}_{\boldsymbol{\lambda}'}.$$

Here, we can again use [26, Theorems 4.1 & 6.2] since estimates for $|\mathbf{a}_{\boldsymbol{\lambda},\boldsymbol{\lambda}'}|$ given there base on a *tensor product argument* applied to estimates in *univariate* settings (cf. [26, Lemma 3.1]). In Proposition 3.3, we have shown that these estimates also hold in unbounded settings when $j_0^{(i)} > -\infty$, $i = 1, \ldots, n$. We infer that the results from [26] can be extended to show that $\mathbf{A}$ is s^*-computable with $s^* = d + 1$.

5.4. **RHS.** We now show how to extend the results from Section 3.2 to the multivariate case when the right-hand side $\mathbf{f}$ is *separable* and when it is *non-separable.* For the sake of brevity, we focus on $j_0^{(i)} > -\infty$, $i = 1, \ldots, n$ and $t = 1$. More detailed results will be given in [21].

5.4.1. *Separable* $\mathbf{f}$. We first analyze the case where f is the product of n univariate functionals and $\mathbf{f} = \mathbf{f}^{(1)} \otimes \cdots \otimes \mathbf{f}^{(n)}$ with $\mathbf{f}^{(i)} \in \ell_2(\mathcal{J}_{j_0^{(i)}})$ for $i = 1, \ldots, n$. Based on the results from Section 3.2, we assume that $\mathbf{f}^{(i)}$ permits computable approximations $\mathbf{f}_J^{(i)}$ such that for some $\widetilde{s} > d-1$ and $c_{\widetilde{s}} > 0$, $\|\mathbf{f}^{(i)} - \mathbf{f}_J^{(i)}\|_{\ell_2} \leq c_{\widetilde{s}} 2^{-\widetilde{s}J}$ with $\#\mathrm{supp}\, \mathbf{f}_J^{(i)} \sim 2^J$ for all $J \geq 0$ and $i = 1, \ldots, n$. Analog to [27, Proposition 8.1], we define for $J \geq 0$, $\ell_1, \ldots, \ell_n \in \mathbb{N}_0$ and $\mathbf{f}_{-1}^{(i)} := \mathbf{0}$ $(i = 1, \ldots, n)$

$$\mathbf{f}_J := \textstyle\sum_{\ell_1+\cdots+\ell_n \leq J} \big(\mathbf{f}_{\ell_1}^{(1)} - \mathbf{f}_{\ell_1-1}^{(1)}\big) \otimes \cdots \otimes \big(\mathbf{f}_{\ell_n}^{(n)} - \mathbf{f}_{\ell_n-1}^{(n)}\big).$$

It holds that $\#\mathrm{supp}\, \mathbf{f}_J \lesssim J^n 2^J$ and one can show that for any $d-1 < \bar{s} < \widetilde{s}$

$$\begin{aligned}\|\mathbf{f} - \mathbf{f}_J\|_{\ell_2} &= \| \textstyle\sum_{\ell_1+\cdots+\ell_n > J} \big(\mathbf{f}_{\ell_1}^{(1)} - \mathbf{f}_{\ell_1-1}^{(1)}\big) \otimes \cdots \otimes \big(\mathbf{f}_{\ell_n}^{(n)} - \mathbf{f}_{\ell_n-1}^{(n)}\big)\|_{\ell_2} \\ &\lesssim c_{\widetilde{s}}^n\, J^n\, 2^{-\widetilde{s}J} \leq D_{\widetilde{s},\bar{s},n}\, c_{\widetilde{s}}^n\, (J^n \cdot 2^J)^{-\bar{s}} \lesssim D_{\widetilde{s},\bar{s},n}\, c_{\widetilde{s}}^n\, (\#\mathrm{supp}\, \mathbf{f}_J)^{-\bar{s}},\end{aligned}$$

where the constant $D_{\widetilde{s},\bar{s},n}$ can be chosen such that $J^{(1+\bar{s})n}\, 2^{-(\widetilde{s}-\bar{s})J} \leq D_{\widetilde{s},\bar{s},n}$ for all $J \geq 0$. Note that $D_{\widetilde{s},\bar{s},n} \to \infty$ for $\widetilde{s} - \bar{s} \to 0$ or $n \to \infty$. Thus, we infer that for $\eta > 0$, we can choose J such that $\|\mathbf{f} - \mathbf{f}_J\|_{\ell_2} \leq \eta$ with $\#\mathrm{supp}\, \mathbf{f}_J \lesssim \eta^{-1/\bar{s}}$. If $\mathbf{f}$ is a sum of such tensors, one only has to apply this technique to each summand.

5.4.2. *Non-separable* $\mathbf{f}$. Now, let $f \in L_2(\mathbb{R}^n)$ be non-separable such that for constants $C_f, C_{R^{(i)}} > 0$ and $R_\eta^{(i)} = C_{R^{(i)}} \eta^{-1/\beta}$, $i = 1, \ldots, n$,

$$\|f - f|_{\square_\eta}\|_{L_2(\mathbb{R}^n)} \leq C_f\, \eta, \quad \square_\eta := [-R_\eta^{(1)}, R_\eta^{(1)}] \times \cdots \times [-R_\eta^{(n)}, R_\eta^{(n)}]. \tag{5.4}$$

The same arguments as in the proof of Lemma 3.8 then permit to show that

$$\|\mathbf{f} - \mathbf{f}|_{\boldsymbol{\nabla}_\eta}\|_{\ell_2} \leq C_f C_{\boldsymbol{\Psi}_{t,\mathbf{j}_0}}^2 c_{\boldsymbol{\Psi}_{t,\mathbf{j}_0}}^{-1} \eta, \quad \boldsymbol{\nabla}_\eta := \{\boldsymbol{\lambda} \in \mathbf{J}_{\mathbf{j}_0} : |\mathrm{supp}\, \boldsymbol{\psi}_{\boldsymbol{\lambda}} \cap \square_\eta| > 0\}.$$

Thus, to approximate $\mathbf{f}|_{\boldsymbol{\nabla}_\eta}$, we only require a bound on the levels $|\boldsymbol{\lambda}|$ in $\boldsymbol{\nabla}_\eta$. To this end, note that when $f \in \otimes_{i=1}^n W^{d,\infty}(\mathbb{R})$, it holds (cf. [15, Eq. (5.24)])

$$|\langle \mathbf{D}_{\boldsymbol{\lambda}}^1 \boldsymbol{\psi}_{\boldsymbol{\lambda}}, f\rangle| \lesssim 2^{-((\frac{1}{2}+d)\mathrm{sum}(|\boldsymbol{\lambda}|)+\max(|\boldsymbol{\lambda}|))}, \quad \forall \boldsymbol{\lambda} \in \mathbf{J}_{\mathbf{j}_0}. \tag{5.5}$$

This estimate leads to the following definition (compare [15, Eq. (5.25)]) for $j \in \mathbb{N}_0$:

$$\boldsymbol{\nabla}_{\eta,j} := \{\boldsymbol{\lambda} \in \boldsymbol{\nabla}_\eta : (\tfrac{1}{2} + d)\, \mathrm{sum}(|\boldsymbol{\lambda}|) + \max(|\boldsymbol{\lambda}|) \leq (\tfrac{1}{2} + d + \tfrac{1}{n}) j\}.$$

With $|\square_\eta| \lesssim \eta^{-\frac{n}{\beta}}$, we have that $\#\boldsymbol{\nabla}_{\eta,j} \lesssim \eta^{-\frac{n}{\beta}} 2^j$ (which can be proven by [15, Proposition 3.3.4]) and $\|\mathbf{f}|_{\boldsymbol{\nabla}_\eta} - \mathbf{f}|_{\boldsymbol{\nabla}_{\eta,j}}\|_{\ell_2} \lesssim \eta^{-\frac{n}{2\beta}}\, 2^{-(d+\frac{1}{n})j}$. In particular, choosing $j \sim \lceil ((1 + \frac{n}{2\beta}) \log_2 \eta)/(d + \frac{1}{n})\rceil$ yields $\|\mathbf{f}|_{\boldsymbol{\nabla}_\eta} - \mathbf{f}|_{\boldsymbol{\nabla}_{\eta,j}}\|_{\ell_2} \lesssim \eta$ and $\#\boldsymbol{\nabla}_{\eta,j} \lesssim \eta^{-\frac{1}{\bar{s}}}$ with $\bar{s} := \frac{\beta(d+\frac{1}{n})}{(d+\frac{1}{2})n+\beta+1}$. Thus, we conclude that if β is sufficiently large, we have $\bar{s} > d-1$. Note that to approximate $\langle \mathbf{D}_{\boldsymbol{\lambda}}^1 \boldsymbol{\psi}_{\boldsymbol{\lambda}}, f\rangle$, one can apply *sparse tensor quadrature rules* as described in [15, 26] with computational cost proportional to $\#\boldsymbol{\nabla}_{\eta,j}$.

However, if f is not sufficiently smooth, then, as in the one-dimensional case, different estimates can be derived. Exemplarily, for $f \in L_1(\mathrm{supp}\, \boldsymbol{\psi}_{\boldsymbol{\lambda}})$, we find that

$$|\langle \mathbf{D}_{\boldsymbol{\lambda}}^1 \boldsymbol{\psi}_{\boldsymbol{\lambda}}, f\rangle| \lesssim \begin{cases} 2^{\frac{1}{2}\mathrm{sum}(|\boldsymbol{\lambda}|)} \|f\|_{L_1(\mathrm{supp}\, \boldsymbol{\psi}_{\boldsymbol{\lambda}})}, & \mathrm{sum}(|\boldsymbol{\lambda}|) < 0, \\ 2^{-\max(|\boldsymbol{\lambda}|)} \|f\|_{L_2(\mathrm{supp}\, \boldsymbol{\psi}_{\boldsymbol{\lambda}})}, & \mathrm{sum}(|\boldsymbol{\lambda}|) \geq 0. \end{cases} \tag{5.6}$$

Thus, if f is smooth except, e.g., in one point x, we can still use (5.5) whenever $x \notin \mathrm{supp}\, \boldsymbol{\psi}_{\boldsymbol{\lambda}}$ and (5.6) otherwise and adopt the above proceeding for smooth f.

5.5. **Numerical examples.** We consider the following bivariate PDE problem: given $f \in H^{-1}(\mathbb{R}^2)$, find $u \in H^1(\mathbb{R}^2)$ such that

$$(\partial_{x_1} v, \partial_{x_1} u)_{L_2(\mathbb{R}^2)} + (\partial_{x_2} v, \partial_{x_2} u)_{L_2(\mathbb{R}^2)} + (v, u)_{L_2(\mathbb{R}^2)} = \langle v, f \rangle, \quad \forall v \in H^1(\mathbb{R}^2). \tag{5.7}$$

We tested both **ADWAV** and **S-ADWAV** in conjunction with the multiwavelet basis constructed in [17] for $d = 2$ where we considered a separable (P4) and a non-separable (P5) reference solution for (5.7):

(P4) $u_4(x_1, x_2) := \exp(-\frac{1}{10}|x_1 - \frac{1}{3}|) \cdot \exp(-\frac{1}{2}|x_2 - \frac{1}{3}|)$,
(P5) $u_5(x_1, x_2) := \exp\big(-\sqrt{(x_1 - \frac{1}{10})^2 + (x_2 - \frac{1}{10})^2}\big)$.

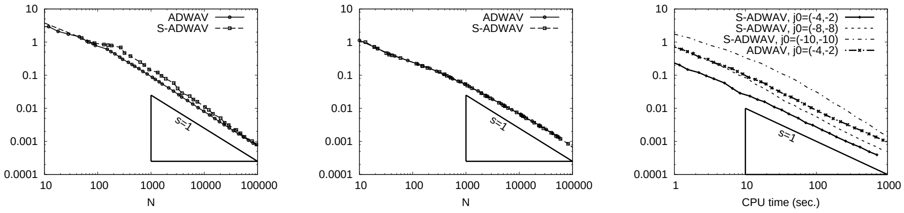
Note that **ADWAV** can be used without further modifications whereas for **S-ADWAV**, the construction of the security zone $\widehat{\mathbf{\Lambda}}$ for a finite index set $\mathbf{\Lambda} \in \mathbf{J}_{\mathbf{j}_0}$ for higher space dimensions is detailed in Algorithm 11.

5.5.1. *Parameters.* For **ADWAV** and **S-ADWAV**, we choose the parameters as in the one-dimensional setting (cf. Sections 3.3.1 and 4.2.1). By computing the Riesz constants $c_{\Psi_{1,j_0^{(i)}}}$ and $C_{\Psi_{1,j_0^{(i)}}}$, $i \in \{1, 2\}$ for the univariate bases as detailed in Section 3.3.1 and using (5.2) and (5.3), we estimated $\kappa(\mathbf{A}) \approx 17.0$ for $\mathbf{j}_0 = (-4, -2)$ and $\kappa(\mathbf{A}) \approx 15.3$ for $\mathbf{j}_0 = (-2, -2)$. The minimal level $\mathbf{j}_0$ was obtained by estimating the largest wavelet coefficient $|\langle \mathbf{D}^1_{\boldsymbol{\lambda}^\star} \boldsymbol{\psi}_{\boldsymbol{\lambda}^\star}, f\rangle|$ using (5.5) and (5.6). With $\boldsymbol{\lambda}^\star = ((j_1^\star, k_1^\star), (j_2^\star, k_2^\star))$, we set the minimal level to $\mathbf{j}_0 := |\boldsymbol{\lambda}^\star|$. The initial index set for **S-ADWAV** is then defined by $\mathbf{\Lambda}^{\text{cand.}}_{1,1} := \{((j_1^\star - 1, k_1^\star), (j_2^\star - 1, k_2^\star))\}$, i.e., $\mathbf{\Lambda}^{\text{cand.}}_{1,1}$ contains only the index associated to the (2d) scaling function $\varphi_{j_1^\star, k_1^\star} \otimes \varphi_{j_2^\star, k_2^\star}$.

Algorithm 11 $\mathbf{C}[\mathbf{\Lambda}, c] \to \widehat{\mathbf{\Lambda}}$

1: $\widehat{\mathbf{\Lambda}} := \emptyset$.
2: **for** $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathbf{\Lambda}$ **do**
3: $\quad\widehat{\mathbf{\Lambda}} := \widehat{\mathbf{\Lambda}} \cup \mathcal{C}(\lambda_1, c) \times \lambda_2 \times \cdots \times \lambda_n \cup \cdots \cup \lambda_1 \times \cdots \times \lambda_{n-1} \times \mathcal{C}(\lambda_n, c)$.
4: **end for**

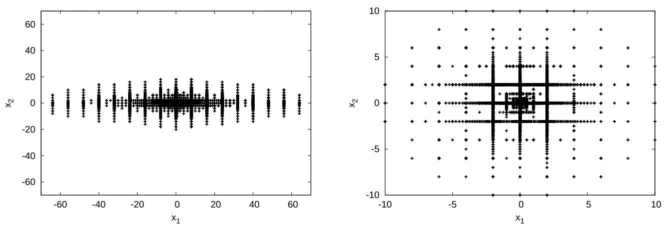
5.5.2. *Convergence rates.* The convergence rates for (P4) and (P5) are shown in Figure 5.1. We observe that the best nonlinear approximation rate from Section 5.2 is asymptotically attained. Problem (P4) has singularities parallel to the coordinate axis. Here, **ADWAV** needs less degrees of freedom which is due to the additional information stored in **RHS** which is not available in **S-ADWAV**. To solve (P5) where the solution is singular in one point, both algorithms nearly need the same number of degrees of freedom. In Figure 5.1 c), we show the computation times for (P4). Observe that **ADWAV** performs within linear complexity and that also here, **S-ADWAV** works very well which is also due to the multiwavelet discretization and the involved near-sparseness of $\mathbf{A}$ (cf. Section 5.3). Nevertheless, *asymptotic* optimality of **S-ADWAV** cannot be guaranteed (cf. Section 4.1.6). As in the one-dimensional setting (cf. Section 3), the choice of $\mathbf{j}_0 = (j_0^{(1)}, j_0^{(2)})$ is important for the performance, e.g., both algorithms (in Figure 5.1 c) exemplarily for **S-ADWAV**) performed worse for too small $\mathbf{j}_0$ (analog for too large $\mathbf{j}_0$). This is why we did not consider $\max(|\mathbf{j}_0|) = -\infty$ where at least one of the minimal levels $j_0^{(1)}, j_0^{(2)}$ is $-\infty$.



(a) (P4) with $\mathbf{j}_0 = (-4,-2)$. (b) (P5) with $\mathbf{j}_0 = (-2,-2)$. (c) (P4) with different $\mathbf{j}_0$

FIGURE 5.1. $H^1(\mathbb{R})$-error for (P4) and (P5).

5.5.3. *Adaptive truncation.* To get an idea of the computational domains computed by **ADWAV** and **S-ADWAV**, we show in Figure 5.2 the centers of the supports of all $\boldsymbol{\psi}_{\boldsymbol{\lambda}}$ with $\boldsymbol{\lambda} \in \operatorname{supp} \mathbf{w}^{(k)}$ for **ADWAV** respectively $\boldsymbol{\lambda} \in \operatorname{supp} \mathbf{u}^{(k,m)}$ for **S-ADWAV**. We observe that the anisotropic decay of u_4 is captured. The same holds true for the point singularity of u_5 which is also detected and resolved.



(a) **S-ADWAV**, $N = 9788$ (b) **ADWAV**, $N = 10599$

FIGURE 5.2. Adaptive truncation for (P4) (left) and (P5) (right).

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APPENDIX A. PROOFS FROM SECTION 3

Proof of Theorem 3.1. Let $\alpha \leq t$. For $|\lambda|, |\lambda'| \in \mathcal{J}$ with $i(\lambda, \lambda' = 1$, (3.5), (3.7) and $|\Box_\lambda| \sim 2^{-|\lambda|}$ yield (cf. [15, Proposition 5.3.3])

$$|b^{(\alpha)}_{\lambda,\lambda'}| \lesssim c_\alpha\, 2^{-\alpha|\lambda|}\, 2^{-\alpha|\lambda'|} \mathrm{diam}(\Box_{\lambda,\lambda'}) |\partial^\alpha \psi_\lambda|\, |\partial^\alpha \psi_{\lambda'}| \lesssim 2^{-\frac{1}{2}\delta(\lambda,\lambda')}, \ \forall \lambda, \lambda' \in \mathcal{J}. \tag{A.1}$$

Moreover, we infer from [15, Remark 5.3.4] that even

$$|b^{(\alpha)}_{\lambda,\lambda'}| \lesssim 2^{-(d-\frac{1}{2}-t)\delta(\lambda,\lambda')}, \quad \forall \lambda, \lambda' \in \mathcal{J}. \tag{A.2}$$

holds. Using that $\mathbf{D}^t_\lambda \lesssim 2^{-\alpha|\lambda|}$ for all $\lambda \in \mathcal{J}$ and $\alpha = 0, \ldots, t$ yields for all $\lambda, \lambda' \in \mathcal{J}$:

$$|a_{\lambda,\lambda'}| \lesssim |b^{(\alpha)}_{\lambda,\lambda'}| \lesssim 2^{-(d-\frac{1}{2}-t)\delta(\lambda,\lambda')}.$$

Next, consider the case where $i(\lambda, \lambda') = 0$ and $|\lambda'| > |\lambda|$. Then, $\Xi_{\lambda,i} \subset \operatorname{supp} \psi_{\lambda'}$ for some $i \in \{1, \ldots, \kappa\}$ and by using the vanishing moments of $\psi_{\lambda'}$, we find that with $p := \psi_\lambda|_{\Xi_{\lambda,i}} \in \mathcal{P}_{d-1}$

$$a^{(\alpha)}_{\lambda,\lambda'} = \mathbf{D}^t_\lambda\, \mathbf{D}^t_{\lambda'}\, c_\alpha \int_{\Box_{\lambda'}} \partial^\alpha p(x)\, \partial^\alpha \psi_{\lambda'}(x)\, \mathrm{d}x = 0. \tag{A.3}$$

Thus, concerning the number of non-zeros in $\mathbf{A}_j$, it suffices to consider indices in $S(\lambda, \ell')$ (cf. (3.9)) for $\ell' \in \{|\lambda| - j, \dots, |\lambda| + j\}$. Since $|\Box_\lambda| \sim 2^{-|\lambda|}$, we have $\#S(\lambda, \ell') \lesssim 1$ uniformly for all $\lambda \in \mathcal{J}$, $\ell' \in \mathbb{Z}$ which yields the first claim. Concerning (3.10), an application of the Schur lemma in conjunction with $|a_{\lambda,\lambda'}| \lesssim 2^{-(d-\frac{1}{2}-t)\delta(\lambda,\lambda')}$, following the lines of the proof of [26, Theorem 4.1], yields the claim. The ∞-compressibility then follows from the fact that we can replace j by 2^j in the definition of $\mathbf{A}_j$. Now, one has $\mathcal{O}(2^j)$ nonzero entries in each column and row of $\mathbf{A}_{2^j}$. For any $s^* > 0$, we have $js^* \le C_{s^*} + (d - \frac{1}{2} - t)2^j$ for all $j \in \mathbb{N}$ where $C_{s^*} := s^*[\log(\frac{s^*}{(d-\frac{1}{2}-t)\log 2})\frac{1}{\log 2} - \frac{1}{\log 2}]$. This yields that $\|\mathbf{A} - \mathbf{A}_{2^j}\| \lesssim 2^{-(d-\frac{1}{2}-t)2^j} \lesssim 2^{C_{s^*}} 2^{-js^*}$. The assertion then follows by using [29, Remark 2.4]. $\Box$

Proof of Theorem 3.2. Let $s^* := t + \widetilde{d}$. We first estimate the number of non-zeros in each row and column of $\mathbf{A}_j$. To this end, we consider a *matrix block* of $\mathbf{A}_j$,

$$\mathbf{A}_{\ell,\ell'} := \begin{cases} (a_{\lambda,\lambda'})_{|\lambda|=\ell,|\lambda'|=\ell'}, & \delta(\lambda,\lambda') \cdot z^{(i(\lambda,\lambda'))} \le j, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

where $z^{(i)}$ is defined in (3.13). Since $a_{\lambda,\lambda'} = 0$ when $|\Box_\lambda \cap \Box_{\lambda'}| = 0$, to estimate the number of non-zeros in a row or column of $\mathbf{A}_{\ell,\ell'}$ for $|\lambda| = \ell$, we only have to consider indices in $S(\lambda, \ell')$ (cf. (3.9)) and $U(\lambda, \ell')$ (cf. (3.11)). As $\#S(\lambda, \ell') \lesssim 1$ (cf. proof of Theorem 3.1) and $\#U(\lambda, \ell') \lesssim 2^{\max\{0,\ell'-\ell\}}$, (cf. also [26, Eq. (4.6)]), these two estimates are sufficient to show that the number of non-zeros in each row or column of $\mathbf{A}_j$ is of order $\mathcal{O}(2^{j/s^*})$ (cf. [26, Lemma 4.2]). The fact that (3.12) implies that $\|\mathbf{A} - \mathbf{A}_j\| \lesssim 2^{-j}$ can then be shown by the Schur Lemma as in [26, Proof of Theorem 4.1]. $\Box$

Proof of Proposition 3.3. Remind that by the transformation of variables $y := 2^{-\ell}x$ with $\ell := \max\{0, -\min\{|\lambda|, |\lambda'|\}\} \ge 0$, we obtain estimate (3.15) which was

$$|b^{(\alpha)}_{\lambda,\lambda'}| \lesssim \begin{cases} 2^{-(\frac{3}{2}+r-\alpha)\delta(\lambda,\lambda')}\, 2^{(r+1-\alpha)\ell}\, \|g_\alpha\|_{W^{r+1-\alpha,\infty}(\mathbb{R})}, & i(\lambda,\lambda') = 1, \\ 2^{-(\frac{1}{2}+\widetilde{d}+\alpha)\delta(\lambda,\lambda')}\, 2^{(\widetilde{d}+\alpha)\ell}\, \|g_\alpha\|_{W^{\widetilde{d}+\alpha,\infty}(\mathbb{R})}, & i(\lambda,\lambda') = 0. \end{cases}$$

First, we note that $\mathbf{D}^t_\lambda \lesssim 2^{-\alpha|\lambda|}$ for $0 \le \alpha \le t$ and for all $\lambda \in \mathcal{J}$ which implies

$$|a^{(\alpha)}_{\lambda,\lambda'}| \lesssim \begin{cases} 2^{-(\frac{3}{2}+r-\alpha)\delta(\lambda,\lambda')}\, 2^{(r+1-\alpha)\ell}\, \|g_\alpha\|_{W^{r+1-\alpha,\infty}(\mathbb{R})}, & i(\lambda,\lambda') = 1, \\ 2^{-(\frac{1}{2}+\widetilde{d}+\alpha)\delta(\lambda,\lambda')}\, 2^{(\widetilde{d}+\alpha)\ell}\, \|g_\alpha\|_{W^{\widetilde{d}+\alpha,\infty}(\mathbb{R})}, & i(\lambda,\lambda') = 0. \end{cases}$$

Since $|\lambda|, |\lambda'| \ge j_0$, we deduce that $\ell \le \max\{0, -j_0\}$. Moreover, we have for $\ell \ge 0$ that $2^{(r+1-\alpha)\ell} \le 2^{(\widetilde{d}+t)\ell}$ as well as $2^{(\widetilde{d}+\alpha)\ell} \le 2^{(\widetilde{d}+t)\ell}$ for all $\alpha \le t$. Thus, we have

$$|a^{(\alpha)}_{\lambda,\lambda'}| \lesssim \begin{cases} 2^{-(\frac{3}{2}+r-\alpha)\delta(\lambda,\lambda')} \max\{1, 2^{-(\widetilde{d}+t)j_0}\}\, \|g_\alpha\|_{W^{r+1-\alpha,\infty}(\mathbb{R})}, & i(\lambda,\lambda') = 1, \\ 2^{-(\frac{1}{2}+\widetilde{d}+\alpha)\delta(\lambda,\lambda')} \max\{1, 2^{-(\widetilde{d}+t)j_0}\}\, \|g_\alpha\|_{W^{\widetilde{d}+\alpha,\infty}(\mathbb{R})}, & i(\lambda,\lambda') = 0, \end{cases}$$

which yields the claim. $\Box$

Proof of Lemma 3.7. By definition of $\mathbf{f}_2^i := (\mathbf{D}_\lambda^t\, c_i\psi_\lambda(x_i))_{\lambda\in\mathcal{J}}$ $(i=1,\dots,m)$, we have that $\mathbf{f}_2 = \sum_{i=1}^m \mathbf{f}_2^i$. Now, (3.18) yields that

$$\begin{aligned}\|\mathbf{f}_2^i - \mathbf{f}_2^i|_{\nabla_{2,\eta}^i}\|_{\ell_2}^2 &= \textstyle\sum_{|\lambda|<-J_2^{(i,-)}(\eta)} |c_i\,\mathbf{D}_\lambda^t\,\psi_\lambda(x_i)|^2 + \sum_{|\lambda|>J_2^{(i,+)}(\eta)} |c_i\,\mathbf{D}_\lambda^t\,\psi_\lambda(x_i)|^2\\ &\le c_i^2 d_t^2 \|\psi\|_{L_\infty}^2 M(\textstyle\sum_{j>J_2^{(i,-)}(\eta)} 2^{-j} + \sum_{j>J_2^{(i,+)}(\eta)} 2^{-(2t-1)j})\\ &\le c_i^2 d_t^2 \|\psi\|_{L_\infty}^2 M(2\,2^{-J_2^{(i,-)}(\eta)} + (1-2^{(1-2t)})^{-1}\,2^{-(2t-1)J_2^{(i,+)}(\eta)}).\end{aligned}$$

By our choice of $J_2^{(i,+)}(\eta)$ and $J_2^{(i,-)}(\eta)$, we get $\|\mathbf{f}_2^i - \mathbf{f}_2^i|_{\nabla_{2,\eta}^i}\|_{\ell_2} \le \frac{\eta}{m}$. By triangle inequality, we obtain $\|\mathbf{f}_2 - \mathbf{f}_{2,\eta}\|_{\ell_2} = \|\sum_{i=1}^m (\mathbf{f}_2^i - \mathbf{f}_2^i|_{\nabla_{2,\eta}^i})\|_{\ell_2} \le \eta$. Finally, it holds that $\#\nabla_{2,\eta} \le \sum_{i=1}^m \#\nabla_{2,\eta}^i \le m\cdot M\cdot\max_i\{J_2^{(i,-)}(\eta)+J_2^{(i,+)}(\eta)+1\} \lesssim 1+|\log_2\eta|$. $\square$

Proof of Lemma 3.8. Let $(\,\cdot\,,\,\cdot\,)_{H^t}$ denote the inner product in $H^t(\mathbb{R})$ and define $u_{f_1}\in H^t(\mathbb{R})$ as the unique solution of the problem finding $u\in H^t(\mathbb{R})$ such that $(v,u)_{H^t} = \langle v, f_1\rangle$ for all $v\in H^t(\mathbb{R})$. It is clear that $\|u_{f_1}\|_{H^t(\mathbb{R})} \le \|f_1\|_{L_2(\mathbb{R})}$. Now, let $u_{f_1} = \mathbf{u}_{f_1}^T\Psi_t$ and $\mathbf{G} := (\Psi_t,\Psi_t)_{H^t}$ the Gramian matrix of Ψ_t in $H^t(\mathbb{R})$. Then, we obtain for any $\lambda\in\mathcal{J}$ that $(\mathbf{D}_\lambda^t\psi_\lambda, u_{f_1})_{H^t} = \langle \mathbf{D}_\lambda^t\psi_\lambda, f_1\rangle$, i.e., $\mathbf{f}_1 = \langle\Psi_t, f_1\rangle = (\Psi_t, \mathbf{u}_{f_1}^T\Psi_t)_{H^t} = \mathbf{G}\mathbf{u}_{f_1}$. In a similar way, we define $u_{f_{1,R_\eta}}$ and $\mathbf{u}_{f_{1,R_\eta}}$ corresponding to $\mathbf{f}_{1,R_\eta} := \langle\Psi_t, f_{1,R_\eta}\rangle$ instead of $\mathbf{f}$. By $\|\mathbf{G}\|\le C_{\Psi_t}^2$ and (3.17), we get

$$\begin{aligned}\|\mathbf{f}_1 - \mathbf{f}_{1,R_\eta}\|_{\ell_2} &= \|\mathbf{G}(\mathbf{u}_{f_1} - \mathbf{u}_{f_{1,R_\eta}})\|_{\ell_2} \le C_{\Psi_t}^2\|\mathbf{u}_{f_1} - \mathbf{u}_{f_{1,R_\eta}}\|_{\ell_2}\\ &\le C_{\Psi_t}^2 c_{\Psi_t}^{-1}\|u_{f_1} - u_{f_{1,R_\eta}}\|_{H^t(\mathbb{R})} \le C_{\Psi_t}^2 c_{\Psi_t}^{-1}\|f_1 - f_{1,R_\eta}\|_{L_2(\mathbb{R})}\\ &\le C_{\Psi_t}^2 c_{\Psi_t}^{-1} C_f R_\eta^{-\beta} \le \eta,\end{aligned}\tag{A.4}$$

where R_η was defined in (3.24). Now, we proceed in two steps. First, we consider separately positive and negative levels. As a second step, we investigate the level bounds $-J_1^-(\eta)$ and $J_1^+(\eta)$. So, let $\bar{\mathbf{f}}_{1,\eta}^+ := \mathbf{f}_1|_{\Delta_{1,\eta}^+}$ and $\bar{\mathbf{f}}_{1,\eta}^- := \mathbf{f}_{1,R_\eta}|_{\Delta_{1,\eta}^-}$ with

$$\Delta_{1,\eta}^+ := \{\lambda\in\mathcal{J}\colon |\Box_\lambda\cap I_\eta|>0, |\lambda|\ge 0\},\ \Delta_{1,\eta}^- := \{\lambda\in\mathcal{J}\colon |\Box_\lambda\cap I_\eta|>0, |\lambda|<0\}.\tag{A.5}$$

Note that $\Delta_{1,\eta}^+$, $\Delta_{1,\eta}^-$ are *not* finite. Since $\mathcal{J}\setminus\Delta_{1,\eta}^+ = \{\lambda\in\mathcal{J} : |\Box_\lambda\cap I_\eta| = 0\}\cup\Delta_{1,\eta}^-$,

$$\begin{aligned}&\|\mathbf{f} - \bar{\mathbf{f}}_{1,\eta}^+ - \bar{\mathbf{f}}_{1,\eta}^-\|_{\ell_2}^2 = \|\mathbf{f}_1|_{\mathcal{J}\setminus\Delta_{1,\eta}^+} - \mathbf{f}_{1,R_\eta}|_{\Delta_{1,\eta}^-}\|_{\ell_2}^2\\ &= \textstyle\sum_{\lambda\in\mathcal{J},|\Box_\lambda\cap I_\lambda|=0} |\langle\mathbf{D}_\lambda^t\psi_\lambda, f_1\rangle|^2 + \sum_{\lambda\in\Delta_{1,\eta}^-}|\langle\mathbf{D}_\lambda^t\psi_\lambda, f_1 - f_{1,R_\eta}\rangle|^2\\ &= \textstyle\sum_{\lambda\in\mathcal{J},|\Box_\lambda\cap I_\lambda|=0} |\langle\mathbf{D}_\lambda^t\psi_\lambda, f_1 - f_{1,R_\eta}\rangle|^2 + \sum_{\lambda\in\Delta_{1,\eta}^-}|\langle\mathbf{D}_\lambda^t\psi_\lambda, f_1 - f_{1,R_\eta}\rangle|^2\\ &\le \textstyle\sum_{\lambda\in\mathcal{J}} |\langle\mathbf{D}_\lambda^t\psi_\lambda, f_1 - f_{1,R_\eta}\rangle|^2 = \|\mathbf{f}_1 - \mathbf{f}_{1,R_\eta}\|_{\ell_2}^2 \le \eta^2,\end{aligned}$$

where we used the fact that $\langle\mathbf{D}_\lambda^t\psi_\lambda, f_{1,R_\eta}\rangle = 0$ for $|\Box_\lambda\cap I_\eta| = 0$ and (A.4). We are now going to consider the effect of restricting the level range, i.e., we estimate the difference of $\bar{\mathbf{f}}_{1,\eta}^+$, $\mathbf{f}_{1,\eta}^+ = \mathbf{f}_1|_{\nabla_{1,\eta}^+}$ on one hand and $\bar{\mathbf{f}}_{1,\eta}^-$, $\mathbf{f}_{1,\eta}^- = \mathbf{f}_{1,R_\eta}|_{\nabla_{1,\eta}^-}$ on the other hand. As a preparation, we consider the following straightforward estimate

$$\mathcal{C}_{j,\eta} := \#\{\lambda\in\mathcal{J} : |\lambda| = j,\ |\Box_\lambda\cap I_\eta|>0\} \le 2^{j+1}R_\eta + 2M,\tag{A.6}$$

i.e., all indices on a fixed level where corresponding wavelets intersect I_η.
Non-negative leves: Let $j\ge 0$. Then, we have for $j > J_1^+(\eta)$ that $\mathcal{C}_{j,\eta} \le 2^{j+1}R_\eta(1+M(R_\eta 2^j)^{-1}) \le 2^{j+2}R_\eta$. Thus, by (3.6) in conjunction with $|\mathbf{D}_\lambda^t| \le d_t 2^{-t|\lambda|}$ for

$|\lambda| \geq 0$ (cf. (3.18)) and the definition of $J_1^+(\eta)$ in (3.25):

$$\begin{aligned}
\|\bar{\mathbf{f}}^+_{1,\eta} - \mathbf{f}^+_{1,\eta}\|^2_{\ell_2} &= \|\mathbf{f}_1|_{\Delta^+_{1,\eta}\setminus\nabla^+_{1,\eta}}\|^2_{\ell_2} = \textstyle\sum_{|\lambda|>J_1^+(\eta), |\Box_\lambda\cap I_\eta|>0} |\langle \mathbf{D}^t_\lambda \psi_\lambda, f_1\rangle|^2 \\
&\leq \textstyle\sum_{j>J_1^+(\eta)} \mathcal{C}_{j,\eta}\, C^2_{\psi,f_1}\, d_t^2\, 2^{-2(\frac{1}{2}+\sigma+t)j} \\
&\leq 4R_\eta C^2_{\psi,f_1}\, d_t^2 \textstyle\sum_{j>J_1^+(\eta)} 2^{-2(\sigma+t)j} \leq D_2\, R_\eta\, 2^{-2(\sigma+t)J_1^+(\eta)} \leq \eta^2. \qquad \text{(A.7)}
\end{aligned}$$

For the cardinality, we have $\#\nabla^+_{1,\eta} \leq \sum_{j=0}^{J_1^+(\eta)} \mathcal{C}_{j,\eta} \lesssim R_\eta 2^{J_1^+(\eta)} \lesssim \eta^{-\frac{\sigma+\beta+t+\frac{1}{2}}{\beta(\sigma+t)}}$.
Negative leves: Let $j = |\lambda| < 0$. Since $\mathbf{D}^t_\lambda \leq 1$, it holds that (compare (3.18))

$$|\langle \mathbf{D}^t_\lambda \psi_\lambda, f_{1,R_\eta}\rangle| \leq 2^{|\lambda|/2} \|\psi\|_{L_\infty} \|f_{1,R_\eta}\|_{L_1} \leq 2^{|\lambda|/2}\|\psi\|_{L_\infty}\|f_1\|_{L_1}. \qquad \text{(A.8)}$$

For $j < -J_1^-(\eta)$, (A.6) yields $\mathcal{C}_{j,\eta} \leq 2^{j+1}R_\eta + 2M \leq 1 + 2M$. Thus, by (A.8)

$$\begin{aligned}
\|\bar{\mathbf{f}}^-_{1,\eta} - \mathbf{f}^-_{1,\eta}|\|^2_{\ell_2} &= \|\mathbf{f}_{1,R_\eta}|_{\Delta^-_{1,\eta}\setminus\nabla^-_{1,\eta}}\|^2_{\ell_2} = \textstyle\sum_{|\lambda|<-J_1^-(\eta), |\Box_\lambda\cap I_\eta|>0} |\langle \mathbf{D}^t_\lambda \psi_\lambda, f_{1,R_\eta}\rangle|^2 \\
&\leq \|\psi\|^2_{L_\infty}\|f_1\|^2_{L_1} \textstyle\sum_{j>J_1^-(\eta)} C_{-j,\eta}\, 2^{-j} \leq D_3\, 2^{-J_1^-(\eta)} \leq \eta^2, \qquad \text{(A.9)}
\end{aligned}$$

by the definition of $J_1^-(\eta)$ in (3.25). Finally, since $J_1^-(\eta) \lesssim \max\{2, \beta^{-1}\}|\log_2\eta|$

$$\#\nabla^-_{1,\eta} \leq \textstyle\sum_{j=1}^{J_1^-(\eta)} \mathcal{C}_{-j,\eta} \leq \sum_{j=1}^{J_1^-(\eta)} (2^{-j+2}R_\eta + 2M) \lesssim R_\eta + 2MJ_1^-(\eta) \lesssim \eta^{-1/\beta}.$$

With (A.4), (A.7) and (A.9), we obtain the claim by $\|\mathbf{f}_1 - \mathbf{f}_{1,\eta}\|_{\ell_2} \leq \|\mathbf{f}_1 - \bar{\mathbf{f}}^+_{1,\eta} - \bar{\mathbf{f}}^-_{1,\eta}\|_{\ell_2} + \|\bar{\mathbf{f}}^+_{1,\eta} - \mathbf{f}^+_{1,\eta}\|_{\ell_2} + \|\bar{\mathbf{f}}^-_\eta - \mathbf{f}^-_{1,\eta}\|_{\ell_2} \leq 3\eta$, and replacing η by $\eta/3$. $\square$

Proof of Proposition 3.10. Let $\mathbf{g}^+_\eta := \mathbf{g}_\eta|_{\nabla^+_{1,\eta}}$, $\widetilde{\mathbf{g}}^+_\eta := \widetilde{\mathbf{g}}_\eta|_{\nabla^+_{1,\eta}}$, $\mathbf{g}^-_\eta := \mathbf{g}_\eta|_{\nabla^-_{1,\eta}}$ and $\widetilde{\mathbf{g}}^-_\eta := \widetilde{\mathbf{g}}_\eta|_{\nabla^-_{1,\eta}}$ First, we consider $|\lambda| \geq 0$. We fix $p > t + \widetilde{d} + d^*$ and define $\varsigma := t + \widetilde{d}$. For computing $\widetilde{\mathbf{g}}_{\eta,\lambda}$ with $|\lambda| = \ell$, we then take $N^+_\ell \sim \lceil 2^{(J^+(\eta)-\ell)\frac{\varsigma}{p}}\rceil$ subintervals with $J_1^+(\eta)$ defined in (3.25). This yields by (3.32) (compare (A.7))

$$\begin{aligned}
\|\mathbf{g}^+_\eta - \widetilde{\mathbf{g}}^+_\eta\|^2_{\ell_2} &\lesssim \textstyle\sum_{\ell=0}^{J_1^+(\eta)} R_\eta\, 2^\ell\, 2^{-2\varsigma(J_1^+(\eta)-\ell)}\, 2^{-2(\frac{1}{2}+p-d^*)\ell} \\
&\lesssim R_\eta\, 2^{-2\varsigma J_1^+(\eta)} \textstyle\sum_{\ell=0}^{J_1^+(\eta)} 2^{-2(p-d^*-\varsigma)\ell} \lesssim R_\eta\, 2^{-2(t+\widetilde{d})J_1^+(\eta)} \lesssim \eta^2.
\end{aligned}$$

Since the cost for computing one entry $\widetilde{\mathbf{g}}^+_{\eta,\lambda}$ in $\widetilde{\mathbf{g}}^+_\eta$are of order $\mathcal{O}(\kappa p N^+_{|\lambda|})$ with κ from (3.4), the overall cost for computing $\widetilde{\mathbf{g}}^+_\eta$ are of order $\mathcal{O}(\#\mathrm{supp}\,\mathbf{g}^+_\eta)$ since

$$\textstyle\sum_{\ell=0}^{J_1^+(\eta)} R_\eta\, 2^\ell\, \kappa p N^+_\ell \sim R_\eta \sum_{\ell=0}^{J_1^+(\eta)} 2^\ell\, 2^{(J^+_\eta-\ell)\frac{\varsigma}{p}} = R_\eta 2^{\frac{\varsigma}{p}J_1^+(\eta)} \sum_{\ell=0}^{J_1^+(\eta)} 2^{\ell(1-\frac{\varsigma}{p})} \sim R_\eta\, 2^{J^+_\eta}.$$

For negative levels, we can proceed in a similar way. Let $p \in \mathbb{N}$ be arbitrary but fixed and N be a constant multiple of $\lceil \eta^{-\frac{3+2p+2\beta}{2p\beta}}\rceil$. Then, by (3.32), it follows that

$$\|\mathbf{g}^-_\eta - \widetilde{\mathbf{g}}^-_\eta\|^2_{\ell_2} \lesssim \#\mathrm{supp}\,\mathbf{g}^-_\eta \cdot (2R_\eta)^{2(1+p)}\, N^{-2p} \lesssim \eta^{-\frac{1}{\beta}}\, \eta^{-\frac{2(1+p)}{\beta}}\, \eta^{\frac{3+2p+2\beta}{\beta}} = \eta^2.$$

The bound for the number of quadrature operations, $\#\nabla^-_{1,\eta} \cdot (\kappa p N) \lesssim p\,\eta^{-\frac{1}{\beta}} \times \eta^{-\frac{3+2p+2\beta}{2p\beta}} = p \cdot \eta^{-\frac{3+4p+2\beta}{2p\beta}}$ is in this case larger than $\#\mathrm{supp}\,\mathbf{g}^-_\eta$. However, for optimality, we only require that $\eta^{-\frac{3+4p+2\beta}{2p\beta}} \lesssim \eta^{-1/\bar{s}}$ which is satisfied if we choose $p \in \mathbb{N}$ such that $2p(\beta - 2\bar{s}) \geq \bar{s}(3 + 2\beta)$ (remind that $\beta > 2\bar{s}$). $\square$

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