An Improved Error Bound for Reduced Basis Approximation of Linear Parabolic Problems

Karsten Urban und Anthony T. Patera

Preprint Series: 2012 - 06

Fakultät für Mathematik und Wirtschaftswissenschaften

UNIVERSITÄT ULM

AN IMPROVED ERROR BOUND FOR REDUCED BASIS APPROXIMATION OF LINEAR PARABOLIC PROBLEMS

KARSTEN URBAN AND ANTHONY T. PATERA

ABSTRACT. We consider a space-time variational formulation for linear parabolic partial differential equations. We introduce an associated Petrov-Galerkin truth finite element discretization with favorable discrete inf-sup constant β_δ, the inverse of which enters into error estimates: β_δ is unity for the heat equation; β_δ decreases only linearly in time for non-coercive (but asymptotically stable) convection operators. The latter in turn permits effective long-time *a posteriori* error bounds for reduced basis approximations, in sharp contrast to classical (pessimistic) exponentially growing energy estimates. The paper contains a full analysis and various extensions for the formulation introduced briefly in [13] as well as numerical results for a model reaction-convection-diffusion equation.

1. INTRODUCTION

The certified reduced basis method (RBM) has been successfully applied to parabolic equations in the case in which the spatial operator is coercive [3, 4]. However, for problems — linear or nonlinear [7] — in which the spatial operator (or linearized spatial operator) is non-coercive, the standard L_2-error bounds based on energy estimates are very pessimistic. In particular, these energy estimates suggest exponential growth in time even for problems which are asymptotically stable and for which the actual error grows at most linearly with time.

In a recent paper [10] space-time adaptive numerical schemes for linear parabolic initial value problems based upon wavelets have been introduced. One key ingredient there is the transformation of the partial differential equation into an equivalent well-conditioned discrete (but still infinite-dimensional) system w.r.t. the wavelet coefficients. In order to show this equivalence, a new proof for the well-posedness of the space-time variational formulation of linear parabolic initial value problems is presented in [10]. This proof contains an explicit lower bound for the inf-sup stability constant. In the context of RBMs, it is well-known that the inverse of the inf-sup-constant multiplied with the (computable) dual norm of the residual form an a-posteriori error estimate in a (Petrov-)Galerkin scheme. A closer investigation and modification of the proof in [10, Theorem 5.1, Appendix A] shows that a space-time inf-sup stability constant — and related appropriate norms — can avoid the "worst–case" energy assumption at each time t (or discrete time level) and instead reflect the coupled temporal behavior over the entire time interval of interest.

Date: 15.06.2012.

1991 *Mathematics Subject Classification.* 35K15, 65M15, 65M60.

We would like to thank Masayuki Yano for helpful comments as well as computational confirmations. A.T.P. was supported by OSD/AFOSR/MURI Grant FA9550-09-1-0613 and by ONR Grant N00014-11-1-0713; K.U. by the Deutsche Forschungsgemeinschaft (DFG) under Ur-63/9 and GrK1100. This paper was partly written while K.U. was Visiting Professor at MIT.

We show in [13] that indeed a space-time formulation can improve reduced basis error bounds: we provide theoretical justification for the symmetric coercive case, and computational evidence for the nonsymmetric noncoercive case. We elaborate here on the brief presentation of [13]: we consider in detail the underlying Petrov-Galerkin discretization and associated Crank-Nicolson interpretation; we provide the proofs of the central propositions; we extend the approach and analysis to primal-dual formulations for the output of interest; and finally, we provide numerical convergence results for the inf-sup constant which plays the crucial role in the reduced basis error bounds.

This paper is organized as follows. In Section 2, we investigate the space-time variational problem, in particular its well-posedness also for long time periods. We also show the main difference of our analysis as opposed to more standard techniques using a temporal transformation. Next, we introduce a space-time discretization which leads to a Petrov-Galerkin scheme whose error is analyzed for the particular case of symmetric coercive spatial operators. Section 3 contains the application of our space-time error analysis for the Reduced Basis Method. We give a posteriori error bounds w.r.t. the residual and discuss various issues concerning the numerical realization. We present numerical results in Section 4, in particular for those cases that are not yet covered by our theory, namely convection-diffusion operators as well as asymptotically unstable equations.

2. Space-Time Truth Solution

2.1. **Space-Time Formulation.** Similar to [10], we consider Hilbert spaces $V \hookrightarrow H \hookrightarrow V'$ with inner products $(\cdot,\cdot)_V$, $(\cdot,\cdot)_H$ and induced norms $\|\cdot\|_V$, $\|\cdot\|_H$, a time interval $I := (0,T]$, $T>0$ and $A \in \mathcal{L}(V,V')$ such that $\langle A\phi,\psi\rangle_{V'\times V} = a(\phi,\psi)$ with a bilinear form $a(\cdot,\cdot) : V \times V \to \mathbb{R}$. We consider the following problem: Given $g \in L_2(I;V')$, determine u such that

$$\dot u(t) + A\,u(t) = g(t) \text{ in } V', \qquad u(0) = 0 \text{ in } H. \tag{2.1}$$

Nonzero initial conditions can easily be treated by slight modifications of the variational form to be introduced next. According to [10] we assume that there exist constants $M_a < \infty$, $\alpha > 0$ and $\lambda \ge 0$ such that for all $\phi,\psi \in V$ we have

$$|a(\psi,\phi)| \le M_a\,\|\psi\|_V\,\|\phi\|_V, \qquad \text{(boundedness)} \tag{2.2}$$

$$a(\psi,\psi) + \lambda\,\|\psi\|_H^2 \ge \alpha\|\psi\|_V^2. \qquad \text{(Gårding inequality)} \tag{2.3}$$

In addition, we consider outputs of the form

$$s := \int_I \ell(u(t))\,dt \quad \text{for some } \ell \in V'. \tag{2.4}$$

The above setting corresponds to the LTI (linear time invariant) case, but we remark that some of our results can be extended to the LTV case as well.

In order to formulate the variational form of (2.1), we need some preparation. We use as trial space

$$\mathcal{X} := \{v \in L_2(I;V) : v, \dot v \in L_2(I;V'), v(0) = 0\} = L_2(I;V) \cap H^1_{(0)}(I;V'),$$

where $H^1_{(0)}(I;V') := \{v \in H^1(I;V') : v(0) = 0\}$ with the norm $\|w\|_{\mathcal{X}}^2 := \|w\|_{L_2(I;V)}^2 + \|\dot w\|_{L_2(I;V')}^2 + \|w(T)\|_H^2$ (note: $\mathcal{X} \hookrightarrow C(I;H)$). The test space is $\mathcal{Y} := L_2(I;V)$ with norm $\|v\|_{\mathcal{Y}} := \|v\|_{L_2(I;V)}$.

Remark 2.1. *At a first glance, it seems to be more standard to use the graph norm* $(\|w\|^2_{L_2(I;V)} + \|\dot{w}\|^2_{L_2(I;V')})^{1/2}$ *on* $\mathcal{X}$. *Obviously,* $\|\cdot\|_{\mathcal{X}}$ *defined above is a stronger norm and also allows the control of the solution at the final time* T.

We will use the following abbreviations: $[w,v]_{\mathcal{H}} := \int_I \langle w(t), v(t)\rangle_{V'\times V} dt$ for $w \in L_2(I;V')$, $v \in L_2(I;V)$ (as well as $[w,v]_{\mathcal{H}} := \int_I (w(t), v(t))_H dt$ for $v, w \in L_2(I;H)$) and $\mathcal{A}[w,v] := \int_I a(w(t), v(t))\, dt$ for $v, w \in L_2(I;V)$. Then, defining

$$b(w,v) := [\dot{w}, v]_{\mathcal{H}} + \mathcal{A}[w,v], \qquad f(v) := [g,v]_{\mathcal{H}}, \tag{2.5}$$

results in the variational formulation

$$\text{find}\, u \in \mathcal{X}: \qquad b(u,v) = f(v) \qquad \forall v \in \mathcal{Y}. \tag{2.6}$$

The output is again given by (2.4) and can also be formulated as

$$s = J(u) \quad \text{where} \quad J \in \mathcal{X}' \quad \text{reads} \quad J(w) := \int_I \ell(w(t))\, dt,\ w \in \mathcal{X}. \tag{2.7}$$

The well-posedness of (2.6) (under the above assumptions) has been shown in [10, Theorem 5.1, Appendix A]. A more detailed investigation of the proof in [10] shows that the arguments used there can also yield an estimate for the inf-sup constant

$$\beta := \inf_{w\in\mathcal{X}} \sup_{v\in\mathcal{Y}} \frac{b(w,v)}{\|w\|_{\mathcal{X}}\, \|v\|_{\mathcal{Y}}}.$$

We define $\varrho := \sup_{0\neq\phi\in V} \frac{\|\phi\|_H}{\|\phi\|_V}$ and $\beta_a^* := \inf_{\phi\in V} \sup_{\psi\in V} \frac{a(\psi,\phi)}{\|\phi\|_V\, \|\psi\|_V}$; we then have

Proposition 2.2 ([13, Proposition 1])**.** *Assume* (2.2) *and* (2.3)*. Then, we obtain the inf-sup lower bound*

$$\beta \geq \beta^{LB} := \frac{\min\{1, (\alpha - \lambda\varrho^2)\min\{1, M_a^{-2}\}\}}{\max\{1, (\beta_a^*)^{-1}\}\sqrt{2}}.$$

Proof. Let $0 \neq w \in \mathcal{X}$ be given and denote by $A^* : V \to V'$ the adjoint of A. Set $z_w := (A^*)^{-1}\dot{w}$ and $v_w := z_w + w \in \mathcal{Y}$. Then, we have

$$\begin{aligned} \|v_w\|^2_{L_2(I;V)} &\leq 2(\|z_w\|^2_{\mathcal{Y}} + \|w\|^2_{\mathcal{Y}}) \leq 2\big((\beta_a^*)^{-2}\|\dot{w}\|^2_{L_2(I;V')} + \|w\|^2_{\mathcal{Y}})\big) \\ &\leq 2\max\{1, (\beta_a^*)^{-2}\}\|w\|^2_{\mathcal{X}}. \end{aligned} \tag{2.8}$$

In order to bound $b(w, v_w)$ we use the following facts $\|\dot{w}(t)\|_{V'} = \|A^* z_w(t)\|_{V'} \leq M_a \|z_w(t)\|_V$ and thus

$$\begin{aligned} \langle \dot{w}(t), z_w(t)\rangle_{V'\times V} &= a(z_w(t), z_w(t)) \geq \alpha\|\|z_w(t)\|^2_V - \lambda\|z_w(t)\|^2_H \\ &\geq (\alpha - \lambda\varrho^2)\|z_w(t)\|^2_V \geq (\alpha - \lambda\varrho^2) M_a^{-2} \|\dot{w}(t)\|^2_{V'}, \end{aligned} \tag{2.9}$$

as well as

$$a(w(t), z_w(t)) = \langle w(t), \dot{w}(t)\rangle_{V\times V'} = \frac{1}{2}\frac{d}{dt}\|w(t)\|^2_H \tag{2.10}$$

to obtain (recalling that $w(0) = 0$)

$$\begin{aligned} b(w, v_w) &= \int_I \langle \dot{w}(t), z_w(t)\rangle_{V'\times V}\, dt + \int_I \langle \dot{w}(t), w(t)\rangle_{V'\times V} dt \\ &\quad + \int_I a(w(t), z_w(t))\, dt + \int_I a(w(t), w(t))\, dt \\ &\geq (\alpha - \lambda\varrho^2) M_a^{-2} \|\dot{w}\|^2_{L_2(I;V')} + \frac{1}{2}\int_0^T \frac{d}{dt}||w(t)||_H^2\, dt \\ &\quad + \frac{1}{2}\int_0^T \frac{d}{dt}||w(t)||_H^2\, dt + (\alpha - \lambda\varrho^2)\|w(t)\|^2_{L_2(I;V)} \\ &\geq (\alpha - \lambda\varrho^2)\min\{1, M_a^{-2}\}(\|\dot{w}\|^2_{L_2(I;V')} + \|w\|^2_{L_2(I;V)}) + \|w(T)\|_H^2 \\ &\geq \min\{(\alpha - \lambda\varrho^2)\min\{1, M_a^{-2}\}, 1\}\|w\|^2_{\mathcal{X}} \geq \beta^{\mathrm{LB}} \|w\|_{\mathcal{X}} \|v_w\|_{\mathcal{Y}}, \end{aligned}$$

where the last step follows from (2.8). □

Remark 2.3. *Note that β^{LB} does not depend on the final time. However, the estimate is only meaningful if $\alpha \geq \lambda\varrho^2$, i.e., if the system is coercive. In the non-coercive case,* (2.1) *is often transformed as described in Section 2.3 below.*

Remark 2.4. *If we would use the graph norm for $\mathcal{X}$, the above proof yields an inf-sup lower bound of $\frac{(\alpha-\lambda\varrho^2)\min\{1,M_a^{-2}\}}{\max\{1,(\beta_a^*)^{-1}\}\sqrt{2}}$.*

2.2. **The heat equation.** The heat equation is a special case of (2.1), where

$$A = -\Delta, \quad V = H_0^1(\Omega), \quad H = L_2(\Omega), \quad \|\phi\|_V^2 = a(\phi,\phi) = \|\nabla\phi\|^2_{L_2(\Omega)}.$$

Thus, we have $M_a = 1$, $\lambda = 0$, $\alpha = 1$ and $\beta_a^* = 1$. Thus, Proposition 2.2 would result in a lower bound of $\frac{1}{\sqrt{2}}$. A slight modification of the proof, however, allows to improve this lower bound.

Corollary 2.5. *For the heat equation, it holds $\beta \geq 1$.*

Proof. Given $0 \neq w \in \mathcal{X}$, we choose as above $v_w := z_w + w \in \mathcal{Y}$ with $z_w := A^{-1}\dot{w}$. Then,

$$\|v_w\|^2_{L_2(I;V)} = \|z_w\|^2_{L_2(I;V)} + \|w\|^2_{L_2(I;V)} + 2\int_I (z_w(t), w(t))_V\, dt.$$

Since $\|z_w\|_{L_2(I;V)} = \|A^{-1}\dot{w}\|_{L_2(I;V)} = \|\dot{w}\|_{L_2(I;V')}$ and recalling that $a(z_w(t), v(t)) = \langle \dot{w}(t), v(t)\rangle_{V'\times V}$ for all $v(t) \in V$, we obtain

$$(z_w(t), w(t))_V = a(z_w(t), w(t)) = \langle \dot{w}(t), w(t)\rangle_{V'\times V} = \frac{1}{2}\frac{d}{dt}\|w(t)\|_H^2,$$

so that

$$\begin{aligned} \|v_w\|^2_{L_2(I;V)} &= \|A^{-1}\dot{w} + w\|^2_{L_2(I;V)} \\ &= \|\dot{w}\|^2_{L_2(I;V')} + \|w\|^2_{L_2(I;V)} + \|w(T)\|_H^2 = \|w\|^2_{\mathcal{X}}. \end{aligned} \tag{2.11}$$

The rest of the proof remains the same so that we arrive at $b(w, v_w) \geq \|w\|^2_{\mathcal{X}} = \|w\|_{\mathcal{X}}\, \|v_w\|_{\mathcal{Y}}$. □

We can go even a step further.

Proposition 2.6. *For the heat equation, it holds $\beta = \gamma = 1$, where γ is the continuity constant defined as $\gamma := \sup_{w\in\mathcal{X}} \sup_{v\in\mathcal{Y}} \frac{b(w,v)}{\|w\|_{\mathcal{X}}\, \|v\|_{\mathcal{Y}}}$.*

Proof. For $w \in \mathcal{X}$ and $v \in \mathcal{Y}$ we have $b(w,v) = \int_I a(A^{-1}\dot{w}(t) + w(t), v(t))\, dt$. Given $v \in \mathcal{Y}$, we have $Av \in L_2(I;V') = \mathcal{Y}'$ and Corollary 2.5 ensures that there exists a unique $z \in \mathcal{X}$ such that $\dot{z} + Az = Av$, i.e, $v = A^{-1}\dot{z} + z$. Then, we have

$$\begin{aligned} \sup_{v\in\mathcal{Y}} \frac{b(w,v)}{\|v\|_{\mathcal{Y}}} &= \sup_{z\in\mathcal{X}} \frac{b(w, A^{-1}\dot{z} + z)}{\|A^{-1}\dot{z} + z\|_{\mathcal{Y}}} \\ &= \sup_{z\in\mathcal{X}} \frac{\int_I a(A^{-1}\dot{w}(t) + w(t), A^{-1}\dot{z}(t) + z(t))\, dt}{\|A^{-1}\dot{z} + z\|_{\mathcal{Y}}} \\ &= \|A^{-1}\dot{w} + w\|_{\mathcal{Y}} = \|w\|_{\mathcal{X}}, \end{aligned}$$

where the last step is shown in (2.11). The claim is thus proven. □

2.3. **Using temporal transformation.** Another possibility to derive a lower inf-sup-bound is the transformation of the initial value problem (2.1) in the following (standard and well-known) way. In view of the Gårding inequality (2.3), setting $\hat{u}(t) := e^{-\lambda t}u(t)$, $\hat{v}(t) := e^{\lambda t}v(t)$ and $\hat{g}(t) := e^{-\lambda t}g(t)$ solves the variational problem

$$\hat{b}(\hat{w}, \hat{v}) = \hat{f}(\hat{v}), \qquad \forall \hat{v} \in \mathcal{Y},$$

where

$$\hat{b}(\hat{w}, \hat{v}) := \int_0^T \langle \frac{d}{dt}\hat{w}(t), \hat{v}(t) \rangle_{V'\times V}\, dt + \int_0^T \hat{a}(\hat{w}(t), \hat{v}(t))\, dt$$

as well as $\hat{a}(\hat{w}(t), \hat{v}(t)) := a(\hat{w}(t), \hat{v}(t)) + \lambda(\hat{w}(t), \hat{v}(t))_H$ and for the right-hand side $\hat{f}(\hat{v}) := \int_I \langle \hat{g}(t), \hat{v}(t) \rangle_{V'\times V}\, dt$. Note, that the form $\hat{a}$ fulfills (2.3) with $\lambda = 0$ which gives rise to the following lower inf-sup-bound

Corollary 2.7. *Under the above assumptions, we get the following lower bound for the inf-sup-constant*

$$\beta \geq \hat{\beta}_{LB} := \frac{e^{-2\lambda T}}{\max\{\sqrt{1 + 2\lambda^2 \varrho^4}, \sqrt{2}\}} \times \frac{\min\{1, \alpha \min\{1, M_a^{-2}\}\}}{\max\{1, (\beta_a^*)^{-1}\}\sqrt{2}}. \tag{2.12}$$

Proof. It is readily seen that $\hat{b}(\hat{w}, \hat{v}) = b(w,v)$ with the above transformations, so that it remains to estimate the norms. It is known from [10, Appendix A] that

$$\|w\|_{\mathcal{X}} \leq e^{\lambda T} \max\{\sqrt{1 + 2\lambda^2 \varrho^4}, \sqrt{2}\} \|\hat{w}\|_{\mathcal{X}}, \quad \|v\|_{\mathcal{Y}} \leq e^{\lambda T} \|\hat{v}\|_{\mathcal{Y}}.$$

This implies that

$$\begin{aligned} \inf_{w\in\mathcal{X}} \sup_{v\in\mathcal{Y}} \frac{b(w,v)}{\|w\|_{\mathcal{X}}\, \|v\|_{\mathcal{Y}}} &= \inf_{w\in\mathcal{X}} \sup_{v\in\mathcal{Y}} \frac{\hat{b}(\hat{w}, \hat{v})}{\|w\|_{\mathcal{X}}\, \|v\|_{\mathcal{Y}}} \\ &\geq e^{-2\lambda T} \max\{\sqrt{1 + 2\lambda^2 \rho^4}, \sqrt{2}\}^{-1} \inf_{w\in\mathcal{X}} \sup_{v\in\mathcal{Y}} \frac{\hat{b}(\hat{w}, \hat{v})}{\|\hat{w}\|_{\mathcal{X}}\, \|\hat{v}\|_{\mathcal{Y}}}. \end{aligned}$$

The result then follows from Proposition 2.2. □

Remark 2.8. *Obviously this approach yields an inf-sup bound that behaves as $e^{-\lambda T}$ — often extremely pessimistic and clearly unsuitable for error estimation in long-time integration.*

2.4. **Petrov-Galerkin Truth Approximation.** Let $\mathcal{X}_\delta \subset \mathcal{X}$, $\mathcal{Y}_\delta \subset \mathcal{Y}$ be finite dimensional subspaces and $u_\delta \in \mathcal{X}_\delta$ the discrete approximation of (2.6), i.e.,

$$b(u_\delta, v_\delta) = f(v_\delta), \qquad \forall v_\delta \in \mathcal{Y}_\delta, \tag{2.13}$$

$s_\delta = \int_0^T \ell(u_\delta(t))\,dt$. Henceforth, we concentrate on the case $H = L_2(\Omega)$, $V = H_0^1(\Omega)$. Let $\mathcal{X}_\delta = S_{\Delta t} \otimes V_h$, $\mathcal{Y}_\delta = Q_{\Delta t} \otimes V_h$, $\delta = (\Delta t, h)$, where $S_{\Delta t}$, V_h are piecewise linear and $Q_{\Delta t}$ piecewise constant finite elements with respect to triangulations $\mathcal{T}_h^{\text{space}}$ in space and $\mathcal{T}_{\Delta t}^{\text{time}} \equiv \{t^{k-1} \equiv (k-1)\Delta t < t \le k\,\Delta t \equiv t^k, 1 \le k \le K\}$ in time for $\Delta t := T/K$.

Let $S_{\Delta t} = \text{span}\{\sigma^1, \dots, \sigma^K\}$, where σ^k is the (interpolatory) hat-function with the nodes t^{k-1}, t^k and t^{k+1} (resp. truncated for $k = K$) and $Q_{\Delta t} = \text{span}\{\tau^1, \dots, \tau^K\}$, where $\tau^k = \chi_{I^k}$, the characteristic function on $I^k := (t^{k-1}, t^k)$. Finally, let $V_h = \text{span}\{\phi_1, \dots, \phi_{n_h}\}$ e.g. be the nodal basis w.r.t. $\mathcal{T}_h^{\text{space}}$. For any given $w_\delta = \sum_{k=1}^K \sum_{i=1}^{n_h} w_i^k \sigma^k \otimes \phi_i \in \mathcal{X}_\delta$ and $v_\delta = \sum_{\ell=1}^K \sum_{j=1}^{n_h} v_j^\ell\, \tau^k \otimes \phi_j$ we obtain

$$\begin{aligned}
b(w_\delta, v_\delta) &= \int_I \big(\langle \dot w_\delta(t), v_\delta(t)\rangle_{V'\times V} + a(w_\delta(t), v_\delta(t))\big)\,dt \\
&= \sum_{k,\ell=1}^{K} \sum_{i,j=1}^{n_h} w_k^i v_\ell^j \big[(\dot\sigma^k, \tau^\ell)_{L_2(I)}\,(\phi_i, \phi_j)_H + (\sigma^k, \tau^\ell)_{L_2(I)}\,a(\phi_i, \phi_j)\big] \\
&= \mathbf{w}_\delta^T \mathbf{B}_\delta \mathbf{v}_\delta,
\end{aligned}$$

where

$$\mathbf{B}_\delta := \mathbf{N}_{\Delta t}^{\text{time}} \otimes \mathbf{M}_h^{\text{space}} + \mathbf{M}_{\Delta t}^{\text{time}} \otimes \mathbf{A}_h^{\text{space}} \tag{2.14}$$

and $\mathbf{M}_h^{\text{space}} := [(\phi_i, \phi_j)_{L_2(\Omega)}]_{i,j=1,\dots,n_h}$, $\mathbf{M}_{\Delta t}^{\text{time}} := [(\sigma^k, \tau^\ell)_{L_2(I)}]_{k,\ell=1,\dots,K}$ are the spatial and temporal mass matrices as well as $\mathbf{N}_{\Delta t}^{\text{time}} := [(\dot\sigma^k, \tau^\ell)_{L_2(I)}]_{k,\ell=1,\dots,K}$ and $\mathbf{A}_h^{\text{space}} := [a(\phi_i, \phi_j)]_{i,j=1,\dots,n_h}$. For our particular spaces we obtain (denoting by $\delta_{k,\ell}$ the discrete Kronecker delta)

$$\begin{aligned}
(\dot\sigma^k, \tau^\ell)_{L_2(I)} &= \delta_{k,\ell} - \delta_{k+1,\ell}, \qquad (\sigma^k, \tau^\ell)_{L_2(I)} = \frac{\Delta t}{2}(\delta_{k,\ell} + \delta_{k+1,\ell}), \\
b(w_\delta, \tau^\ell \otimes \phi_j) &= \sum_{i=1}^{n_h} \Big[(w_i^\ell - w_i^{\ell-1})(\phi_i, \phi_j)_H + \frac{\Delta t}{2}(w_i^\ell + w_i^{\ell-1})\,a(\phi_i, \phi_j)\Big] \\
&= \Delta t\Big[\mathbf{M}_h^{\text{space}} \frac{1}{\Delta t}(\mathbf{w}^\ell - \mathbf{w}^{\ell-1}) + \mathbf{A}_h^{\text{space}} \mathbf{w}^{\ell-1/2}\Big],
\end{aligned}$$

where $\mathbf{w}^\ell := (w_i^\ell)_{i=1,\dots,n_h}$, $w_i^{\ell-1/2} := \frac{1}{2}(w_i^\ell + w_i^{\ell-1})$ and $\mathbf{w}^{\ell-1/2}$ accordingly. If we use a trapezoidal approximation of the right-hand side temporal integration

$$\begin{aligned}
f(\tau^\ell \otimes \phi_j) &= \int_0^T \langle g(t), \tau^\ell \otimes \phi_j\rangle_{V'\times V}\,dt \\
&\approx \frac{\Delta t}{2}\langle g(t^{\ell-1}) + g(t^\ell), \phi_j\rangle_{V'\times V} = \frac{\Delta t}{2}(\mathbf{g}^{\ell-1} + \mathbf{g}^\ell)_j = \Delta t\ \mathbf{g}_j^{\ell-1/2},
\end{aligned}$$

where $\mathbf{g}^\ell = (\langle g(t^\ell), \phi_j\rangle_{V'\times V})_{j=1,\dots,n_h}$, then we can rewrite (2.13) as

$$\frac{1}{\Delta t}\mathbf{M}_h^{\text{space}}(\mathbf{w}^\ell - \mathbf{w}^{\ell-1}) + \mathbf{A}_h^{\text{space}} \mathbf{w}^{\ell-1/2} = \mathbf{g}^{\ell-1/2}, \qquad \mathbf{w}^0 := 0, \tag{2.15}$$

which is nothing else then the well-known Crank–Nicolson (CN) scheme; hence, we can derive error bounds for the CN scheme via our space-time formulation.

For the analysis we introduce a different norm on $\mathcal{X}$: For $w \in \mathcal{X}$ set $\bar{w}^k := (\Delta t)^{-1} \int_{I^k} w(t)\, dt \in V$ and $\bar{w} := \sum_{k=1}^K \chi_{I^k} \otimes \bar{w}^i \in L_2(I;V)$; then, set

$$|||w|||^2_{\mathcal{X},\delta} := \|\dot{w}\|^2_{L_2(I;V')} + \|\bar{w}\|^2_{L_2(I;V)} + \|w(T)\|^2_H$$

and the inf-sup parameter as well as the stability parameter

$$\beta_\delta := \inf_{w_\delta \in \mathcal{X}_\delta} \sup_{v_\delta \in \mathcal{Y}_\delta} \frac{b(w_\delta, v_\delta)}{|||w_\delta|||_{\mathcal{X},\delta}\, \|v_\delta\|_{\mathcal{Y}}}, \qquad \gamma_\delta := \sup_{w_\delta \in \mathcal{X}_\delta} \sup_{v_\delta \in \mathcal{Y}_\delta} \frac{b(w_\delta, v_\delta)}{|||w_\delta|||_{\mathcal{X},\delta}\, \|v_\delta\|_{\mathcal{Y}}}.$$

Proposition 2.9 ([13, Proposition 3])**.** *Let $a(\cdot,\cdot)$ be symmetric, bounded and coercive and set $\|\phi\|^2_V := a(\phi,\phi)$, $\phi \in V$; then we have $\beta_\delta = \gamma_\delta = 1$.*

Proof. Since $v_\delta \in \mathcal{Y}_\delta$ is piecewise constant in time, we have $\int_I a(w(t), v_\delta(t))\, dt = \int_I a(\bar{w}(t), v_\delta(t))\, dt$ for all $w \in \mathcal{X}$. Hence, $b(w_\delta, v_\delta) = \int_I a(A_h^{-1}\dot{w}_\delta(t) + \bar{w}_\delta(t), v_\delta(t))\, dt$, where $z_\delta(t) := A_h^{-1}\dot{w}_\delta(t)$ is defined by $a(z_\delta(t), \phi_h) = \langle \dot{w}_\delta(t), \phi_h\rangle_{V'\times V}$ for all $\phi_h \in V_h$. Note that for $\tilde{v} \in V'$ we have $\|A_h^{-1}\tilde{v}\|^2_V = a(A_h^{-1}\tilde{v}, A_h^{-1}\tilde{v}) = \langle \tilde{v}, A_h^{-1}\tilde{v}\rangle_{V'\times V} = \|\tilde{v}\|^2_{V'}$. We will prove later that for all $v_\delta \in \mathcal{Y}_\delta$ there exists a unique $z_\delta \in \mathcal{X}_\delta$ such that

$$\int_I a(A_h^{-1}\dot{z}_\delta(t) + \bar{z}_\delta(t), q_\delta(t))\, dt = \int_I a(v_\delta(t), q_\delta(t))\, dt \qquad \forall q_\delta \in \mathcal{Y}_\delta. \tag{2.16}$$

Note that $v_\delta := A_h^{-1}\dot{z}_\delta + \bar{z}_\delta \in \mathcal{Y}_\delta$ for $z_\delta \in \mathcal{X}_\delta$. Hence,

$$\begin{aligned}
\sup_{v_\delta \in \mathcal{Y}_\delta} \frac{b(w_\delta, v_\delta)}{\|v_\delta\|_{\mathcal{Y}}} &= \sup_{z_\delta \in \mathcal{X}_\delta} \frac{b(w_\delta, A_h^{-1}\dot{z}_\delta + \bar{z}_\delta)}{\|A_h^{-1}\dot{z}_\delta + \bar{z}_\delta\|_{\mathcal{Y}}} \\
&= \sup_{z_\delta \in \mathcal{X}_\delta} \frac{\int_I a(A_h^{-1}\dot{w}_\delta + \bar{w}_\delta, A_h^{-1}\dot{z}_\delta + \bar{z}_\delta)\, dt}{\|A_h^{-1}\dot{z}_\delta + \bar{z}_\delta\|_{\mathcal{Y}}} = \|A_h^{-1}\dot{w}_\delta + \bar{w}_\delta\|_{\mathcal{Y}}
\end{aligned}$$

by the Cauchy-Schwarz inequality and choosing $z_\delta = w_\delta$. Next,

$$\begin{aligned}
\|A_h^{-1}\dot{w}_\delta + \bar{w}_\delta\|^2_{\mathcal{Y}} &= \|A_h^{-1}\dot{w}_\delta\|_{\mathcal{Y}} + \|\bar{w}_\delta\|_{\mathcal{Y}} + 2\int_I \langle \dot{w}_\delta(t), \bar{w}_\delta(t)\rangle_{V'\times V}\, dt \\
&= \|\dot{w}_\delta\|^2_{L_2(I;V')} + \|\bar{w}_\delta\|^2_{L_2(I;V)} + \|w_\delta(T)\|^2_H = |||w_\delta|||^2_{\mathcal{X},\delta},
\end{aligned}$$

so that $\sup_{v_\delta \in \mathcal{Y}_\delta} \frac{b(w_\delta, v_\delta)}{\|v_\delta\|_{\mathcal{Y}}} = |||w_\delta|||_{\mathcal{X},\delta}$ which implies $\beta_\delta = \gamma_\delta = 1$.

It remains to prove (2.16). Let $\lambda_j > 0$, $e_j \in \mathbb{R}^{n_h}$, $j = 1, \dots, n_h$, be the eigenvalues and normalized eigenvectors of A_h, i.e.,

$$a(e_j, \phi_h) = \lambda_j\, (e_j, \phi_h)_H \qquad \forall \phi_h \in V_h, \qquad \|e_j\|_H = 1,\ 1 \le j \le n_h.$$

Given $\mathcal{Y}_\delta \ni v_\delta = \sum_{k=1}^K v^k\, \tau^k$, $v^k = \sum_{j=1}^{n_h} v_j^k\, e_j \in V_h$, determine ζ_j^k, $k = 1, \dots, K$, $j = 1, \dots, n_h$ as the unique solution of the iteration

$$\zeta_j^0 = 0, \qquad \frac{1}{\Delta t}(\zeta_j^k - \zeta_j^{k-1}) + \frac{\lambda_j}{2}(\zeta_j^k + \zeta_j^{k-1}) = \lambda_j\, v_j^k,\ k = 1, \dots, K. \tag{2.17}$$

Then, define $z_\delta := \sum_{k=1}^K \sum_{j=1}^{n_h} \zeta_j^k\, e_j\, \zeta^k \in \mathcal{X}_\delta$, so that

$$\bar{z}_\delta = \sum_{k=1}^K \bar{z}_\delta^k\, \chi_{I^k} = \sum_{k=1}^K \frac{\Delta t}{2}(z^k + z^{k-1})\tau^k, \quad z^k := z_\delta(t^k),$$

since z_δ is piecewise linear in time. Then we obtain for any $q_\delta \in \mathcal{Y}_\delta$, $q_\delta = \sum_{k=1}^K q^k \tau^k$, $q^k = q_\delta(t^k)$

$$\begin{aligned}
\int_I a(v_\delta(t), q_\delta(t))\, dt &= \sum_{k,\ell=1}^K a(v^k, q^\ell) \int_I \tau^k(t)\, \tau^\ell(t)\, dt = \sum_{k=1}^K \Delta t\; a(v^k, q^k) \\
&= \sum_{k=1}^K \sum_{j=1}^{n_h} \Delta t\, v_j^k\, \lambda_j\, (e_j, q^k)_H = \sum_{k=1}^K \sum_{j=1}^{n_h} \Delta t\, v_j^k\, \lambda_j\, (e_j, q^k)_H \\
&= \sum_{k=1}^K \sum_{j=1}^{n_h} \Delta t\, (e_j, q^k)_H \Big[\frac{1}{\Delta t}(\sigma_j^k - \sigma_j^{k-1}) + \lambda_j\, \sigma_j^{k-1/2} \Big] \\
&= \sum_{k=1}^K \Big(\sum_{j=1}^{n_h} (\sigma_j^k - \sigma_j^{k-1}) e_j, q^k \Big)_H + \Delta t \sum_{k=1}^K \sum_{j=1}^{n_h} a(\sigma_j^{k-1/2} e_j, q^k) \\
&= \int_I \langle \dot{z}_\delta(t), q_\delta(t) \rangle_{V' \times V}\, dt + \int_I a(z_\delta(t), q_\delta(t))\, dt.
\end{aligned}$$

This proves the existence in (2.16). The uniqueness is seen as follows. Let $z_\delta, w_\delta \in \mathcal{X}_\delta$ be two solutions of (2.16), then

$$\int_I a(A_h^{-1}(\dot{z}_\delta(t) - \dot{w}_\delta(t)) + \bar{z}_\delta(t) - \bar{w}_\delta(t), q_\delta(t))\, dt = 0 \quad \forall q_\delta \in \mathcal{Y}_\delta.$$

By using the first argument as test function we arrive at $\|\dot{z}_\delta - \dot{w}_\delta\|_{L_2(I;V')} + \|\bar{z}_\delta - \bar{w}_\delta\|^2_{L_2(I;V)} = 0$, which shows the uniqueness in $\mathcal{X}$. □

Remark 2.10. *We may rephrase Proposition 2.9 also in the following way:*

$$\sup_{v_\delta \in \mathcal{Y}_\delta} \frac{b(w_\delta, v_\delta)}{\|v_\delta\|_{\mathcal{Y}}} = |||w_\delta|||_{\mathcal{X},\delta}, \quad w_\delta \in \mathcal{X}_\delta. \tag{2.18}$$

Moreover, the proof also shows that

$$\forall\, 0 \neq w_\delta \in \mathcal{X}_\delta \quad \exists v_\delta \in \mathcal{Y}_\delta : \qquad \frac{b(w_\delta, v_\delta)}{\|v_\delta\|_{\mathcal{Y}}} = |||w_\delta|||_{\mathcal{X},\delta} \neq 0. \tag{2.19}$$

Remark 2.11. *Proposition 2.9 also shows the well-posedness of the discrete problem with continuity and inf-sup constant being unity.*

For later purpose, we consider also the dual inf-sup parameter defined as

$$\beta_\delta^* := \inf_{v_\delta \in \mathcal{Y}_\delta} \sup_{w_\delta \in \mathcal{X}_\delta} \frac{b(w_\delta, v_\delta)}{|||w_\delta|||_{\mathcal{X},\delta}\, \|v_\delta\|_{\mathcal{Y}}}.$$

Proposition 2.12. *Under the hypotheses of Proposition 2.9, we have $\beta_\delta^* = \beta_\delta = 1$.*

Proof. We use Nečas' theorem [6, Theorem 3.3] that shows that (2.18) and (2.19) are equivalent to $\beta_\delta^* = \beta_\delta = 1$. □

3. The Reduced Basis Method (RBM)

3.1. **Parameter-dependence.** Now, let $\mu \in \mathcal{D} \subseteq \mathbb{R}^P$ be a parameter vector and $A = A(\mu)$ a parameter-dependent linear partial differential operator. It is fairly standard to assume that $A(\mu)$ is induced by a bilinear form $a(\cdot, \cdot; \mu)$ that is affine w.r.t. the parameter, i.e., there exist functions θ_q^a and bilinear forms $a_q(\cdot,\cdot)$ such that

$$a(\psi,\phi;\mu) = \sum_{q=1}^{Q^a} \theta_q^a(\mu)\, a_q(\psi,\phi), \qquad \mu\in\mathcal{D},\ \psi,\phi\in V. \tag{3.1}$$

We obtain the parameter-dependent space-time bilinear form

$$b(w,v;\mu) = [\dot{w},v;\mu]_{\mathcal{H}} + \mathcal{A}[w,v;\mu], \quad \text{where } \mathcal{A}[w,v;\mu] = \int_I a(w(t),v(t);\mu)\,dt,$$

and $[\cdot,\cdot;\mu]_{\mathcal{H}}$ is a parameter-dependent version of $[\cdot,\cdot]_{\mathcal{H}}$ with a similar expansion as in (3.1), so that we derive an affine decomposition according to

$$b(w,v;\mu) = \sum_{q=1}^{Q} \theta_q(\mu)\, b_q(w,v).$$

Also the right-hand side may depend on the parameter and is also assumed to be affine in functions of the parameter, i.e.,

$$f(v;\mu) = \sum_{q=1}^{Q} \theta_q^f(\mu)\, f_q(v), \qquad \mu\in\mathcal{D},\ v\in\mathcal{Y}. \tag{3.2}$$

If (3.1,3.2) are not satisfied, it is fairly standard to construct an approximation via the *Empirical Interpolation Method* (EIM), [1, 12].

The parameter-dependent version of (2.6) then reads

$$u(\mu)\in\mathcal{X}: \qquad b(u(\mu),v;\mu) = f(v;\mu)\ \ \forall v\in\mathcal{Y}. \tag{3.3}$$

The output reads $s(\mu) := \int_I \ell(u(t;\mu))\,dt$. The truth approximations are then fairly standard, i.e.,

$$u_\delta(\mu)\in\mathcal{X}_\delta: \qquad b(u_\delta(\mu),v_\delta;\mu) = f(v_\delta;\mu)\ \ \forall v_\delta\in\mathcal{Y}_\delta, \tag{3.4}$$

and the output reads $s_\delta(\mu) := \int_I \ell(u_\delta(t;\mu))\,dt = J(u_\delta(\mu))$. Defining

$$\gamma_\delta(\mu) := \sup_{w_\delta\in\mathcal{X}_\delta}\sup_{v_\delta\in\mathcal{Y}_\delta} \frac{b(w_\delta,v_\delta;\mu)}{|||w_\delta|||_{\mathcal{X},\delta}\,\|v_\delta\|_{\mathcal{Y}}}, \quad \beta_\delta(\mu) := \inf_{w_\delta\in\mathcal{X}_\delta}\sup_{v_\delta\in\mathcal{Y}_\delta} \frac{b(w_\delta,v_\delta;\mu)}{|||w_\delta|||_{\mathcal{X},\delta}\,\|v_\delta\|_{\mathcal{Y}}} \tag{3.5}$$

it is well-known (see also [10]) from the Babuška-Aziz theorem that (3.3) is well-posed for all $\mu\in\mathcal{D}$ provided that the following three properties hold

(i) $\gamma_\delta(\mu)\le\gamma_\delta^{\mathrm{UB}}<\infty$, (ii) $\beta_\delta(\mu)\ge\beta_\delta^{\mathrm{LB}}>0$, (iii) $b(\cdot,\cdot;\cdot)$ is surjective.

3.2. **RB error bounds.** We introduce a standard Reduced Basis (RB) approximation [3, 8, 9] for the Crank–Nicolson interpretation (2.15) of our discrete problem. Let $V_N := \mathrm{span}\{\xi_1,\dots,\xi_N\}\subset V_h$ be an RB space provided for example by the POD-Greedy procedure of [4]. Then, set $\mathcal{X}_{\Delta t,N} := S_{\Delta t}\otimes V_N$, $\mathcal{Y}_{\Delta t,N} := Q_{\Delta t}\otimes V_N$ and let $u_N(\mu)\in\mathcal{X}_{\Delta t,N}$ denote the unique solution of

$$b(u_N(\mu),v_N;\mu) = f(v_N;\mu) \qquad \forall v_N\in\mathcal{Y}_{\Delta t,N}. \tag{3.6}$$

The RB output is then given by

$$s_N(\mu) := J(u_N(\mu)) = \int_I \ell(u_N(t;\mu))dt (= \int_I \ell(\bar{u}_N(t;\mu))dt).$$

(It is possible, alternatively, to consider a space–time RB approximation as well [11].)

We define the common RB-quantities, namely the *error* $e_N(\mu) := u_\delta(\mu) - u_N(\mu)$, the *residual*

$$r_N(v;\mu) := f(v;\mu) - b(u_N(\mu), v; \mu) = b(e_N(\mu), v; \mu), \quad v \in \mathcal{Y},$$

the *Riesz representation* $\hat{r}_N(\mu) \in \mathcal{Y}$ (not in $\mathcal{X}$!) as

$$(\hat{r}_N(\mu), v)_{\mathcal{Y}} = r_N(v;\mu), \quad v \in \mathcal{Y}$$

and $\|\hat{r}_N(\mu)\|_{\mathcal{Y}} = \|r_N(\mu)\|_{\mathcal{Y}'}$. The 'truth dual norm' on $\mathcal{X}'$ is defined as

$$|||\tilde{J}|||_{\mathcal{X}',\delta} := \sup_{w \in \mathcal{X}} \frac{\tilde{J}(w)}{|||w|||_{\mathcal{X},\delta}}, \qquad \tilde{J} \in \mathcal{X}'.$$

It is then simple [9] to demonstrate

Proposition 3.1. *The following estimates hold*

(a) $|||u_\delta(\mu) - u_N(\mu)|||_{\mathcal{X},\delta} \le \frac{\|r_N(\mu)\|_{\mathcal{Y}'}}{\beta_\delta^{\mathrm{LB}}}$;
(b) $|s_\delta(\mu) - s_N(\mu)| \le \frac{\sqrt{T}}{\beta_\delta^{LB}} \|\ell\|_{V'} \|r_N(\mu)\|_{\mathcal{Y}'}$;
(c) $|s_\delta(\mu) - s_N(\mu)| \le \frac{1}{\beta_\delta^{\mathrm{LB}}} |||J|||_{\mathcal{X}',\delta} \; \|r_N(\mu)\|_{\mathcal{Y}'}$.

Proof. The proof follows standard arguments

$$\beta_\delta^{\mathrm{LB}} |||u_\delta(\mu) - u_N(\mu)|||_{\mathcal{X},\delta} \le \sup_{v_\delta \in \mathcal{Y}_\delta} \frac{b(e_N(\mu), v_\delta(\mu))}{\|v_\delta\|_{\mathcal{Y}}} = \sup_{v_\delta \in \mathcal{Y}_\delta} \frac{r_N(v_\delta;\mu)}{\|v_\delta\|_{\mathcal{Y}}} = \|r_N(\mu)\|_{\mathcal{Y}'}$$

as well as

$$\begin{aligned} |s_\delta(\mu) - s_N(\mu)| &\le \int_I |\ell(u_\delta(t;\mu)) - \ell(u_N(t;\mu))|\, dt \\ &\le \int_I \|\ell\|_{V'} \, \|u_\delta(t;\mu) - u_N(t;\mu)\|_V \, dt \\ &\le \|\ell\|_{V'} \sqrt{T} \|e_N(\mu)\|_{\mathcal{Y}} \le \sqrt{T} \|\ell\|_{V'} \, |||e_N(\mu)|||_{\mathcal{X},\delta} \end{aligned}$$

which –combined with (a)– proves (b). Finally, (c) follows standard lines since $s_\delta(\mu) - s_N(\mu) = J(e_N(\mu))$ and using (a). □

Remark 3.2. *The estimates in Proposition 3.1 (b) and (c) differ in the fact that (b) explicitly involves the time on the right-hand side so that the error estimate grows with increasing time T. Even though the right-hand side of (c) is formally independent of T, one would expect that the dual norm of J grows with increasing T since J is the integral over a time period of length T. However, the estimate in (c) still should be sharper since it avoids the application of one additional Cauchy-Schwarz inequality.*

The utility of these *a posteriori* error bounds is critically dependent on the dependence of β_δ as a function of the parameter μ and final time T, $\beta_\delta(\mu; T)$. We will investigate this dependence in our numerical experiments described in Section 4 below.

Primal-dual formulation. The estimate (b) in Proposition 3.1 is not completely satisfying since the error estimator grows with respect to time. In order to overcome this issue, we consider a dual problem. The original, truth and RB dual problem, respectively, read

$$\text{find } z(\mu) \in \mathcal{Y}: \qquad b(w, z(\mu); \mu) = -J(w) \qquad \forall\, w \in \mathcal{X}, \tag{3.7}$$

$$\text{find } z_\delta(\mu) \in \mathcal{Y}_\delta: \qquad b(w_\delta, z_\delta(\mu); \mu) = -J(w_\delta) \qquad \forall\, w_\delta \in \mathcal{X}_\delta, \tag{3.8}$$

$$\text{find } z_{\tilde N}(\mu) \in \tilde{\mathcal{Y}}_{\Delta t,\tilde N}: \ b(w_{\tilde N}, z_{\tilde N}(\mu); \mu) = -J(w_{\tilde N}) \quad \forall\, w_{\tilde N} \in \tilde{\mathcal{X}}_{\Delta t,\tilde N}, \tag{3.9}$$

where $\mathcal{X}_{\Delta t,\tilde N} := S_{\Delta t} \otimes V_{\tilde N}$, $\mathcal{Y}_{\Delta t,\tilde N} := Q_{\Delta t} \otimes V_{\tilde N}$ and $V_{\tilde N} \subset V_h$ is a spatial RB-space also possibly different from V_N. The dual RB residual is defined as $\tilde r_{\tilde N}(w;\mu) := -J(w) - b(w, z_{\tilde N}(\mu);\mu)$ for $w \in \mathcal{X}$, i.e., $\tilde r_{\tilde N}(\mu) := \tilde r_{\tilde N}(\cdot;\mu) \in \mathcal{X}'$ and the dual error as $\tilde e_{\tilde N}(\mu) := z_\delta(\mu) - z_{\tilde N}(\mu)$. Finally, we define the RB output in this primal-dual setting as

$$s_N(\mu) := J(u_N(\mu)) - r_N(z_{\tilde N}(\mu)).$$

Then, standard RB-arguments yield:

Proposition 3.3. *The following estimates hold*

(a) $\|z_\delta(\mu) - z_{\tilde N}(\mu)\|_{\mathcal{Y}} \le \frac{1}{\beta_{LB}} |||\tilde r_{\tilde N}(\mu)|||_{\mathcal{X}',\delta}$;

(b) $|s_\delta(\mu) - s_N(\mu)| \le \frac{1}{\beta_{LB}} \|r_N(\mu)\|_{\mathcal{Y}'}\, |||\tilde r_{\tilde N}(\mu)|||_{\mathcal{X}',\delta}$.

Proof. Since $\beta^*_{\mathrm{LB}} = \beta_{\mathrm{LB}}$, we obtain

$$\beta_{\mathrm{LB}}\, \|\tilde e_{\tilde N}(\mu)\|_{\mathcal{Y}} \le \sup_{w_\delta \in \mathcal{X}} \frac{b(w_\delta, \tilde e_{\tilde N}(\mu);\mu)}{|||w_\delta|||_{\mathcal{X},\delta}} = \sup_{w_\delta \in \mathcal{X}} \frac{\tilde r_{\tilde N}(w_\delta;\mu)|}{|||w_\delta|||_{\mathcal{X},\delta}} = |||\tilde r_{\tilde N}(\mu)|||_{\mathcal{X}',\delta},$$

which proves (a). In order to show (b), we first note that

$$\begin{aligned} s_\delta(\mu) - s_N(\mu) &= J(e_N(\mu)) + r_N(z_{\tilde N}(\mu)) = J(e_N(\mu)) + b(e_N(\mu), z_{\tilde N}(\mu);\mu) \\ &= -\tilde r_{\tilde N}(e_N(\mu);\mu) \end{aligned}$$

so that $|s_\delta(\mu) - s_N(\mu)| \le |||\tilde r_{\tilde N}(\mu)|||_{\mathcal{X}',\delta}\, \|e_N(\mu)\|_{\mathcal{X}}$ so that (b) follows by Proposition 3.1, (a). □

Remark 3.4. *Note that both estimates in Proposition 3.3 do not depend on the time T. Again, however, one expects that the space-time norms of the residuals will show T-dependence, which is due to the nature of the evolution problem, not because of the discretization.*

Let us comment on the numerical realization of (3.8). We are looking for $z_\delta(\mu) = \sum_{\ell=1}^K \sum_{j=1}^{n_h} z_j^\ell\, \tau^\ell \otimes \phi_j \in \mathcal{Y}_\delta$, $\mathbf{z}_\delta^\ell := (z_j^\ell)_{j=1,\dots,n_h}$. Then, for $1 \le i \le n_h$, we obtain

$$\begin{aligned} b(\sigma^K \otimes \phi_i) &= \sum_{\ell=1}^K \sum_{j=1}^{n_h} z_j^\ell \left[(\dot\sigma^K, \tau^\ell)_{L_2(I)}\, (\phi_i, \phi_j)_H + (\sigma^K, \tau^\ell)_{L_2(I)}\, a(\phi_i, \phi_j) \right] \\ &= \sum_{j=1}^{n_h} \left(z_j^K\, (\phi_i, \phi_j)_H + \frac{\Delta t}{2} z_j^K\, a(\phi_i, \phi_j) \right) = [(\mathbf{M}_h^{\text{space}} + \frac{\Delta t}{2} \mathbf{A}_h^{\text{space}}) \mathbf{z}^K]_i \end{aligned}$$

and $J(\sigma^K \otimes \phi_i) = \frac{\Delta t}{2} \ell(\phi_i)$, so that $\mathbf{z}_\delta^K(\mu)$ can be computed via the solution of

$$(\mathbf{M}_h^{\text{space}} + \frac{\Delta t}{2} \mathbf{A}_h^{\text{space}}) \mathbf{z}_\delta^K(\mu) = -\frac{\Delta t}{2} \mathbf{l}, \tag{3.10}$$

where $\mathbf{l} := (\ell(\phi_i))_{i=1,\dots,n_h}$. Correspondingly, we obtain for $k = K-1,\dots,1$

$$\begin{aligned} b(\sigma^k \otimes \phi_i) &= \sum_{\ell=1}^{K}\sum_{j=1}^{n_h} z_j^\ell \,[(\dot\sigma^k,\tau^\ell)_{L_2(I)}\,(\phi_i,\phi_j)_H + (\sigma^k,\tau^\ell)_{L_2(I)}\,a(\phi_i,\phi_j)] \\ &= \sum_{j=1}^{n_h}[(z_k^k - z_j^{k+1})(\phi_i,\phi_j)_H + \frac{\Delta t}{2}(z_j^k + z_j^{k+1})\,a(\phi_i,\phi_j)] \\ &= [\mathbf{M}_h^{\text{space}}(\mathbf{z}_\delta^k(\mu) - \mathbf{z}_\delta^{k+1}(\mu)) + \frac{\Delta t}{2}\mathbf{A}_h^{\text{space}}(\mathbf{z}_\delta^k(\mu) + \mathbf{z}_\delta^{k+1}(\mu))]_i \end{aligned}$$

as well as $J(\sigma^k\otimes\phi_i) = \Delta t\,\ell(\phi_i)$, so that for $k = K-1,\dots,1$

$$(\mathbf{M}_h^{\text{space}} + \frac{\Delta t}{2}\mathbf{A}_h^{\text{space}})\mathbf{z}_\delta^k(\mu) = -\Delta t\,\mathbf{l} + (\mathbf{M}_h^{\text{space}} - \frac{\Delta t}{2}\mathbf{A}_h^{\text{space}})\mathbf{z}_\delta^{k+1}(\mu). \tag{3.11}$$

This means that (3.10) and (3.11) is an iterative procedure for computing the dual truth solution very similar to a backward Crank-Nicholson scheme. We do not need to solve a coupled space-time problem.

3.3. **Numerical realization.** We are now going to consider the quantities that we have to determine while numerically approximating terms like the inf-sup-constants.

Norms. Let $w_\delta = \sum_{i=1}^r \sum_{k=1}^{n_h} w_k^i\,\sigma^i\otimes\phi_k \in \mathcal{X}_\delta$, $\mathbf{w}_\delta := (w_k^i)_{i,k}$. Then,

$$\begin{aligned} \|w_\delta\|^2_{L_2(I;V)} &= \int_I \|w_\delta(t)\|_V^2\,dt = \sum_{k,\ell=1}^{K}\sum_{i,j=1}^{n_h} w_k^i w_\ell^j \int_I \sigma^k(t)\,\sigma^\ell(t)\,(\phi_i,\phi_j)_V\,dt \\ &= \mathbf{w}_\delta^T(\mathbf{M}_{\Delta t}^{\text{time}}\otimes\mathbf{V}_h^{\text{space}})\mathbf{w}_\delta, \end{aligned}$$

where $\mathbf{M}_{\Delta t}^{\text{time}}$ is the temporal mass matrix and $\mathbf{V}_h^{\text{space}} = [(\phi_k,\phi_l)_V]_{k,l}$ the spatial matrix w.r.t. the V-inner product. For the discrete norm $|||\cdot|||_{\mathcal{X},\delta}$, we need $|||\bar w_\delta|||_{L_2(I;V)}$. We obtain

$$\begin{aligned} \|\bar w_\delta\|^2_{L_2(I;V)} &= \sum_{k=1}^{K}\int_{I^k}(\bar w^k(t),\bar w^k(t))_V\,dt = \frac{1}{\Delta t}\sum_{k=1}^{K}\Big(\int_{I^k} w(t)\,dt, \int_{I^k} w(s)\,ds\Big)_V \\ &= \Delta t\sum_{k=1}^{K}\sum_{i,j=1}^{n_h} w_i^k w_j^k(\phi_i,\phi_j)_V = \Delta t\,\mathbf{w}_\delta^T(\mathbf{I}_{\Delta t}^{\text{time}}\otimes\mathbf{V}_h^{\text{space}})\mathbf{w}_\delta. \end{aligned}$$

The second part of the $\mathcal{X}$-norm, $\|\dot w_\delta\|_{L_2(I;V')}$, is a little bit more involved due to the appearance of the V'-norm. Given $\tilde v_h = \sum_{k=1}^{n_h}\tilde v_k\,\phi_k \in V_h$, $\tilde{\mathbf{v}}_h = (\tilde v_k)_k$, we need the Riesz representation $R_h\tilde v_h = \sum_{k'=1}^{n_h} r_{k'}\,\phi_{k'}$, $\mathbf{r}_h = (r_{k'})_{k'}$ (since we know from the Riesz representation theorem that $\|R_h\tilde v_h\|_V = \|\tilde v_h\|_{V'}$), which is determined by the condition

$$(R_h\tilde v_h,\phi_\ell)_V = \sum_{k'=1}^{n_h} r_{k'}(\phi_{k'},\phi_\ell)_V = \sum_{k=1}^{n_h}\tilde v_k(\phi_k,\phi_\ell)_H = (\tilde v_h,\phi_\ell)_H \quad \forall \ell = 1,\dots,n_h,$$

or in condensed form $\mathbf{V}_h^{\text{space}}\mathbf{r}_h = \mathbf{M}_h^{\text{space}}\tilde{\mathbf{v}}_h$, i.e., $\mathbf{r}_h = (\mathbf{V}_h^{\text{space}})^{-1}\mathbf{M}_h^{\text{space}}\tilde{\mathbf{v}}_h$ for the coefficients. Then,

$$\begin{aligned} \|\tilde v_h\|^2_{V'} &= \|R_h\tilde v_h\|_V^2 = \mathbf{r}_h^T\mathbf{V}_h^{\text{space}}\mathbf{r}_h \\ &= ((\mathbf{V}_h^{\text{space}})^{-1}\mathbf{M}_h^{\text{space}}\tilde{\mathbf{v}}_h)^T\,\mathbf{V}_h^{\text{space}}(\mathbf{V}_h^{\text{space}})^{-1}\mathbf{M}_h^{\text{space}}\tilde{\mathbf{v}}_h \\ &= \tilde{\mathbf{v}}_h^T\,\mathbf{M}_h^{\text{space}}(\mathbf{V}_h^{\text{space}})^{-1}\mathbf{M}_h^{\text{space}}\,\tilde{\mathbf{v}}_h. \end{aligned}$$

Using this, we get

$$\begin{aligned}\|\dot{w}_\delta\|^2_{L_2(I;V')} &= \sum_{k,\ell=1}^{K}\sum_{i,j=1}^{n_h} w_i^k w_j^\ell \int_I \dot\sigma^k(t)\,\dot\sigma^\ell(t)(\mathbf{M}_h^{\text{space}}(\mathbf{V}_h^{\text{space}})^{-1}\mathbf{M}_h^{\text{space}})_{i,j}\,dt\\ &= \mathbf{w}_\delta^T\big(\mathbf{V}_{\Delta t}^{\text{time}}\otimes(\mathbf{M}_h^{\text{space}}(\mathbf{V}_h^{\text{space}})^{-1}\mathbf{M}_h^{\text{space}})\big)\mathbf{w}_\delta,\end{aligned}$$

where $\mathbf{V}_{\Delta t}^{\text{time}} = [(\dot\sigma^k,\dot\sigma^\ell)_{L_2(I)}]_{k,\ell}$ is the temporal matrix of the derivatives. As for the last part, we obtain by $\sigma^k(T)=\delta_{k,K}$

$$\|w_\delta(T)\|_H^2 = \sum_{i,j=1}^{n_h} w_i^K\,w_j^K(\phi_i,\phi_j)_H = (\mathbf{w}_\delta^K)^T\mathbf{V}_h^{\text{space}}\mathbf{w}_\delta^K.$$

Consequently, we obtain for the norm $\|w_\delta\|^2_{\mathcal{X}} = \mathbf{w}_\delta^T\mathbf{X}_\delta\mathbf{w}_\delta + (\mathbf{w}_\delta^K)^T\mathbf{V}_h^{\text{space}}\mathbf{w}_\delta^K$ with

$$\mathbf{X}_\delta := \mathbf{M}_{\Delta t}^{\text{time}}\otimes\mathbf{V}_h^{\text{space}} + \mathbf{V}_{\Delta t}^{\text{time}}\otimes(\mathbf{M}_h^{\text{space}}(\mathbf{V}_h^{\text{space}})^{-1}\mathbf{M}_h^{\text{space}}). \tag{3.12}$$

For the discrete norm, we just need to modify $\mathbf{X}_\delta$ to $\mathbf{X}_\delta^{|||\cdot|||} := \mathbf{I}_{\Delta t}^{\text{time}}\otimes\mathbf{V}_h^{\text{space}} + \mathbf{V}_{\Delta t}^{\text{time}}\otimes(\mathbf{M}_h^{\text{space}}(\mathbf{V}_h^{\text{space}})^{-1}\mathbf{M}_h^{\text{space}})$.

For $v_\delta = \sum_{k=1}^K\sum_{i=1}^{n_h} v_i^k\,\tau^k\otimes\phi_i\in\mathcal{Y}_h$ we can use very similar arguments and get $\|v_\delta\|^2_{\mathcal{Y}} = \mathbf{v}_\delta^T\mathbf{Y}_\delta\mathbf{v}_\delta$ with

$$\mathbf{Y}_\delta := \mathbf{G}_{\Delta t}^{\text{time}}\otimes\mathbf{V}_h^{\text{space}} \tag{3.13}$$

and $\mathbf{G}_{\Delta t}^{\text{time}} = [(\tau^k,\tau^\ell)_{L_2(I)}]_{k,\ell}$ being the mass matrix of the $Q_{\Delta t}$-basis functions. In our case of piecewise constants, this coincides with $\Delta t\ \mathbf{I}_{\Delta t}^{\text{time}}$.

Bilinear form. We have already seen that $b(w_\delta,v_\delta) = \mathbf{w}_\delta^T\mathbf{B}_\delta\mathbf{v}_\delta$ with $\mathbf{B}_\delta$ given by (2.14).

Supremizing operator. Finally, we determine the supremizing operator for the bilinear form b, i.e., $T_\delta w_\delta = \arg\sup_{v_\delta\in\mathcal{Y}_\delta}\frac{b(w_\delta,v_\delta)}{\|v_\delta\|_{\mathcal{Y}}}$ for given $w_\delta\in\mathcal{X}_\delta$. It is well-known that $T_\delta w_\delta\in\mathcal{Y}_\delta$ is the solution of $(T_\delta w_\delta, v_\delta)_{\mathcal{Y}} = b(w_\delta,v_\delta)$ for all $v_\delta\in\mathcal{Y}_\delta$. The coefficients $\mathbf{t}_\delta$ of $T_\delta w_\delta$ are then given by $\mathbf{t}_\delta = \mathbf{Y}_\delta^{-1}\mathbf{B}_\delta^T\mathbf{w}_\delta$. Finally, it is also well known that

$$\beta_\delta = \inf_{w_\delta\in\mathcal{X}_\delta}\frac{\|Tw_\delta\|_{\mathcal{Y}}}{\|w_\delta\|_{\mathcal{X}}}$$

and we get

$$\frac{\|T_\delta w_\delta\|^2_{\mathcal{Y}}}{\|w_\delta\|^2_{\mathcal{X}}} = \frac{\mathbf{t}_\delta^T\mathbf{Y}_\delta\mathbf{t}_\delta}{\mathbf{w}_\delta^T\mathbf{X}_\delta\mathbf{w}_\delta} = \frac{\mathbf{w}_\delta^T\mathbf{B}_\delta\mathbf{Y}_\delta^{-1}\mathbf{B}_\delta^T\mathbf{w}_\delta}{\mathbf{w}_\delta^T\mathbf{X}_\delta\mathbf{w}_\delta}$$

with the involved matrices defined in (3.12), (3.13) and (2.14). Thus, we need to determine the square root of the smallest eigenvalue of the generalized eigenvalue problem $\mathbf{B}_\delta\mathbf{Y}_\delta^{-1}\mathbf{B}_\delta^T\mathbf{v} = \lambda\,\mathbf{X}_\delta\mathbf{v}$.

Error estimators. Since the computation of lower bounds for the inf-sup parameters has already been described, it remains to detail numerical schemes for the dual norms of the residuals, i.e., $\|r_N(\mu)\|_{\mathcal{Y}'}$ and $|||\tilde r|||_{\mathcal{X}',\delta}$. We have already seen that $\|r_N(\mu)||_{\mathcal{Y}'} = \|\hat e_N(\mu)\|_{\mathcal{Y}}$ with the Riesz representation $\hat e_N(\mu\in\mathcal{Y}_\delta$ which is given by $(\hat e_N(\mu), v_\delta)_{\mathcal{Y}} = f(v_\delta;\mu) - b(u_N(\mu), v_\delta;\mu)$ for all $v_\delta\in\mathcal{Y}_\delta$. In matrix-vector form for the coefficients this reads

$$\mathbf{Y}_\delta\hat{\mathbf{e}}_N(\mu) = \mathbf{f}_\delta(\mu) - \mathbf{B}_\delta^T\mathbf{u}_N(\mu),$$

where as above $\mathbf{Y}_\delta = \mathbf{G}^{\text{time}}_{\Delta t} \otimes \mathbf{V}^{\text{space}}_h$, $\mathbf{f}_\delta(\mu) = (f(\sigma^k \otimes \phi_i;\mu))_{k=1,\dots,K;\ i=1,\dots,n_h}$, $\mathbf{B}_\delta = \mathbf{N}^{\text{time}}_{\Delta t} \otimes \mathbf{M}^{\text{space}}_h + \mathbf{M}^{\text{time}}_{\Delta t} \otimes \mathbf{A}^{\text{space}}_h$ and $\mathbf{u}_N(\mu)$ being the vector of expansion coefficients of the RB-solution. Finally, for the right-hand side using the affine assumption (3.2) and defining $q_G \in \mathcal{Y}'$ by $[g_q, v]_{\mathcal{H}} = f_q(v)$, $v \in \mathcal{Y}$, we get

$$\begin{aligned} f(\sigma^k \otimes \phi_i;\mu) &= \sum_{q=1}^{Q} \theta^f_q(\mu)\, f_q(\sigma^k \otimes \phi_i) = \sum_{q=1}^{Q} \theta^f_q(\mu)\, [g_q, \sigma^k \otimes \phi_i]_{\mathcal{H}} \\ &= \sum_{k=1}^{K}\sum_{q=1}^{Q} \theta^f_q(\mu) \langle g_q(t^k), \phi_i\rangle_{V'\times V}, \end{aligned}$$

where we used the fact that σ^k are piecewise linear and are thus integrated exactly by a trapezoidal rule. This shows that expanding $g_q(t^k)$ in any appropriate basis gives rise to a tensor-product representation of $\mathbf{f}_\delta(\mu)$. Hence, the Riesz representation calculation is reduced to a sequence of K uncoupled spatial problems in V — just as in the non-space-time case.

The situation is different for $|||\tilde r_{\tilde N}(\mu)|||_{\mathcal{X}',\delta} = |||\hat{\tilde e}_{\tilde N}(\mu)|||_{\mathcal{X},\delta}$, where the Riesz representation $\hat{\tilde e}_{\tilde N} \in \mathcal{X}_\delta$ is defined by $(\hat{\tilde e}(\mu), w_\delta)_{\mathcal{X},\delta} = -J(w_\delta) - b(w_\delta, z_{\tilde N}(\mu);\mu)$ and the truth inner product is defined as

$$(v_\delta, w_\delta)_{\mathcal{X},\delta} := (\dot v_\delta, \dot w_\delta)_{V'} + (\bar v_\delta, \bar w_\delta)_V + (v_\delta(T), w_\delta(T))_H, \quad v_\delta, w_\delta \in \mathcal{X}_\delta.$$

In general, this sum can *not* be written as one tensor product. Thus, in practice this represents a space-time coupled problem and hence is rather expensive; however, at least in the primal-only formulation these calculations are restricted to the offline stage.

4. Numerical Results

Now, let $\mu = (\mu_1, \mu_2) \in \mathcal{D} := \mathbb{R}^2$ be a parameter vector and $A = A(\mu) := -\Delta u + \mu_1\, \boldsymbol{\beta}(x)\cdot \nabla u + \mu_2\, u$, i.e., a diffusion-convection-reaction operator with convection field $\boldsymbol{\beta}$. We report numerical results for the Crank–Nicolson scheme for various choices of the parameters μ_1, μ_2 as well as for different time steps Δt and uniform mesh sizes h. For simplicity, we consider the univariate case (in space) $\Omega = (0,1)$ and choose $\boldsymbol{\beta}(x) = x - \frac{1}{2}$. Let us denote by $\beta_\delta(\mu;T)$, $\gamma_\delta(\mu;T)$ the numerical values for the truth inf-sup and continuity constants, respectively, corresponding to parameter μ and final time T.

We start by confirming Proposition 2.9. Thus, we choose $\mu_1 = \mu_2 = 0$; for several values of T, h, and Δt we invariantly obtain 1.000 for both $\beta_\delta(\mu;T)$ and $\gamma_\delta(\mu;T)$, as must be the case.

The next issue is that we want to confirm the independence of $\beta_\delta(\mu;T)$ with respect to the discretization parameters $\delta = (\Delta t, h)$. In Table 1 we consider the case $\mu = (50, 10)$ with the final time $T = 0.2$. We clearly see the rapid convergence for $\Delta t \to 0$ as well as for $h \to 0$. This behavior has been observed for various choices of the parameters and final time.

Next, we investigate the case of convection, $\mu_2 = 0$, in which case a is coercive only for $\mu_1 < 2\pi^2$. We are particularly interested in the long-time behavior. The results are displayed in Table 2 for the choice $N_s = 19$ and $N_t = 10$ per time interval of length 0.2. The displayed numbers, however, are relatively invariant for sufficiently small h and Δt. We observe numerically an overall behavior of $\beta_\delta((\mu_1, 0);T) \sim (\mu_1 T)^{-1}$ and $\gamma_\delta((\mu_1, 0);T) \sim \mu_1$ (the latter is readily proven, but

N_t; N_s	9	14	19	24	29
10	5.7242e-02	5.8419e-02	5.8863e-02	5.9073e-02	5.9188e-02
15	5.7459e-02	5.8631e-02	5.9072e-02	5.9281e-02	5.9395e-02
20	5.7535e-02	5.8704e-02	5.9145e-02	5.9353e-02	5.9467e-02
25	5.7570e-02	5.8739e-02	5.9179e-02	5.9387e-02	5.9501e-02
30	5.7589e-02	5.8757e-02	5.9197e-02	5.9405e-02	5.9519e-02
35	5.7600e-02	5.8768e-02	5.9208e-02	5.9416e-02	5.9530e-02
40	5.7608e-02	5.8775e-02	5.9216e-02	5.9423e-02	5.9537e-02

TABLE 1. Inf-sup parameter $\beta_\delta((50,10);0.2)$ for various choices of $\delta = (\frac{1}{N_s}, \frac{0.2}{N_t})$.

		β_δ		
N_t	T	$\mu_1 = 50$	$\mu_1 = 100$	$\mu_1 = 150$
10	0.200000	2.081838e-01	9.189784e-02	5.605419e-02
20	0.400000	1.164954e-01	4.767668e-02	2.858245e-02
30	0.600000	8.062734e-02	3.200346e-02	1.911024e-02
40	0.800000	6.187347e-02	2.405788e-02	1.434315e-02
50	1.000000	5.040255e-02	1.926570e-02	1.147687e-02
60	1.200000	4.267737e-02	1.606301e-02	9.564429e-03
70	1.400000	3.712638e-02	1.377228e-02	8.197915e-03
80	1.600000	3.294756e-02	1.205285e-02	7.172878e-03
90	1.800000	2.968954e-02	1.071484e-02	6.375585e-03
100	2.000000	2.707910e-02	9.644058e-03	5.737750e-03

TABLE 2. Long time-behavior in the convection case $\mu = (\mu_1, 0)$.

not the former). Note $T = \mathcal{O}(1)$ is effectively a "long time" in convective units, $1/\mu_1$. We emphasize that although the problem is non-coercive, the problem is asymptotically stable in the sense that all eigenvalues σ of $-a(\psi,\phi) = \sigma\langle\psi,\phi\rangle_{V'\times V}$ lie in the left-hand plane; this stability is reflected in the inf-sup behavior. In contrast, a standard energy approach [5] gives effective inf-sup constants on the order of $e^{-\mu_1 T}$ (here about 10^{-8}). Hence, the traditional method fails to provide useful results, whereas our new approach, which reflects the true time-coupled properties of the system, yields relatively sharp error bounds.

Finally, we consider the case $\mu_1 = 0$ which gives rise to an asymptotically unstable (and non-coercive) system for $\mu_2 < -\pi^2$. This means that any error estimate *must* grow exponentially with the final time T. We observe this for our estimator as well, as Table 3 shows, the values are in the order of $e^{\mu_2 T}$.

REFERENCES

[1] M. Barrault, Y. Maday, Y., N.C. Nguyen, and A.T. Patera. An 'empirical interpolation' method: application to efficient reduced-basis discretization of partial differential equations. *C. R. Math. Acad. Sci. Paris*, 339(9), 667–672.

[2] R. Dautray and J.-L. Lions. Mathematical analysis and numerical methods for science and technology. Vol. 5. Springer-Verlag, Berlin, 1992. Evolution problems I.

		β_δ			
N_t	T	$N_s = 19$	$N_s = 24$	$N_s = 29$	$N_s = 34$
10	0.200000	1.328157e-01	1.327088e-01	1.326507e-01	1.326157e-01
20	0.400000	1.747513e-02	1.743612e-02	1.741498e-02	1.740224e-02
30	0.600000	2.297580e-03	2.289068e-03	2.284460e-03	2.281686e-03
40	0.800000	3.020714e-04	3.005078e-04	2.996622e-04	2.991535e-04
50	1.000000	3.971441e-05	3.945054e-05	3.930789e-05	3.922218e-05

TABLE 3. Long time-behavior in the asymptotically unstable case $\mu = (0, -20)$ for different spatial resolution.

[3] M. Grepl and A.T. Patera. A posteriori error bounds for reduced-basis approximations of parametrized parabolic partial differential equations. M2AN Math. Model. Numer. Anal. **39** (2005), no. 1, 157-181.

[4] B. Haasdonk and M. Ohlberger. Reduced basis method for finite volume approximations of parametrized linear evolution equations. M2AN Math. Model. Numer. Anal. **42** (2008), 277-302.

[5] D.J. Knezevic, N.C. Nguyen, and A.T. Patera. Reduced basis approximation and a posteriori error estimation for the parametrized unsteady Boussinesq equations. Math. Mod. Meth. Appl. Sci., **21** no. 7 (2011), 1415-1442.

[6] J. Necas. Sur une méthode pour resoudre les équations aux dérivées partielles du type elliptique, voisine de la variationnele. Ann. Sc. Norm. Super. Pisa, Sci. Fis. Mat., III. Ser. **16** (1962), 305-326.

[7] N.C. Nguyen, G. Rozza, and A.T. Patera. Reduced basis approximation and a posteriori error estimation for the time-dependent viscous Burgers' equation. Calcolo, **46**(3):157-185, 2009. (doi: 10.1007/s10092-009-0005-x)

[8] D.V. Rovas, L. Machiels, and Y. Maday. Reduced-basis output bound methods for parabolic problems. IMA J. Numer. Anal. **26** no. 3 (2006), 423-445.

[9] G. Rozza, D.B.P. Huynh, and A.T. Patera. Reduced basis approximation and a posteriori error estimation for affinely parametrized elliptic coercive partial differential equations — Application to transport and continuum mechanics. Arch. Comp. Meth. Eng., **15** no. 3 (2008), 229-275.

[10] C. Schwab and R. Stevenson. Space-time adaptive wavelet methods for parabolic evolution problems. Math. Comp. **78** (2009), 1293-1318.

[11] K. Steih and K. Urban. Space-time reduced basis methods for time-periodic parabolic problems.University of Ulm, Preprint 2012. www.uni-ulm.de/mawi/fakultaet/forschung/preprint-server.html

[12] T. Tonn. Reduced-Basis Method (RBM) for Non-Affine Elliptic Parametrized PDEs (Motivated by Optimization in Hydromechnanics). Ph.D. thesis, Ulm University, Germany, 2012.

[13] K. Urban and A.T. Patera. A New Error Bound for Reduced Basis Approximation of Parabolic Partial Differential Equations. C.R. Acad. Sci. Paris Series I, 350(3-4), 203-207 (2012).

[14] S. Vallaghé, A. Le-Hyaric, M. Fouquemberg, and C. Prud'homme. A successive constraint method with minimal offline constraints for lower bounds of parametric coercivity constant. Preprint: hal-00609212, hal.archives-ouvertes.fr.

UNIVERSITY OF ULM, INSTITUTE FOR NUMERICAL MATHEMATICS, HELMHOLTZSTR. 18, 89081 ULM (GERMANY), karsten.urban@uni-ulm.de

MECHANICAL ENGINEERING DEPARTMENT, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, 77 MASSACHUSETTS AVE., CAMBRIDGE, MA 02139-4307 (USA), patera@mit.edu