Singular Sturmian Theory for Linear Hamiltonian Differential Systems

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Abstract. We establish a Sturmian type theorem comparing the number of focal points of any conjoined basis of a nonoscillatory linear Hamiltonian differential system with the number of focal points of the principal solution. We also present various extensions of this statement.

1. Introduction

In this paper we consider the linear Hamiltonian differential system

$$z' = \mathcal{J}H(t)z, \quad \mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad t \in \mathbb{R}, \tag{1}$$

where $z \in \mathbb{R}^{2n}$ and

(H) $\quad H \in \mathbb{R}^{2n\times 2n}$ is a symmetric matrix of piecewise continuous entries.

It is known, see, e.g., [9], that the classical Sturmian theory for the second order differential equation

$$(r(t)x')' + p(t)x = 0 \tag{2}$$

can be extended to (1). Some statements of this theory we will recall in the next section. Our research is motivated by the paper [1], where Sturmian type theorems for a pair of equations (2) on an interval with possibly *singular* endpoints are established. In our paper we show, among others, that some results of [1] can be extended to (1).

Our paper is organized as follows. In the next section we present essentials of the oscillation theory of (1), including the concept of the principal solution of this system and inequalities for the minimal solution of the associated Riccati matrix differential equation. The main results of the paper are given in the Sections 3 and 4. In the first one we deal with one system, while in the Section 4 we prove a result comparing two Hamiltonian systems.

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2. Linear Hamiltonian systems

If we split the matrix H in (1) into $n \times n$ blocks $H = \begin{pmatrix} -C & A^T \\ A & B \end{pmatrix}$ and $z = \binom{x}{u}$ with $x, u \in \mathbb{R}^n$, then the system (1) can be written in the form

$$x' = A(t)x + B(t)u, \quad u' = C(t)x - A^T(t)u. \tag{1'}$$

Along with (1'), we also consider its matrix version (referred to again as (1) or (1'))

$$X' = A(t)X + B(t)U, \quad U' = C(t)X - A^T(t)U$$

with $n \times n$ matrices X, U.

A $2n \times n$ matrix solution $Z = \binom{X}{U}$ is said to be a *conjoined basis* of (1) if

$$Z^T(t)\mathcal{J}Z(t) = X^T(t)U(t) - U^T(t)X(t) = 0 \quad \text{and} \quad \operatorname{rank} Z(t) = n. \tag{3}$$

Note that if (3) holds for one t, then it holds everywhere. Throughout the paper, in addition to (H), we suppose that

(C) the system (1) is *completely controllable* on $\mathbb{R}$

(an alternative terminology is *identically normal*). This means that the trivial solution $z(t) = \binom{x(t)}{u(t)} \equiv 0$ is the only solution for which $x(t) \equiv 0$ on a nondegenerate interval $J \subset \mathbb{R}$. We also suppose that (1) satisfies the so-called *Legendre condition* which means that we suppose that

(B) the matrix $B(t)$ is nonnegative definite for all $t \in \mathbb{R}$.

Two points $t_1 < t_2$ are said to be *conjugate* relative to (1) if there exists a nontrivial solution $z = \binom{x}{u}$ such that $x(t_1) = 0 = x(t_2)$. A point $t \in \mathbb{R}$ is said to be a *focal point* of a conjoined basis $\binom{X}{U}$ if $\det X(t) = 0$ and then $m = \operatorname{def} X(t) = \dim \operatorname{Ker} X(t)$ is called the *multiplicity* of this focal point. The controllability assumption (C) implies that the focal points of any conjoined basis are *isolated*. The system (1) is said to be *disconjugate* in an interval I if this interval contains no pair of conjugate points relative to (1), and (1) is said to be *nonoscillatory* at ∞ if there exists $T \in \mathbb{R}$ such that (1) is disconjugate on $[T, \infty)$, in the opposite case it is said to be *oscillatory*. (Non)oscillation of (1) at $-\infty$ is defined analogously. Nonoscillation of (1) both at ∞ and $-\infty$ means that any conjoined basis of this system has only finitely many focal points in $[T, \infty)$ and $(-\infty, T]$ for every $T \in \mathbb{R}$.

A conjoined basis $\tilde{Z} = \binom{\tilde{X}}{\tilde{U}}$ of the system (1), which is nonoscillatory at ∞, is said to be the *principal solution* at ∞ if

$$\lim_{t \to \infty} X^{-1}(t)\tilde{X}(t) = 0 \tag{4}$$

for any other conjoined basis $Z = \binom{X}{U}$ for which the (constant) "Wronskian matrix"

$$W = Z^T(t)\mathcal{J}\tilde{Z}(t) = U^T(t)\tilde{X}(t) - X^T(t)\tilde{U}(t)$$

is invertible. The principal solution $\hat Z = \binom{\hat X}{\hat U}$ at $-\infty$ is defined analogously. Note that (4) is (for controllable systems satisfying the Legendre condition (B)) equivalent to

$$\lim_{t\to\infty} \left(\int^t \tilde X^{-1}(s) B(s) \tilde X^{T-1}(s)\, ds \right)^{-1} = 0. \tag{5}$$

The principal solution at $-\infty$ can be characterized analogously. Note also that the principal solution of (1) at ∞ exists whenever this system is controllable and nonoscillatory at ∞, see [9] and also [2, 3, 5], where various properties of principal solutions are investigated.

The "standard" Sturmian separation theorem for (1) (see, e.g., [4, 8, 9]) claims that the numbers of focal points of any pair of conjoined bases in any interval differ by at most n. The comparison theorem says, roughly speaking, that if a Hamiltonian system

$$z' = \mathcal{J}\mathcal{H}(t) z, \tag{6}$$

is a *majorant system* to (1) on an interval I, i.e.

$$\mathcal{H}(t) \geq H(t) \quad \text{on } I \tag{7}$$

(this inequality means that the matrix $\mathcal{H} - H$ is nonnegative definite in I) and if $Z = \binom{X}{U}$, $\mathcal{Z} = \binom{\mathcal{X}}{\mathcal{U}}$ are solutions of (1) and (6), respectively, given by the initial condition $Z(a) = \mathcal{Z}(a) = \binom{0}{I}$, then $\mathcal{Z}$ has in the interval $(a, T]$ not less focal points than Z.

We finish this preparatory section with the relationship between (1) and the associated Riccati matrix equation

$$Q' - C(t) + A^T(t) Q + Q A^T(t) + Q B(t) Q = 0, \tag{8}$$

which is related to (1) by the Riccati substitution $Q = UX^{-1}$. If (1) is nonoscillatory at ∞, and if $\binom{\tilde X}{\tilde U}$ is its principal solution at ∞ with $\tilde Q = \tilde U \tilde X^{-1}$, then

$$Q(t) \geq \tilde Q(t) \quad \text{for large } t,$$

see [4, Theorem 8, p. 54] or [10, Theorem 8.4, p. 159], where $Q = UX^{-1}$ is a solution of (8) corresponding to any other conjoined basis of (1). The solution $\tilde Q$ is called the *distinguished* or *eventually minimal* solution of (8). Concerning the solution of (8) corresponding to the principal solution $\hat Z = \binom{\hat X}{\hat U}$ at $-\infty$, we have

$$\hat Q(t) = \hat U(t) \hat X^{-1}(t) \geq Q(t) = U(t) X^{-1}(t) \quad \text{for } t \text{ sufficiently close to } -\infty,$$

where $\binom{X}{U}$ is any other conjoined basis of (1). If we consider along with (1) its majorant system (6), then we have the inequality

$$\tilde{\mathcal{Q}}(t) \geq \tilde Q(t) \quad \text{for large } t$$

for eventually minimal solutions of the associated Riccati equations. Similarly, for solutions corresponding to the principal solutions at $-\infty$, we have the inequality $\hat{\mathcal{Q}}(t) \leq \hat{Q}(t)$ for t sufficiently close to $-\infty$.

Finally, we recall (in a modified form suitable for our purposes) Theorem 7.3.1 of [6] which plays the crucial role in the proofs of our main results. To formulate it, together with (1) consider the system (6) on some interval $I = [a, b]$ and suppose that it is a majorant system to (1), i.e., (7) holds on I.

Proposition 1. *Let $Z = \binom{X}{U}$ and $\mathcal{Z} = \binom{\mathcal{X}}{\mathcal{U}}$ be conjoined bases of* (1) *and of its majorant system* (6)*, respectively, and let $P(a, b)$ and $\mathcal{P}(a, b)$ denote the numbers of focal points in (a, b) of Z and $\mathcal{Z}$ (including multiplicities), respectively. Denote $Q = UX^{-1}$, $\mathcal{Q} = \mathcal{U}\mathcal{X}^{-1}$. Then we have*

$$\begin{cases} P(a,b) - \mathcal{P}(a,b) \leq & \operatorname{ind}(\mathcal{Q} - Q)(b-) - \operatorname{ind}(\mathcal{Q} - Q)(a+), \\ P(a,b) - \mathcal{P}(a,b) \leq & \operatorname{ind}(Q - \mathcal{Q})(a+) - \operatorname{ind}(Q - \mathcal{Q})(b-). \end{cases} \tag{9}$$

Moreover, if $\mathcal{H}(t) = H(t)$ for $t \in [a, b]$, then the previous inequalities are satisfied as equalities.

3. Focal points and principal solution

In this section we present our main results. We always assume (H), (B), and (C) in this section. The first result shows that the principal solution at ∞ behaves like a solution having at ∞ a focal point of multiplicity n. More precisely, if b is a *finite* (and regular) right endpoint of an interval $[a, b]$ and $Z_b = \binom{X_b}{U_b}$ is the conjoined basis of (1) given by the initial condition $X_b(b) = 0$, $U_b(b) = -I$ (thus, Z_b has a focal point of multiplicity n at $t = b$), then as a direct consequence of the fact that the numbers of focal points of conjoined bases differ by at most n we have the following "regular" inequality for the number of focal points.

Proposition 2. *Let $P_b(T, b)$ denote the number of focal points of the solution Z_b of* (1) *in the interval $[T, b)$. If Z is any other conjoined basis of* (1) *and if $P(T, b)$ denotes the number of its focal points in $[T, b)$, then $P(T, b) \geq P_b(T, b)$.*

The singular version of the previous statement reads as follows.

Theorem 1. *Let $Z = \binom{\tilde{X}}{\tilde{U}}$ be the principal solution at ∞ of the nonoscillatory system* (1) *and let $Z = \binom{X}{U}$ be any other conjoined basis of* (1)*. Denote by $\tilde{P}(T)$ and $P(T)$ the number of focal points in $[T, \infty)$ of $\tilde{Z}$ and Z (including multiplicities), respectively. Then*

$$P(T) \geq \tilde{P}(T). \tag{10}$$

Proof. Let $P(a,b)$ and $\tilde P(a,b)$ denote the number of focal points of Z and $\tilde Z$ in an interval $(a,b) \subset \mathbb{R}$ and let $Q(t) = U(t)X^{-1}(t)$, $\tilde Q(t) = \tilde U(t)\tilde X^{-1}(t)$. Then by Proposition 1 (equality case $H = \mathcal{H}$) we have that

$$\tilde P(a,b) - P(a,b) = \operatorname{ind} D(b-) - \operatorname{ind} D(a+),$$

where $D(t) = Q(t) - \tilde Q(t)$. Now, let $T_1 > T$ be so large that both Z and $\tilde Z$ have no focal point for $t \geq T_1$ and $D(t) \geq 0$ for $t \geq T_1$. Moreover, let $\varepsilon > 0$ be sufficiently small such that

$$P(T) = P(a,b), \quad \tilde P(T) = P(a,b) \quad \text{for } a = T - \varepsilon,\ b = T_1 + \varepsilon.$$

We have

$$\operatorname{ind} D(b-) = 0 \quad \text{and} \quad \operatorname{ind} D(a+) \geq 0,$$

hence

$$\begin{aligned}\tilde P(T) - P(T) = \tilde P(a,b) - P(a,b) &= \operatorname{ind} D(b-) - \operatorname{ind} D(a+) \\ &= -\operatorname{ind} D(a+) \leq 0.\end{aligned}$$

Consequently, $P(T) \geq \tilde P(T)$ what we needed to prove. □

Remark 1. If $\hat Z = \binom{\hat X}{\hat U}$ is the principal solution at $-\infty$ of (1) and $\hat P_{-\infty}(T)$ denotes the number of focal points (including multiplicities) of $\hat Z$ in the interval $(-\infty, T]$, then $\hat P_{-\infty}(T) \leq P_{-\infty}(T)$, where $P_{-\infty}(T)$ is the number of focal points of any other conjoined basis in $(-\infty, T]$.

The previous results, applied to the pair of principal solutions $\hat Z$ and $\tilde Z$ at $\mp\infty$, respectively, give the following statement. In this statement, we suppose that (1) is nonoscillatory both at ∞ and $-\infty$.

Corollary 1. *Let $\hat Z = \binom{\hat X}{\hat U}$, $\tilde Z = \binom{\tilde X}{\tilde U}$ be the principal solutions of* (1) *at $-\infty$ and ∞, respectively, and let $\hat P$, $\tilde P$ denote the number of focal points in $\mathbb{R}$ of $\hat Z$ and $\tilde Z$. Then*

$$\hat P = \tilde P.$$

Proof. Nonoscillation of (1) both in $-\infty$ and ∞ implies that there exist $T_0 < T_1$ such that

$$\hat Q(t) - \tilde Q(t) \geq 0 \quad \text{for} \quad t \in (-\infty, T_0] \cup [T_1, \infty).$$

The argument from the previous theorem implies that $\hat P(T) \geq \tilde P(T)$ for any $T \in \mathbb{R}$, $\hat Z$ plays here the role of any conjoined basis of (1) from Theorem 1. On the other hand, from Remark 1, $\hat P_{-\infty}(T) \leq \tilde P_{-\infty}(T)$ for any $T \in \mathbb{R}$, where $\tilde Z$ plays the role of any conjoined basis of (1). Combining these inequalities, since $\hat P = \hat P(T)$, $\tilde P = \tilde P(T)$ for sufficiently large T, and $\hat P = \hat P_{-\infty}(T)$, $\tilde P = \tilde P_{-\infty}(T)$ for T sufficiently small, we have the required statement. □

The statement of Corollary 1 extends the result of [8, Theorem 7.2, p. 360], where (1) is considered on a compact interval $[a,b]$.

4. Singular comparison theorem

In this section we establish a singular Sturm *comparison* theorem. In this section we suppose throughout (H) for both systems (1) and (6) and (B) and (C) for (1). We show that the principal solution of the minorant system (1) has in any interval of the form $[T, \infty)$ not more focal points than any conjoined basis of the majorant system (6).

Theorem 2. *Let $\binom{\tilde X}{\tilde U}$ be the principal solution at ∞ of the nonoscillatory system* (1) *and let $\mathcal{Z} = \binom{\mathcal{Z}}{\mathcal{U}}$ be any conjoined basis of the majorant system* (6)*, i.e.* (7) *holds for $t \in \mathbb{R}$. For any $T \in \mathbb{R}$, denote by $\mathcal{P}(T)$ and $\tilde P(T)$ the number of focal points in $[T, \infty)$ (including multiplicities) of $\mathcal{Z}$ and $\tilde Z$, respectively. Then*

$$\mathcal{P}(T) \geq \tilde P(T). \tag{11}$$

Proof. Note that the condition $\mathcal{H}(t) \geq H(t)$, $t \in \mathbb{R}$, implies that the majorant system also satisfies the Legendre condition (B). Moreover, by [6, Proposition 5.1.5], the system (6) satisfies (C) too. Hence, there exists $\varepsilon > 0$ such that

$$\tilde X(t) \text{ and } \mathcal{X}(t) \text{ are invertible for } t \in [T-\varepsilon, T). \tag{12}$$

Let Z denote the solution of (1) given by the initial condition

$$X\big(T-\varepsilon/2\big) = \mathcal{X}\big(T-\varepsilon/2\big), \quad U\big(T-\varepsilon/2\big) = \mathcal{U}\big(T-\varepsilon/2\big).$$

Then, by [6, Theorem 5.1.2], $X(t)$ is invertible for all $t \in [T-\varepsilon/2, T)$ and

$$Q(t) = U(t)X^{-1}(t) \geq \mathcal{Q}(t) = \mathcal{U}(t)\mathcal{X}^{-1}(t) \quad \text{for all } t \in [T-\varepsilon/2, T). \tag{13}$$

Let $P(T)$ be defined as in Theorem 1. Then by this theorem $P(T) \geq \tilde P(T)$. Now, we want to show that $\mathcal{P}(T) \geq P(T)$. Without loss of generality we may suppose that $\mathcal{P}(T) < \infty$ (otherwise there is nothing to prove). Then there exists $T_1 > T$ such that $\mathcal{X}(t)$ and $X(t)$ are invertible for $t \geq T_1$. Hence, by what we have already shown in (12) and (13),

$$P(T-\delta, T_1) = P(T) \quad \text{and} \quad \mathcal{P}(T-\delta, T_1) = *\mathcal{P}(T).$$

for all $0 < \delta \leq \frac{\varepsilon}{2}$. It follows from (9) that for $0 < \delta < \varepsilon/2$

$$P(T-\delta, T_1) - \mathcal{P}(T-\delta, T_1) \leq \operatorname{ind}(Q-\mathcal{Q})((T-\delta)+) - \operatorname{ind}(Q-\mathcal{Q})(T_1-)$$

and $\operatorname{ind}(Q-\mathcal{Q})((T-\delta)+) = 0$ by (13). Hence

$$P(T) \leq \mathcal{P}(T) - \operatorname{ind}(Q-\mathcal{Q})(T_1-) \leq \mathcal{P}(T).$$

Consequently, (11) holds. □

Remark 2. (i) In our paper we suppose that the considered Hamiltonian system (1) is controllable. In this case, the construction of the principal solution is relatively simple and this solution exists whenever the Hamiltonian system under consideration is nonoscillatory.

(ii) The situation is considerably more complicated when the controllability assumption is dropped. Nonoscillation theory of Hamiltonian systems without the controllability assumption is elaborated in [7]. As for the existence and properties of principal solutions of such systems, we refer to [11] and to [9, Sec. V.12]. However, there are still many open problems in this direction which are subject of the present investigation. We refer to [12].

(iii) The singular points of (1) are $\mp\infty$ in our treatment. However, they can be replaced by any finite singularities $a < b$ (i.e., points where the unique solvability of (1) is violated). Principal solutions at finite singularities can be defined analogously to the case of singularities at $\pm\infty$. Then the results of our paper, properly modified, remain to hold.

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