Optimal insurance contract design: effects of mortality

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OPTIMAL INSURANCE CONTRACT DESIGN UNDER MORTALITY

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Abstract. Some recent literature on optimal pension/life insurance design studies whether contracts with guarantees can be preferred by a utility-maximizing policyholder. In the absence of mortality risk, the main result in the literature (e.g. Døskeland and Nordahl (2008)) is that expected utility theory (EUT) fails to interpret demand for any forms of guarantees, while cumulative prospect theory (CPT) is able to support the demand for guarantees. In the present paper, we incorporate mortality in life/pension insurance contracts and investigate the effects on EUT- and CPT-investors. Our results show that an EUT investor might prefer products with guarantees. For CPT we propose two possibilities to include mortality in the analysis and examine their influence in a simulation based approach.

Keywords: Cumulative prospect theory, Mortality, Life and pension insurance, Portfolio choice

JEL: G11, G13, G22

1. Introduction

The optimal design of pension and life insurance contracts has attracted a lot of attention in the recent literature. In particular, the question whether insurance products with guarantees can be favored is investigated. Related to this question, there are mainly two streams of literature. The first tries to solve the optimization problem for the optimal insurance contracts or the optimal investment strategy, see for example Yaari (1965), Merton (1971), Richard (1975), Raviv (1979), Pliska and Ye (2007), Nielsen and Steffensen (2008), Huang et al. (2008), Bruhn and Steffensen (2011) and Pirvu and Zhang (2012). The second stream considers some specific insurance products and tries to find the one (among the given products) that delivers the highest utility or value in a given setup, see for example Dichtl and Drobetz (2011), Ebert et al. (2012) and Døskeland and Nordahl (2008). The

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main common result in these papers is: expected utility theory (EUT, taking power utility as an example) is unable to interpret the demand for financial/insurance products which provide a guaranteed payment. However, under Cumulative Prospect Theory (CPT, introduced by Tversky and Kahneman (1992)), the demand for products with guarantees is explainable. Our paper follows the second stream of literature and in particular the analysis performed by Døskeland and Nordahl (2008).

In their setup they consider four different life/pension insurance contracts, where three contain a guaranteed rate of return and one is the Merton portfolio (see Merton (1971)). The products with guarantees are: an implicit put with a roll-up guarantee, a simple life contract with a roll-up guarantee and a possible terminal default, as well as a product with an annual guarantee, including the possibility of an annual default. They show that the Merton portfolio outperforms the products with guarantees in terms of expected utility, whereas within CPT, the products with guarantees turn out to be a better choice for the investors. The results of Døskeland and Nordahl (2008) are certainly very interesting. However, in the analysis of demand for insurance contracts with guarantees, they purely consider the financial risks incorporated in these products and neglect another important source of risk: mortality risk. The present paper aims to find out whether by incorporating this risk, the results stated in Døskeland and Nordahl (2008) can be confirmed.

To add mortality risk in the analysis of demand for guarantees, we extend the model of Døskeland and Nordahl (2008) by a factor for the remaining life time of the liability holder and thus account for payouts in the case of premature death. In other words, we consider an endowment insurance contract. We then analyze the considered products with EUT and CPT and compare the resulting utility/value levels. For the analysis with CPT we propose two ways to include mortality in the calculation of the CPT value which has, to our best knowledge, not been done in literature so far. We find that an EUT investor is, depending on his risk aversion, indifferent between products with and without guarantees. In the case of a premature death the investor even favors the products with guarantees over a pure investment into the available assets. In the CPT case, we confirm the existing results, that a CPT investor prefers products with guarantees. However, we also find, that in the situation of the Tversky and Kahneman (1992) setup, a CPT investor actually favors the pure Merton investment over the guaranteed products.

The remainder of this paper is organized as follows. In Section 2, we introduce our model setup and particularly specify the payoffs of the contracts. Section 3 introduces the expected utility theory (EUT) and cumulative prospect theory (CPT) on which the liability holder bases his decision. Section 4 provides numerical results and shows the impact of key parameters on the optimal choice. Our conclusions are set out in Section 5.

2. Model setup

We consider an endowment insurance contract of a single liability holder. We assume the insurance contract is issued at time $t_0 = 0$. At time 0, the liability holder provides an upfront premium payment of L_0 to the insurance company. The company receives an amount of initial contributions E_0 from the equity holder at time 0. Consequently, the initial asset value of the company is given by the sum of the contributions from both the liability holder and the equity holder, i.e. $A_0 = L_0 + E_0$. Furthermore, we denote the upfront premium as a fraction α of the total assets, i.e. $L_0 = \alpha A_0$ with $\alpha \in (0,1)$. The insurance company invests the proceeds in a diversified portfolio of risky and non-risky assets.

The contract has a finite maturity date T, which can be e.g. considered as the retirement date of the liability holder. In case of an endowment life insurance contract, the liability holder receives payments both upon survival of the maturity date T and upon premature death. We use Φ_L to denote the terminal contract payoff of the liability holder:

$$\Phi_L = \mathbf{1}_{\{\tau_x > T\}} \Psi_L(A_T) + \sum_{t=1}^{T} \mathbf{1}_{\{t-1<\tau_x \le t\}} \cdot \Psi_L(A_t) \cdot e^{r(T-t)}, \tag{1}$$

where τ_x denotes the remaining life time of the contract holder (with an age x at the contract-issuing time) and $\mathbf{1}_A$ denotes the indicator function which is 1 if A occurs and 0 otherwise. If the contract holder survives the maturity date T, he receives $\Psi_L(A_T)$. If the liability holder dies within $(t-1, t]$, $t = 1, \ldots, T$, the contract payoff $\Psi_L(A_t)$ follows at the end of the period (time t). To make payments comparable, we assume that all the death payments will be invested in the risk-free asset until maturity, as in Bauer et al. (2008). Figure 1 illustrates the time line of the payment upon premature death between $(t-1, t]$. Furthermore, we allow both the premature and maturity payoffs to depend on the asset evolution of the insurance company's assets. These payoffs will be specified later.

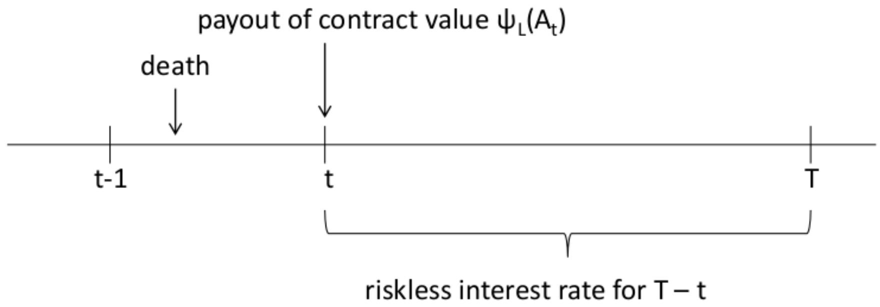

FIGURE 1. Timeline in the event of death between $t-1$ and t

2.1. **Underlying financial and demographic risks.** Assume a financial market in which there are two traded investment opportunities: a risky and a riskfree asset (bank account). The traded risky asset S satisfies

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

where W_t is a standard Brownian motion on our probability space $(\Omega, \mathcal{F}, \mathbb{P})$ ($\mathbb{P}$ is the market probability measure), i.e. this asset follows Black-Scholes dynamics with an instantaneous rate of return $\mu > 0$ and a constant volatility $\sigma > 0$. Also assume the existence of a riskfree asset R which satisfies

$$dR_t = rR_t dt$$

for a deterministic riskfree rate r. The insurance company invests in these two assets in a self-financing way starting with initial wealth A_0. The insurance companies' wealth process A_t is given by the following stochastic differential equation (SDE)

$$dA_t = A_t(r + \theta(\mu - r))dt + \theta\,\sigma\,A_t\,dW_t. \tag{2}$$

The parameter θ denotes the fraction invested in the risky asset S and the remainder (1-θ) is invested in the riskless asset R. We assume short-selling is not allowed, i.e. $\theta \in [0, 1]$. For a given θ, the solution to the SDE is given by

$$A_t(\theta) = A_0 \exp\left\{\left(r + \theta(\mu - r) - \frac{\theta^2\sigma^2}{2}\right)t + \sigma\theta W_t\right\}. \tag{3}$$

We assume the mortality risk is independent of financial market risk. Other than that, we do not impose any restrictions on the underlying mortality model. As we will see in Section 3, the sole relevant quantities for the derivation of EUT- and CPT- values are annual survival and death probabilities. In the following we use standard actuarial notations to denote the survival and death probabilities. The probability that an x-year old person survives time t is denoted by

$$ {}_tp_x = P(\tau_x > t). \tag{4} $$

Similarly, $P(t-1 < \tau_x \leq t) = {}_{t-1}p_x \cdot q_{x+t-1}$ denotes the probability that an x-year old person dies between $(t-1, t]$.

2.2. **Contract specifications.** In order to better compare our results with existing literature in this field, we take the four contract types used in Døskeland and Nordahl (2008) as an example to specify our $\Psi_L(A_t)$, $\forall\, t = 1, \cdots T$.[1] The four contract types are: Merton, implicit put, simple life insurance and annual guarantees. In case of a Merton-type contract, the liability holder obtains at time t

$$ \Psi_L(A_t) = \alpha A_t. \tag{5} $$

In other words, the liability holder bears all the financial market risk and no guarantees are provided.

In case of an implicit put, the liability holder's payoffs at time t are given by

$$ \Psi_L(A_t) = L_0 e^{gt} + \alpha\delta \left[A_t - \frac{1}{\alpha} L_0 e^{gt} \right]^+ . \tag{6} $$

The beneficiary obtains a guaranteed amount with an interest rate $g < r$ and an indexed bonus payment contingent on the insurance company's assets. Given that the assets are sufficiently large ($A_t > \frac{1}{\alpha} L_0 e^{gt}$), the liability holder is allowed to participate in the surplus of the company ($A_t - \frac{1}{\alpha} L_0 e^{gt}$) with a participation rate δ, where $\delta \in [0, 1]$ is the surplus distribution parameter.

In a simple life insurance contract, the contract payoff can be expressed as

$$ \Psi_L(A_t) = L_0 e^{gt} + \alpha\delta \left[A_t - \frac{1}{\alpha} L_0 e^{gt} \right]^+ - \left[L_0 e^{gt} - A_t \right]^+ . \tag{7} $$

Compared to the implicit put contract, the life insurance contract does not provide an absolute guaranteed amount and takes into account the shortfall possibility of the company.

[1] Our performed analysis is general. We can also analyze other (equity-linked) life insurance contracts.

It is characterized by the default put option. When the company performs extremely bad ($A_t < L_0 e^{gt}$), the liability holder receives A_t smaller than the guaranteed amount.

The last contract is so-called annual guarantees which dates back to Miltersen and Persson (2003) and Grosen and Jørgensen (2000). This contract contains a bonus account (B_t) which serves as an additional buffer. The proportion declared to this account is denoted by b. The liabilities L_t at time t can be specified as follows

$$L_t = \begin{cases} A_t & \text{if } A_t \leq L_{t-1}e^g \\ L_{t-1}e^g & \text{if } L_{t-1}e^g < A_t \leq L_{t-1}e^g + E_{t-1}e^g + B_{t-1} \\ L_{t-1}e^g + \delta\alpha(1-b) \cdot & \\ \cdot \left(A_t - (L_{t-1}e^g + E_{t-1}e^g + B_{t-1})\right) & \text{if } A_t > L_{t-1}e^g + E_{t-1}e^g + B_{t-1} \end{cases}$$

where the bonus account B_t and the equity account E_t evolve according to

$$B_t = \begin{cases} 0 & \text{if } A_t \leq L_{t-1}e^g + E_{t-1} \\ A_t - L_{t-1}e^g - E_{t-1} & \text{if } L_{t-1}e^g + E_{t-1} < A_t \leq L_{t-1}e^g + E_{t-1} + B_{t-1} \\ B_{t-1} & \text{if } L_{t-1}e^g + E_{t-1} + B_{t-1} < A_t \leq L_{t-1}e^g + E_{t-1}e^g + B_{t-1} \\ B_{t-1} + \delta\alpha b\Big[A_t - & \\ (L_{t-1}e^g + E_{t-1}e^g + B_{t-1})\Big] & \text{if } A_t > L_{t-1}e^g + E_{t-1}e^g + B_{t-1} \end{cases}$$

$$E_t = A_t - L_t - B_t.$$

The contract payoff $\Psi_L(A_t)$ at time t consists of the liabilities at time t and the surplus accumulated in the bonus account up to time t, i.e. $\Psi_L(A_t) = L_t + B_t$. In this contract, when the insurance company performs relatively well, it provides the liability holder with annual guaranteed amounts and annual bonuses. When it performs moderately, only a periodic guaranteed payment is distributed to the liability holder. When there is a premature default, the liability holder ends up with what remains in the insurance company. Given the random time of default $\widetilde{t} < t$, $\widetilde{t} \in \{1, 2, \ldots, T\}$, we have (at time t):

$$E_t = 0, B_t = 0, \Psi_L(A_t) = A_{\widetilde{t}} \cdot e^{r(t-\widetilde{t})}, \text{ for } \widetilde{t} \leq t \leq T. \tag{8}$$

3. Utility- and CPT-based comparison

In order to analyze the optimal insurance contract, we assume the liability holder bases his optimal contract decision on expected utility theory (EUT) or cumulative prospect theory (CPT).

Under EUT, we assume that the liability holder is risk averse and bases his optimal insurance contract decision on maximizing the following expected utility:

$$E[U(X)] = E\left[\frac{X^{1-\gamma}}{1-\gamma}\right] \text{ with } \gamma \neq 1$$

where γ is the coefficient of relative risk aversion. We choose the power utility with CRRA, since it is the mostly accepted utility function and commonly used in settings like ours. A higher value of γ corresponds to a more risk-averse liability holder and $\gamma = 0$ gives the special case that the beneficiary is risk-neutral. According to the definition of Φ_L as in (1), the expected utility of the terminal payoff Φ_L of the liability holder is given by

$$\begin{aligned}
E[U(\Phi_L)] =& E\left[U\left(\mathbf{1}_{\{\tau_x>T\}} \cdot \Psi_L(A_T) + \sum_{t=1}^{T} \mathbf{1}_{\{t-1<\tau_x\leq t\}} \cdot \Psi_L(A_t) \cdot e^{r(T-t)}\right)\right] \\
=& E\left[\mathbf{1}_{\{\tau_x>T\}} \cdot U\left(\Psi_L(A_T)\right) + \sum_{t=1}^{T} \mathbf{1}_{\{t-1<\tau_x\leq t\}} \cdot U\left(\Psi_L(A_t) \cdot e^{r(T-t)}\right)\right] \\
=& E\left[\mathbf{1}_{\{\tau_x>T\}} \cdot U\left(\Psi_L(A_T)\right)\right] + \sum_{t=1}^{T} E\left[\mathbf{1}_{\{t-1<\tau_x\leq t\}} \cdot U\left(\Psi_L(A_t) \cdot e^{r(T-t)}\right)\right] \\
=& {}_Tp_x \cdot E[U\left(\Psi_L(A_T)\right)] + \sum_{t=1}^{T} {}_{t-1}p_x \cdot q_{x+t-1} \cdot E\left[U\left(\Psi_L(A_t) \cdot e^{r(T-t)}\right)\right].
\end{aligned}$$

The second equality follows because all the events in the indicator functions are disjoint events. Furthermore, we have used the fact that mortality and financial market risk are independent. For the first three contracts, it is possible to compute the expectations $E\left[U\left(\Psi_L(A_t) \cdot e^{r(T-t)}\right)\right]$ analytically, based on the cumulative distribution function of the normal distribution (see Appendix A).

In the numerical section, we will derive the certainty equivalents from

$$E[U(\Phi_L)] = U(\text{CEQ})$$

to ease our comparison.

CPT dates back to Tversky and Kahneman (1992) and can be considered as an extension of the usual EUT. CPT uses a value function with different shapes around a so called reference point Γ. Above the reference point the value function is concave, whereas it is convex below the reference point. More specifically, in Tversky and Kahneman (1992) the following value function is considered

$$V(X) = \begin{cases} V_+(X) = (X - \Gamma)^\beta, & X \geq \Gamma \\ V_-(X) = -\lambda(\Gamma - X)^\beta, & X < \Gamma \end{cases}$$

where $\lambda > 1$ is the so called loss aversion parameter. Figure 2 shows the s-shaped form of the value function.

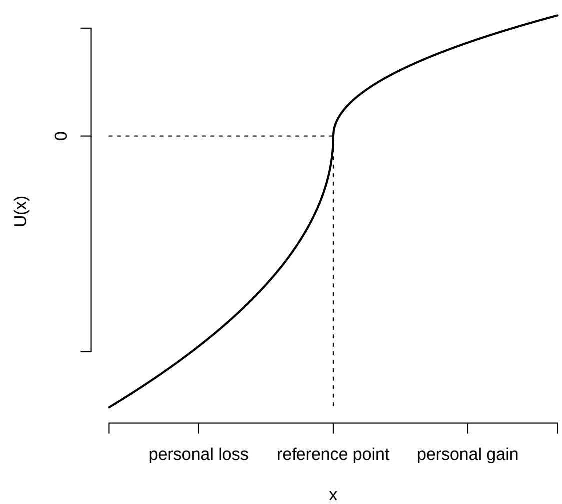

FIGURE 2. Value function in CPT with different shapes for gains and losses with $\beta = 0.5$ and $\lambda = 2.25$.

For the calculation of the expected value, real world probabilities are transformed with a weighting function w. It assigns to each real world probability a personal probability which reflects the attitude of the investor (see Figure 3). Different weighting functions have been used in the literature. In order to obtain results which are comparable to those found in the literature, we use the weighting function proposed by Prelec (1998), which is a one-parametric weighting function with the following form:

$$w(p) = e^{-(-\log(p))^\varphi}.$$

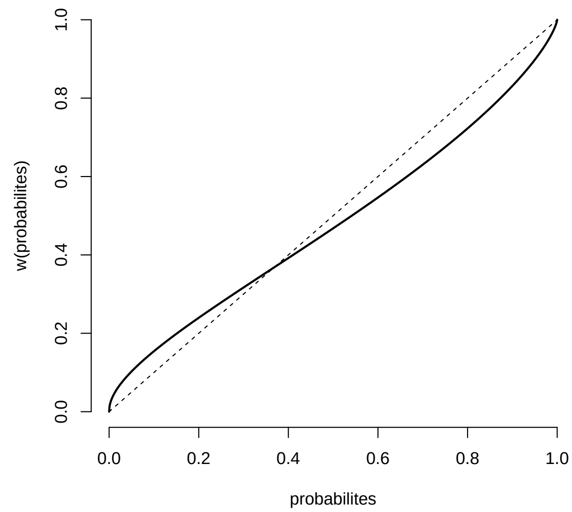

FIGURE 3. Probability weighting function $w(p) = e^{-(-\log p)^{\varphi}}$ in CPT with $\varphi = 0.75$.

Where $\varphi \in (0, 1)$ controls the shape of the probability distortion, a detailed discussion of the properties of $w(p)$ can be found in Prelec (1998). Considering n possible outcomes for the evolution of the risky asset, we get n different contract payoffs $\Psi_L^i(A_T)$ with corresponding real-world probabilities $p_i, i = 1, \dots, n$. Applying the parametrization of Tversky and Kahneman (1992) in our context to evaluate the ordered n outcomes of $\Psi_L^1(A_T) < \Psi_L^2(A_T) < \dots < \Psi_L^n(A_T)$, we obtain

$$CPT(\Psi_L(A_T)) = \sum_{i=1}^{n} \pi_i V(\Psi_L^i(A_T))$$

with $\pi_i := w(p_1 + \dots + p_i) - w(p_1 + \dots + p_{i-1})$.[2] Similarly, we can also compute for the ordered n outcomes of $\Psi_L^1(A_t) < \Psi_L^2(A_t) < \dots < \Psi_L^n(A_t)$:

$$CPT(\Psi_L(A_t e^{r(T-t)})) = \sum_{i=1}^{n} \pi_i V(\Psi_L^i(A_t) e^{r(T-t)}).$$

To further study the effects of mortality we will analyze the CPT value under two different assumptions. On the one hand, we assume that the investor is able to anticipate the real world distribution of his remaining life time. In other words, we only apply the transformation of the probabilities onto the distribution of the assets and thus on the

[2]With $w(p_1 + p_0) := 0$.

distribution of the liabilities and not onto the distribution of the remaining life time τ_x. Thus the total CPT-value for Φ_L is given by

$$CPT(\Phi_L) = {}_Tp_x \cdot CPT(\Psi_L(A_T)) + \sum_{t=1}^{T} {}_{t-1}p_x \cdot q_{x+t-1} \cdot CPT(\Psi_L(A_t \cdot e^{r(T-t)})).$$

On the other hand we consider the occurrence of death and the state of the economy as one single combined outcome, with n possible realizations. We denote the random time of death of the individual in the i-th state by τ_x^i and use the following representation:

$$\tau_x^i = \begin{cases} T & \text{if } \zeta_j^i \neq 0 \, \forall j = 0, \ldots, T \\ \min\{j \in \{0, \ldots, T\} : \zeta_j^i = 0\} & \text{otherwise} \end{cases}$$

where $\zeta_j^i \sim \text{Bernoulli}({}_jp_x \cdot q_{x+j})$.

By distorting the probabilities for the financial outcome and the occurrence of death, we account for the fact, that individuals may not anticipate the real world probabilities of both financial and mortality risk. The CPT value under this assumption is of the form

$$CPT(\Phi_L) = \sum_{i=1}^{n} \pi_i V(\Psi_L^i(A_{\tau_x^i}) e^{r(T-\tau_x^i)}).$$

In the numerical section, we will focus on determining the certainty equivalent of the CPT value:

$$CEV(\Phi_L) = \begin{cases} CPT(\Phi_L)^{1/\beta} + \Gamma & \text{if } CPT(\Phi_L) \geq 0 \\ -\left(\frac{-CPT(\Phi_L)}{\lambda}\right)^{1/\beta} + \Gamma & \text{if } CPT(\Phi_L) < 0. \end{cases}$$

4. Numerical results

This section focuses on the effect of mortality on the optimal insurance problem of the liability holder. Specifically, we want to answer the following questions:

- What is the optimal choice (among the four contracts specified above) of an expected-utility-maximizing liability holder?
- What is the optimal choice of a CPT-maximizing liability holder?
- Is EUT really not able to interpret demand for life/pension insurance contracts with guarantees?
- Can we identify main driving factors influencing the optimal choice under EUT and CPT?

In order to investigate these questions, we follow the considerations of Døskeland and Nordahl (2008) and fix the parameters for our benchmark case as follows:

$$L_0 = 4.75,\ A_0 = L_0/\alpha,\ r = 4\%,\ \mu = 6.5\%, \sigma = 15\%,\ g = 2\%,\ \alpha = 90\% \text{ and } b = 20\%.$$

Yet, we choose a different time of maturity $T = 25$, in order to account for the long term nature of pension insurance contracts and to be able to observe the effects of premature death. The calculation of the expected utility for the Merton investment, the implicit put and the simple life contract can be done with the formulas in Appendix A using numerical integration. For the annual guarantees and the values for the cumulative prospect theory for all four products, we performed Monte Carlo simulations with 100 000 paths.

4.1. **German life table.** Our model considers a representative liability-holder. We assume this person is aged 40 at the contract-issuing time, i.e. would be 65 at the maturity of the contract, which is a realistic age for a retiree. For the corresponding death and survival probabilities we use the standard mortality table DAV2008T from the German Actuarial Society (DAV) (see German Actuarial Society (2008)). The table is designed for contracts where the main source of risk is premature death. The resulting death probabilities ${}_tp_x \cdot q_{x+t}$ are between 0.1% and 1% and the probability to survive the whole period of 25 years is about 85.79% as seen in Figure 4.

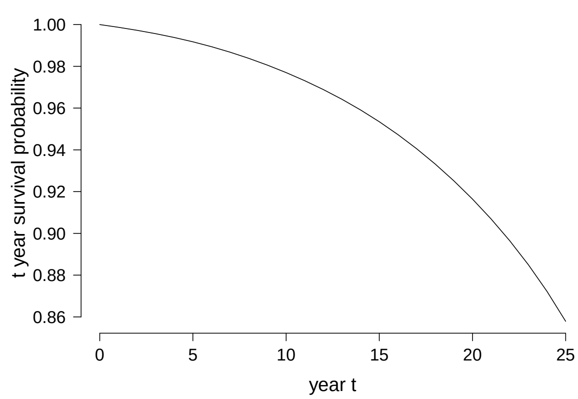

FIGURE 4. t year survival probability (${}_tp_x$) for an $x = 40$ year old for mortality data from German Actuarial Society (2008).

4.2. **Fair contract specification.** Comparison between different contracts makes only sense when the initial investments in these contracts are identical. Therefore, we base our comparison on the fair contract principle. A contract is called fair if the initial investment of the liability holder equals the expected discounted value of the contract payoff under the risk-neutral pricing measure Q. I.e. from the perspective of the liability holder, it holds

$$L_0 = e^{-rT} E^Q [\Phi_L] = {}_T p_x E^Q[e^{-rT} \Psi_L(A_T)] + \sum_{t=1}^{T} {}_{t-1}p_x \cdot q_{x+t-1} \cdot E^Q[e^{-rt} \Psi_L(A_t)]. \quad (9)$$

For equation (9) we have assumed independence of mortality and financial risk under the risk neutral measure Q. Following Dhaene et al. (2013) this assumption makes sense in our simple model setup.

Here, we choose to determine the fair surplus participation rate δ implicitly through equality (9). Since the right-hand side of the equality boils down to determining some standard option values in a Black-Scholes economy, we will jump to the fair participation rate.

In the case of implicit put, the fair participation rate is given by

$$\delta_{IP} = \frac{1 - {}_T p_x \cdot e^{(g-r)T} - \sum_{t=1}^{T} {}_{t-1}p_x \cdot q_{x+t-1} e^{(g-r)t}}{{}_T p_x \cdot (\Phi(d_1^{(T)}) - e^{(g-r)T} \Phi(d_2^{(T)})) + \sum_{t=1}^{T} {}_{t-1}p_x \cdot q_{x+t-1} \cdot (\Phi(d_1^{(t)}) - e^{(g-r)t} \Phi(d_2^{(t)}))}$$

with $d_1^{(t)} = \dfrac{(r - g + \frac{(\theta\sigma)^2}{2}) \cdot t}{\theta\sigma\sqrt{t}}, \qquad d_2^{(t)} = d_1^{(t)} - \theta\sigma\sqrt{t}.$

In case of a simple life contract, the fair participation rate can be expressed as

$$\delta_{SL} = \frac{\alpha - {}_T p_x \cdot \Big(1 - (\Phi(\widehat{d}_1^{(T)}) - \alpha e^{(g-r)T} \Phi(\widehat{d}_2^{(T)}))\Big) - \sum_{t=1}^{T} {}_{t-1}p_x q_{x+t-1} \Big(1 - (\Phi(\widehat{d}_1^{(t)}) - \alpha e^{(g-r)t} \Phi(\widehat{d}_2^{(t)}))\Big)}{{}_T p_x \Big(\alpha \Phi(d_1^{(T)}) - \alpha e^{(g-r)T} \Phi(d_2^{(T)})\Big) + \sum_{t=1}^{T} {}_{t-1}p_x q_{x+t-1} \Big(\alpha \Phi(d_1^{(t)}) - \alpha e^{(g-r)t} \Phi(d_2^{(t)})\Big)}$$

with $\widehat{d}_1^{(t)} = \dfrac{-\log(\alpha)}{\theta\sigma\sqrt{t}} + d_1^{(t)}, \qquad \widehat{d}_2^{(t)} = \widehat{d}_1^{(t)} - \theta\sigma\sqrt{t},$

$d_1^{(t)}$ and $d_2^{(t)}$ take the same values as in the implicit put case. In case of annual guarantees, we need to rely on numerical methods to determine the fair participation rate. In Figure 5, δ is plotted as a function of the proportion invested in the risky asset for a fixed guarantee value of $g = 2\%$ on the left side and on the right side as a function of the minimum interest guarantee g for $\theta = 40\%$.

Under all contracts including guarantees, a higher proportion invested into the risky asset and a higher guarantee rate lead to a lower surplus participation δ. For the same g, the annual guarantee case delivers the lowest δ, which is followed by the implicit put case, and the resulting δ in the simple life case is the highest. It is due to the fact that for the same parameters (particularly same g and δ), the annual guarantee delivers the highest contract payoff, and the contract payoff in the implicit put is always larger than in the simple life case (because of the shorted default option).

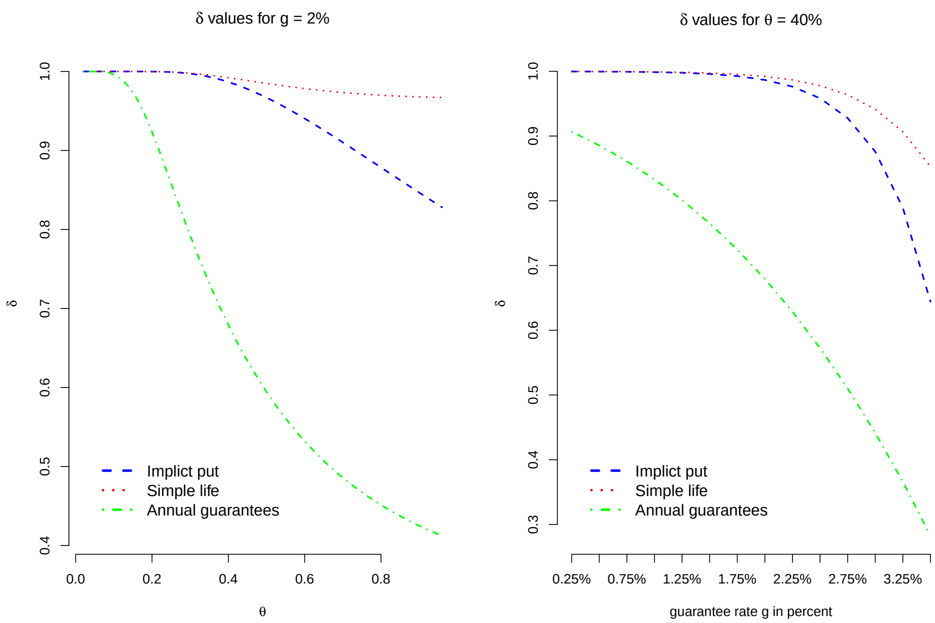

FIGURE 5. Values of δ for the three products with $\mu = 6.5\%$, $r = 4\%$, left: different values of θ, for $g = 2\%$, right: different values of g, for $\theta = 40\%$

4.3. **Expected utility.** We start comparing the four contracts using expected utility theory, with the previously described utility function. For the coefficient of risk aversion γ we consider the values $\gamma = 0.5$, $\gamma = 3$ and $\gamma = 6$ to compare different risk attitudes of the investor. In Figure 6 we plot the certainty equivalents for the expected utility for the three different values of γ and observe that we get clearly different results depending on the chosen risk aversion. When considering the case $\gamma = 0.5$ we get the same result as Døskeland and Nordahl (2008), namely that the Merton investment is clearly preferred over the other three products. The Merton investment delivers for low values of θ a CEQ equal to that of the other products, but has a significantly higher CEQ at the optimal θ of 100%. This also shows that an investor with a relatively small risk aversion prefers a high expected payout over any form of guarantee and seeks the complete investment into the risky asset. But when the risk aversion is larger than one, this is not necessarily the case

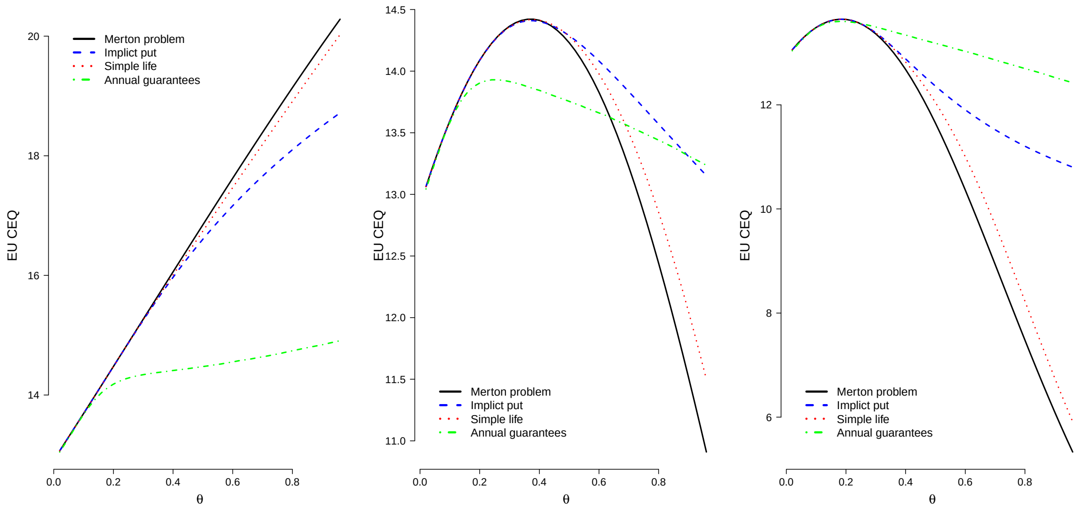

FIGURE 6. CEQ for $E[U(\Phi_L)]$ left: $\gamma = 0.5$, middle: $\gamma = 3$, right: $\gamma = 6$ and all other parameters as described before.

anymore. For an investor with risk aversion of $\gamma = 3$, the optimal portion of risky assets in case of the Merton investment, the implicit put and the simple life contract is between 30% and 40%. This investor does not see much difference between these three products at the optimal θ. Yet, as soon as the proportion invested into the risky asset is larger than the optimal value, the products including simple guarantees, namely the implicit put and the simple life, deliver a higher CEQ than the Merton investment. The annual guarantee mostly delivers the least CEQ in this case and only outperforms the other products, when the investment in the risky asset is very high, i.e. a lot of risk is contained in the portfolio. When the risk aversion is further increased to $\gamma = 6$, the optimal θ for all four products is between 15% and 20%. Now we even observe that for the optimal θ the Merton investment, the implicit put and the simple life contract yield the exact same CEQ and the annual guarantees have a slightly smaller CEQ than the other three products. Comparing the products for a proportion invested in the risky asset above the optimal one, the Merton investment delivers a CEQ below the one of the other three products. Now even the annual guarantees deliver the highest CEQ for a value of θ above 30%. This shows that when the risk in the investment is increasing the products offering a guarantee deliver a higher CEQ than the pure Merton investment. This increased risk may either result from a higher proportion invested into the risky asset, or for example a higher volatility σ of the risky asset. Hence, for an investor with a high risk aversion, the products including guarantees are more favorable than a pure investment into the portfolio without any sort of guarantee. When the risk aversion is high and the risk contained in the portfolio is large enough, even an annual guarantee is preferred over the other simple types of guarantee.

The above result is mainly driven by the expected utility upon premature death. Recall the expected utility of the liability holder is given by

$$E[U(\Phi_L)] = {}_Tp_x \cdot E[U(\Psi_L(A_T))] + \sum_{t=1}^{T} {}_{t-1}p_x \cdot q_{x+t-1} \cdot E\left[U\left(\Psi_L(A_t) \cdot e^{r(T-t)}\right)\right].$$

The Merton portfolio delivers the highest expected utility upon survival, i.e. $E[U(\Psi_L(A_T))]$, for a constant proportion invested in the risky asset (see Merton (1971)). However, it may no longer be optimal when it comes to the expected utility upon premature death. For a single payout prior to maturity at time $t < T$, the expected utility is given by $E[U(\Psi_L(A_t) \cdot e^{r(T-t)})]$. Hence a constant proportion in risky and riskless assets is held up to the time of death and after that the proceedings will be invested completely in the riskless asset until the maturity date. This payout might not be optimal anymore compared to the other products including guarantees. In Figure 7 we plot the expected utility of the payout in case of death in year $t = 15$ and observe that the Merton portfolio is no longer optimal, but depending on the risk aversion γ, one of the other products becomes optimal. This is also the case for other values of $t < T$, up to a certain time point t^*. From that point on, the closer t gets to T, the better the Merton case becomes compared to the other products. The value of t^* depends on the chosen parameters and especially the size of the risk aversion.

The previous observations show that, depending on the risk aversion of the individual, the products including a guarantee, especially the implicit put and the simple life contract, become favorable in the sense, that they give the same or a higher CEQ than the Merton investment for any value of θ. In the following we consider the relation between the size of the included guarantee and the risk aversion of the investor.

In Table 1 we show different CEQ values for combinations of risk aversion γ and guarantee g at the optimal θ and a θ value above the optimal one. When the risk aversion is below 1, e.g. $\gamma = 0.5$, the Merton investment delivers a CEQ equal or above the other products for all θ no matter how small the value of the guarantee is (even for $g = 0.01\%$). Hence we do not show CEQ values for this risk aversion in the table.

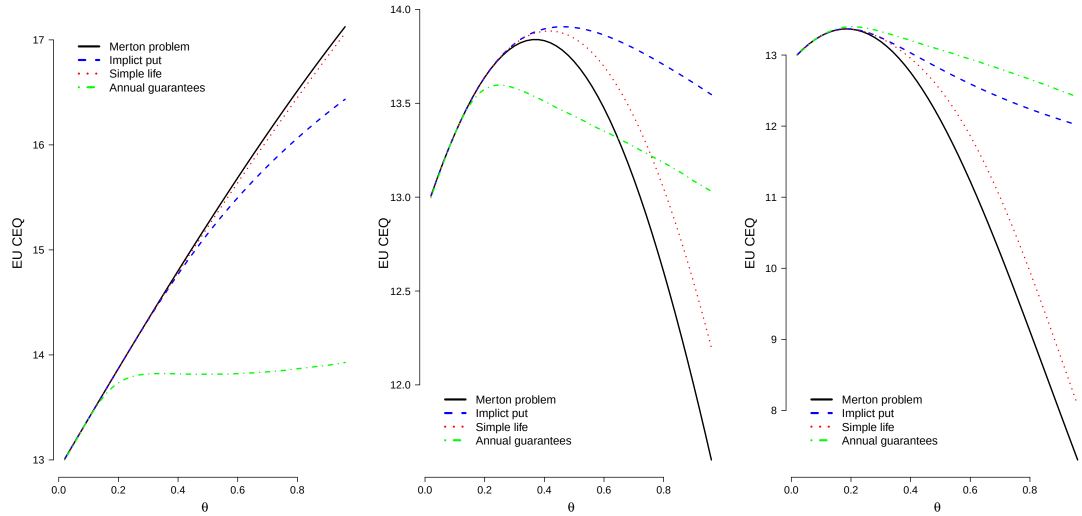

FIGURE 7. Expected utility from premature payment ($E[U(\Psi_L(A_t) \cdot e^{r(T-t)})]$) at time $t = 15$, left: $\gamma = 0.5$, middle: $\gamma = 3$, right: $\gamma = 6$ and all other parameters as described before.

For a risk aversion of $\gamma = 3$, we see that for a low guarantee the CEQ at the optimal θ are nearly the same for the Merton investment, implicit put and simple life. When the guarantee is too high, the Merton investment yields the higher CEQ. When the risk aversion is increased up to $\gamma = 6$, even higher guarantees are preferable over the Merton investment. For a risk aversion of $\gamma = 12$, even a guarantee of $g = 3.5\%$ is favored by the investor over the pure investment into the portfolio.

4.4. **Cumulative prospect theory.** In this section we analyze the preferences of CPT-investors with varying assumptions about their risk awareness (i.e. the way we incorporate mortality) and varying contract parameters (e.g. the guarantee rate). To make results comparable we fix the market-parameters as described at the beginning of this chapter. For the CPT parameters we choose for the risk aversion $\beta = 0.5$ and the loss aversion $\lambda = 2.25$ following Døskeland and Nordahl (2008). As they have already mentioned there are multiple estimations of probability weighting functions and their parameters, hence we follow them by applying the weighting by Prelec (1998) (with $\phi = 0,75$), for reasons of comparability. For the reference point Γ, Barberis (2012) argues that the choice of adequate reference points is not trivial and, in general, can not be answered definitively. We choose $\Gamma = L_0$ to reflect the initial investment, which is a standard choice in the literature. Furthermore, it seems natural to consider $\Gamma = L_0 e^{gT}$ since investors might

	Merton portfolio		Implicit put		Simple life		Annual guarantee	
$\gamma = 3$								
	θ^*	$\theta = 60\%$	θ^*	$\theta = 60\%$	θ^*	$\theta = 60\%$	θ^*	$\theta = 60\%$
$g = 1.0\%$	14.4214	13.8279	14.4207	13.9327	14.4213	13.8902	14.1572	13.7333
$g = 2.0\%$	14.4214	13.8279	14.4094	14.0822	14.4183	13.9779	13.9297	13.6609
$\gamma = 6$								
	θ^*	$\theta = 40\%$	θ^*	$\theta = 40\%$	θ^*	$\theta = 40\%$	θ^*	$\theta = 40\%$
$g = 2.0\%$	13.6454	12.679	13.6454	12.8825	13.6454	12.8221	13.6037	13.342
$g = 3.0\%$	13.6454	12.679	13.6389	13.2955	13.6404	13.0749	13.4334	13.3293
$\gamma = 12$								
	θ^*	$\theta = 30\%$	θ^*	$\theta = 30\%$	θ^*	$\theta = 30\%$	θ^*	$\theta = 30\%$
$g = 3.5\%$	13.2717	11.5776	13.2692	12.9638	13.2695	12.4418	13.179	13.1275

TABLE 1. CEQ of the total payoff for an EU investor for different γ and different guarantee values g and the previously described parameters.

expect to earn at least the guarantee rate, as there are products available which promise this return. Another choice could for example be the expected payout of the Merton investment. But all other products mostly deliver a payout below this reference point and are thus considered losses and are therefore dominated by the Merton investment. Therefore we do not consider this reference point in the following analysis.

In the following we first calculate the CPT values under the assumption that the investor anticipates the real world death probabilities. Hence, the weighting is only applied to the financial risk. The resulting CEV's for the two above mentioned reference points are shown in Figure 8. We see that, considering the maximal achievable certainty equivalents (CEV), we get in both cases that the implicit put delivers the highest value, followed by the simple life contract, the Merton investment and finally the annual guarantees. This shows that CPT reflects the demand for guarantees with the inclusion of mortality.

Considering a CPT investor with a lower risk aversion, i.e. a higher value of β (e.g. $\beta = 0.88$) as proposed by Tversky and Kahneman (1992), we get the resulting CEVs

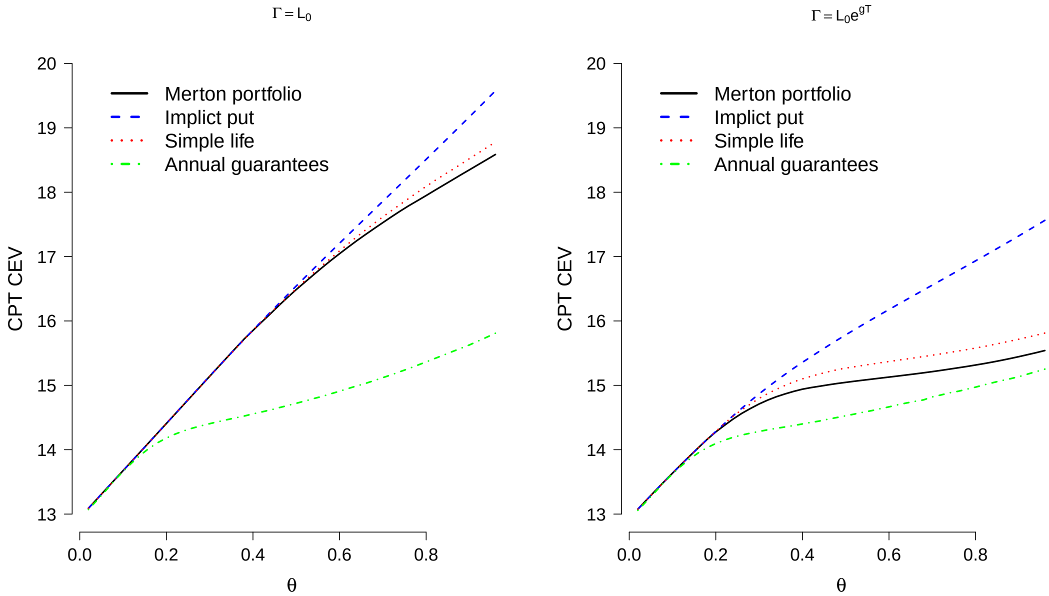

FIGURE 8. CEV for CPT with $\Gamma = L_0$ (left) and $\Gamma = L_0 \cdot e^{gT}$ (right) with $\beta = 0.5$ and $\varphi = 0.75$

as shown in Figure 9.[3] We see that the investor would prefer the Merton portfolio over the products including guarantees. This result is similar to the previous EUT analysis, where we see that guarantees are only preferred when the risk aversion is large enough. When the risk aversion parameter β is lowered to e.g. $\beta = 0.35$, we even see that for the optimal θ, although the implicit put still gives the highest CEV, it is followed by the annual guarantees, which deliver a CEV higher than the simple life contract and the Merton investment. Hence, we observe, exactly as in the EUT case, that an investor with a high risk aversion benefits more from guarantees included in these products.

These preferences of a CPT investor seem to result from the so called skewness preference, a major driving factor for decision making for CPT-investors, which is among others described in Ebert and Strack (2012). They state that a CPT investor prefers right-skewed risks over others. We compare the expected total payoff and the skewness of each contract at a fixed value of θ in Table 2. We see that for each guarantee value, the Merton investment delivers the highest expected total payout. But on the other hand all four products have a higher skewness than the pure Merton investment. The expected value

[3]We want to note at this point, that using $\beta = 0.88$, but the weighting function suggested by Prelec (1998), we are not back at the original setting of Tversky and Kahneman (1992). However, when applying their weighting function, the results turn out to be similar to the ones presented above.

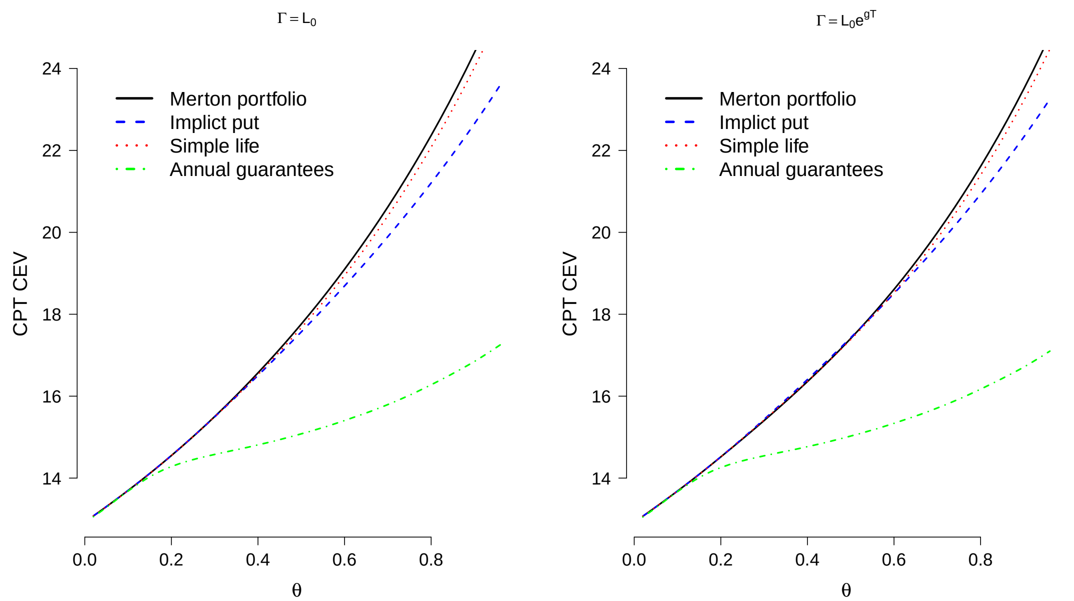

FIGURE 9. CEV for CPT with $\Gamma = L_0$ (left) and $\Gamma = L_0 \cdot e^{gT}$ (right) with $\beta = 0.88$ and $\varphi = 0.75$

of the payoff of the Merton investment, implicit put and simple life are quite close, but since the implicit put and the simple life contract have a higher positive skewness, they are preferred by the CPT investor, when $\beta = 0.5$. Although the annual guarantees have the highest skewness, their expected payoff is much smaller than the payoff of the other three products, therefore it is not the favored contract.

	Merton portfolio		Implicit put		Simple life		Annual guarantee	
	Expected value	Skewness	Expected value	Skewness	Expected value	Skewness	Expected value	Skewness
$g = 1.0\%$	15.4542	0.670703	15.4531	0.671164	15.4532	0.671131	14.9586	0.847502
$g = 2.0\%$	15.4542	0.670703	15.4358	0.676369	15.4401	0.675472	14.4333	0.939844
$g = 3.0\%$	15.4542	0.670703	15.1651	0.759176	15.2761	0.733827	13.8055	0.88429

TABLE 2. Expected value and skewness of the total payout at $\theta = 30\%$ for all four contracts and all other parameters as before.

When the investment in the risky asset is further increased, i.e. θ is increased, the skewness of the payouts also increases. Since the CPT investor prefers right-skewness, he desires a high investment into the risky asset. Hence, we observe in the above two figures, that $\theta = 100\%$ is optimal for all four considered products.

So far we have focused on a CPT-investor who can asses his own survival probabilities realistically. This assumption may not seem realistic, in particular since Andersson and Lundborg (2007) show that individuals tend to under estimate their mortality risk. Hence, we examine those CPT investors where the probability of a premature death is incorporated into the cash flow, as described in Section 3, using the weighting function suggested by Prelec (1998).[4] However the resulting CEV values are almost the same as in the previous case, where we did not apply the weighting on the death probabilities. This suggests, that the distortion of the mortality risk might be different than the ones of the financial risk, to account for the individual perception of the different risks. To our knowledge there have not been made any suggestions yet, on how to weigh the death probabilities within the CPT framework.

We thus observe that a CPT investor might favor products including guarantees over the Merton investment, as long as the risk aversion is large enough. To compare the results for different guarantee values, we show in Table 3 the CEV for a CPT investor with $\beta = 0.5$, $\lambda = 2.25$ and $\phi = 0.75$ for different values of g at the optimal value of θ. Comparing the CEV for one single product and different guarantee values, we see that for all products and for both reference points, the general level of the CEV decreases with increasing guarantee. Hence, we see that, other than an EUT investor, a CPT investor in favors, independet of the reference point, a lower guarantee, no matter which guarantee product.

	Merton portfolio		Implicit put		Simple life		Annual guarantee	
	$\Gamma = L_0$	$\Gamma = L_0 e^{gT}$	$\Gamma = L_0$	$\Gamma = L_0 e^{gT}$	$\Gamma = L_0$	$\Gamma = L_0 e^{gT}$	$\Gamma = L_0$	$\Gamma = L_0 e^{gT}$
$g = 1.0\%$	18.5862	17.1742	20.042	18.8915	18.8354	17.4228	16.3281	16.1074
$g = 2.0\%$	18.5862	15.5406	19.5751	17.563	18.7799	15.8121	15.8096	15.2521
$g = 3.0\%$	18.5862	14.1571	18.0094	15.6614	18.6226	14.3638	15.2241	13.7536

TABLE 3. CEV of the total payoff for a CPT investor at the optimal θ, $\beta = 0.5$, $\lambda = 2.25$, $\phi = 0.75$, different guarantee values g and the previously described parameters.

[4]We also applied the weighting function suggested by Tversky and Kahneman (1992), but got similar results. Hence we do not show these results here in further detail.

5. Conclusion

In the present paper, we extend optimal life/pension insurance design literature by incorporating mortality risk, hence analyzing more realistic contracts. We assess the value of these contracts within expected utility theory and cumulative prospect theory. For the latter we propose two possibilities to incorporate mortality in the calculation of the CPT value. Our main findings in the present paper are:

- By including mortality in the contract design, we show that EUT can actually prefer products with guarantees or be indifferent regarding guarantees, depending on the risk aversion.
- We can confirm existing literature that CPT can explain the demand for products with guarantees. But we also find examples where CPT does not reflect the demand for guarantees, especially in the original Tversky and Kahneman (1992) setup, which is in contrast to existing literature.
- In general, we see that products including (simple) roll-up guarantees are mostly favored over the contract including annual guarantees.

Our analysis shows that EUT and CPT may yield a similar ranking regarding the considered contracts. But where the preferences of an EUT investor clearly depend on the level of the risk aversion γ, the application of CPT contains a lot of uncertainty resulting from the wide variety of choices for the CPT parameters. The literature offers several diverse parameters, which may lead to different results, like in our setup. For instance, we encounter similar problems as described in Barberis (2012), i.e. the correct choice of the reference point.

Furthermore, the correct application of CPT with financial and mortality risk is not completely clear. Our proposed solutions assume that either individuals are able to assess their mortality rates in a realistic manner or distort them the same way as the financial risk. The second seems not appropriate to reflect the individual perception of mortality risk, but to our knowledge, there have been no studies so far on how to apply the CPT framework of probability weighting to the death probabilities.

In our paper, we show that both EUT and CPT can succeed as well as fail to reflect the investor's preference for products with guarantees. A possible extension of our analysis could include more realistic aspects: stochastic interest rates due to the long-term nature of life & pension contracts, periodic beneficiaries' contributions and consumption, etc. Further research might also consider including a dynamic reference point in a dynamic optimization setting or a different weighting of the mortality risk in the evaluation of the CPT values.

Appendix A. Expected utility of Merton investment, implicit put and simple life contract

The expected utility of one payout at time t can be calculated in closed form for the Merton investment and up to an integral for the implicit put contract and the simple life contract. For shorter description we assume the following notation:

$$\bar{\mu} = \left(r + \theta(\mu - r) - \frac{\theta^2\sigma^2}{2} \right) t \text{ and } \bar{\sigma}^2 = \theta^2\sigma^2 t$$

From the properties of the asset process we know that $\frac{A_t}{A_0} \sim LN(\bar{\mu}, \bar{\sigma}^2)$ and hence can get the expected utility of a payout of the Merton investment at time t:

$$E\left[U(\alpha A_t(\theta)e^{r(T-t)})\right] = \frac{L_0^{1-\gamma}}{1-\gamma} \cdot e^{r(T-t)(1-\gamma)} \cdot \exp\left(\bar{\mu}(1-\gamma) + \frac{\bar{\sigma}^2(1-\gamma)^2}{2} \right)$$

To get the expected utility of the implicit put we first rewrite the payout the following way:

$$\begin{aligned}
\Psi_L(A_t) &= L_0 e^{gt} + \alpha\delta \max\left\{ A_t - \frac{1}{\alpha} L_0 e^{gt}, 0 \right\} \\
&= L_0 \left((\delta \frac{A_t}{A_0} + (1-\delta)e^{gt}) \mathbf{1}_{\frac{A_t}{A_0} > e^{gt}} + e^{gt} \cdot \mathbf{1}_{\frac{A_t}{A_0} \le e^{gt}} \right)
\end{aligned}$$

Then we get for the expected utility:

$$E[U(\Psi_L(A_t)e^{r(T-t)})] = \frac{L_0^{1-\gamma}}{1-\gamma} \cdot e^{r(T-t)(1-\gamma)} \cdot \left(\int_{e^{gt}}^{\infty} (\delta z + (1-\delta)e^{gt})^{1-\gamma} f_{\frac{A_t}{A_0}}(z) dz + e^{gt(1-\gamma)} \cdot F_{\frac{A_t}{A_0}}(e^{gt}) \right)$$

To get the expected utility of the simple life contract we rewrite the payout the following way:

$$\begin{aligned}
\Psi_L(A_t) &= L_0 e^{gt} + \alpha\delta \left(A_t - \frac{1}{\alpha} L_0 e^{gt}, 0 \right)^+ - \left(L_0 e^{gt} - A_t \right)^+ \\
&= e^{gt} \mathbf{1}_{\alpha e^{gt} \le \frac{A_t}{A_0} \le e^{gt}} + (\delta \frac{A_t}{A_0} + (1-\delta)e^{gt}) \mathbf{1}_{e^{gt} < \frac{A_t}{A_0}} + \frac{1}{\alpha} \frac{A_t}{A_0} \mathbf{1}_{\frac{A_t}{A_0} < \alpha e^{gt}}
\end{aligned}$$

and we hence get for the expected utility:

$$
\begin{aligned}
E[U(\Psi_L(A_t)e^{r(T-t)})] =& \frac{L_0^{1-\gamma}}{1-\gamma} \cdot e^{r(T-t)(1-\gamma)} \cdot \Bigg(+ e^{gt(1-\gamma)} \cdot \left(F_{\frac{A_t}{A_0}}(e^{gt}) - F_{\frac{A_t}{A_0}}(\alpha e^{gt}) \right) \\
& + \int_{e^{gt}}^{\infty} (\delta z + (1-\delta) e^{gt})^{1-\gamma} f_{\frac{A_t}{A_0}}(z) dz \\
& + \left(\frac{1}{\alpha}\right)^{1-\gamma} \cdot \exp\left((1-\gamma)\bar{\mu} + \frac{1}{2}(1-\gamma)^2 \bar{\sigma}^2 \right) \cdot \Phi\left(\frac{\log(\alpha e^{gt}) - \bar{\mu}}{\bar{\sigma}} - (1-\gamma)\bar{\sigma} \right) \Bigg)
\end{aligned}
$$

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