Numerical Investigations of an Error Bound for Reduced Basis Approximations of Non-Coercice Variational Inequalities

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Numerical Investigations of an Error Bound for Reduced Basis Approximations of Noncoercice Variational Inequalities ⋆

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Abstract: We consider variational inequalities with different trial and test spaces and a possibly noncoercive bilinear form. Well-posedness has been shown under general conditions that are e.g. valid for the space-time formulation of parabolic variational inequalities. Fine discretizations for such problems resolve in large scale problems and thus in long computing times. To reduce the size of these problems, we use the Reduced Basis Method (RBM). Combining the RBM with the space-time formulation, a residual based error estimator has been derived in [Glas and Urban (2014)]. In this paper, we provide corresponding numerical results for a parametrized heat inequality model. Particularly, we perform two experiments concerning the error estimator. In the first one, we focus on rigor and efficiency of the error estimator depending on the specific method used for the basis generation and on the shape of the obstacle. In the second one, we show the quantitative reduction using the RBM in this setting.

Keywords: Variational Inequalities, Parabolic Problems, Error Estimates, Obstacle Problems, Space-Time Discretizations, Petrov-Galerkin Formulations, Reduced Basis Methods

1. INTRODUCTION

Parabolic variational inequalities often arise in industrial or financial applications, e.g. as time-dependent obstacle problems or the valuation of American Options. Instead of using a time stepping scheme, we formulate the problem as a space-time variational inequality, where we use the time as an additional variable in the variational formulation of the problem. Existence and uniqueness for this formulation has been shown under mild assumptions in [Glas and Urban (2014)].

Discretizing, e.g. with finite elements or finite volume methods, and solving the arising large scale Petrov-Galerkin parabolic variational inequality lead to enormous computational costs. Particularly, if one has a multi-query-setting, i.e. solving the same problem several times with different values of parameters, e.g. calibration of volatility or other parameters or fitting unknown material parameters to measurements. One option to significantly decrease the size of the problem is the concept of Reduced Basis Methods (RBM) [Patera and Rozza (2006)]. The objective of the RBM is to efficiently build a fast reduced model which is a good surrogate for the high dimensional costly model. Problems are considered where not only a single solution is needed, but solutions are wanted for a whole range of different parameter configurations.

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In the context of variational inequalities, RBMs have initially been applied to the elliptic case, [Haasdonk et al. (2012a)]. Based upon this, [Haasdonk et al. (2012b)] combines RBMs with parabolic variational inequalities for American Options, but does not provide error estimators. Using the space-time framework based upon [Urban and Patera (2014)], error estimators for parabolic variational inequalities have recently been derived in [Glas and Urban (2014)]. After finishing this paper, we got aware of [Burkovska et al. (2014)], where a time stepping error estimator has been presented.

The main focus of this paper is a quantitative numerical investigation of the mentioned space-time error estimator. To show the rigor of the error bound, a parameterized heat inequality model is considered. We show that the choice of primal and dual basis functions as well as the shape of the obstacle are crucial for the effectivity of the error bound. Note that this observation is also relevant in the elliptic case and is by no means induced by the parabolic space-time setting.

In Section 2, we formulate a parabolic variational inequality first in space and then transfer it into a space-time form. We recall its well-posedness. In Section 3, we introduce the Reduced Basis Method also for a saddle point formulation. Section 4 consists of a brief summary of the derivation of the error estimator. Section 5 is the main part of the paper, namely the numerical investigation of the error bound. We show numerically that this error estimator is efficient, as well as discuss in which cases it can be effective at all.

2. SPACE-TIME FORMULATION OF PARABOLIC VARIATIONAL INEQUALITIES

We start with the derivation of a space-time variational inequality introducing the trial and test spaces and forms. Afterwards, we recall existence and uniqueness of a solution for this problem.

2.1 Spaces

Let $V \hookrightarrow H \hookrightarrow V'$ be a Gelfand triple with V, H being Hilbert spaces with corresponding inner products $(\cdot,\cdot)_V$ respectively $(\cdot,\cdot)_H$ and let $I := (0,T)$ be a time interval with final time $T > 0$. By

$$\rho := \sup_{\phi \in V} \frac{\|\phi\|_V}{\|\phi\|_H},$$

we denote the embedding constant of V in H. In order to derive a space-time variational formulation, we require the definition of Bochner Spaces for any normed space U, i.e., $L_2(I;U) := \{v : I \to U : \|v\|^2_{L_2(I;U)} := \int_0^T \|v\|_U \, dt < \infty\}$, [Evans (2010)]. In the following we set

$$X := \{v \in L_2(I;V) : v_t \in L_2(I;V'), v(0) = 0\},$$
$$Y := \{v \in L_2(I;V)\},$$

with X being densely embedded in Y. Note that $v(0)$ and $v(T)$ are well defined in H because $X \hookrightarrow C(\bar{I};H)$. With these spaces, we define the following norms

$$\|v\|_Y := \|v\|_{L_2(I;V)}, \qquad [\![v]\!]_X^2 := \|v\|_Y^2 + \|v(T)\|_H^2,$$
$$\|v\|_X^2 := \|v\|_Y^2 + \|v_t\|_{Y'}^2, \; |\!|\!|v|\!|\!|_X^2 := \|v_t\|_{Y'}^2 + [\![v]\!]_X^2.$$

In comparison to the standard norm $\|\cdot\|_X$, $|\!|\!|\cdot|\!|\!|_X$ is obviously stronger, but allows control at the final time T.

2.2 Forms

Now, we derive a space-time variational formulation for a parabolic variational inequality. Let $c : V \times V \to \mathbb{R}$ be the bilinear and $f : V \to \mathbb{R}$ the linear form corresponding to the weak form in space. Let $K \subset Y$ be a closed convex set, i.e. $K(t) \subset V$ for a.e. $t \in I$. We consider the parabolic variational inequality: Find $u \in H^1(I;H) \cap C(\bar{I};V)$, such that $u(t) \in K(t)$, $t \in I$ a.e., and

$$(u_t(t), v(t) - u(t))_H + c(u(t), v(t) - u(t)) \qquad (1)$$
$$\geq \langle f(t), v(t) - u(t)\rangle_{V' \times V}, \; \forall v(t) \in K(t),$$
$$u(0) = 0 \text{ in } H.$$

The solution u, if it exists, is called *strong solution* according to [Ito and Kunisch (2006)]. In this book, it is also analyzed under which conditions such a strong solution exists. We now define the space-time forms

$$a(u,v) := \int_I (u_t(t), v(t))_H dt + \int_I c(u(t), v(t)) dt$$
$$f(v) := \int_I \langle f(t), v(t)\rangle_{V' \times V} \, dt,$$

and arrive at the space-time formulation of (1): Find $u \in X \cap K$ with $u(t) \in K(t)$ for all $t \in I$ a.e. and

$$a(u, v - u) \geq f(v - u) \; \forall v \in K. \qquad (2)$$

2.3 Existence and Uniqueness

For formulating the well-posedness analysis from [Glas and Urban (2014)], we need the concept of symmetrically boundedness and weak coercivity, which we now explain.

Definition 1. A bilinear form $a : X \times Y \to \mathbb{R}$, $X \subset Y$, is called

(a) *weakly coercive* [1], if there exists a constant $\alpha_w > 0$ such that $a(v,v) \geq \alpha_w [\![v]\!]_X^2$ for all $v \in X$;
(b) *symmetrically bounded* [2], if there exists a constant $0 \leq \gamma_s < \infty$ such that $|a(v,w)| \leq \gamma_s [\![v]\!]_X |\!|\!|w|\!|\!|_X$ for all $v, w \in X$.

One can prove that the space-time bilinear form fulfills these properties.

Proposition 2. If the bilinear form $c(\cdot,\cdot)$ is bounded with constant γ_c and satisfies a Gårding inequality, i.e. there exist constants $\alpha_c > 0$ and $\lambda_c \geq 0$ such that

$$c(u(t), u(t)) + \lambda_c \|u(t)\|_H^2 \geq \alpha_c \|u(t)\|_V^2, \; \forall u(t) \in V$$

with $\alpha_c - \lambda_c \rho^2 > 0$, then the bilinear form $a(\cdot,\cdot)$ is bounded, symmetrically bounded and weakly coercive.

Proof. See [Glas and Urban (2014), Proposition 4.2].

Now, we can pose the announced well-posedness result.

Theorem 3. If the assumptions of Proposition 2 hold, the space-time variational inequality (2) has a solution which is unique w.r.t. $[\![\cdot]\!]_X$.

Proof. See [Glas and Urban (2014), §§2.4.1, 2.4.2].

3. THE REDUCED BASIS METHOD (RBM)

We first introduce a parametric setting of the previous non-parametric forms. We refer to e.g. [Patera and Rozza (2006)] for a survey on RBM.

Let $\mu \in \mathcal{D} \subset \mathbb{R}^p$ be a parameter, i.e., consider a parametric bilinear and linear form. Then, the space-time formulation of the parametric parabolic variational inequality reads: For $\mu \in \mathcal{D}$, find $u(\mu) \in X(\mu) \cap K(\mu)$ with $u(t;\mu) \in K(t;\mu)$ for all $t \in I$ a.e. and

$$a(u, v - u; \mu) \geq f(v - u; \mu) \; \forall v \in K.$$

Next, we assume the availability of a detailed discretization (sometimes also called "truth") of possibly large dimension $\mathcal{N}$ such that the detailed solution $u^{\mathcal{N}}(\mu)$ is a sufficiently good approximation of the real solution $u(\mu)$.

To apply the RBM, a property called affine decomposition in the parameter (also known as parameter-separability) is crucial. This characteristics allows to split the parameter dependent bilinear (linear) form, into parameter dependent functions, which are independent of the high dimension $\mathcal{N}$, and parameter-independent bilinear (linear) forms:

$$a(u,v;\mu) = \sum_{q=1}^{Q_a} \Theta_a^q(\mu) a^q(u,v), \quad f(v;\mu) = \sum_{q=1}^{Q_f} \Theta_f^q(\mu) f^q(v).$$

In the offline phase one assembles the computational costly system matrices to $a^q(u,v)$, $f^q(v)$ with complexity depending on $\mathcal{N}$, whereas online only problems of small size $N \ll \mathcal{N}$ have to be solved.

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[1] Only coercivity with respect to the smaller space X and the weaker semi-norm $[\![\cdot]\!]_X$ has to be fulfilled.

[2] The name corresponds to the switched norms on the righthand side of the estimate. Again, only with respect to the space X.

3.1 Saddle-Point Formulation

It is quite common to transfer a variational inequality into a saddle point problem. Therefore, we introduce a Hilbert space $W := L(I; \tilde{W})$ for the dual variable. It is required that the convex set K has the particular structure

$$K = \{v \in V : b(v, q) \le g(q)\ \forall q \in M\},$$

where $b : Y \times W \to \mathbb{R}$ is a bilinear form $g : W \to \mathbb{R}$ and $M \subset W$ denotes a convex cone (the *dual cone*).

The parametric saddle point problem then is the following: For $\mu \in \mathcal{D}$, find $(u(\mu), \lambda(\mu)) \in X \times W$ such that

$$a(u, v; \mu) + b(v, \lambda(\mu)) = f(v; \mu), \qquad v \in Y, \tag{3a}$$
$$b(u(\mu), \eta - \lambda(\mu)) \le g(\eta - \lambda(\mu); \mu),\ \eta \in M. \tag{3b}$$

At a first glance, this looks complicated, but it has the advantage that functions in the full function spaces X, Y can be used instead of a convex set for the primal variable. We denote the detailed solutions by $u^{\mathcal{N}}(\mu)$, $\lambda^{\mathcal{N}}(\mu)$.

3.2 Reduced Basis Generation

For the primal reduced basis, we use a standard greedy procedure [Patera and Rozza (2006)] using the Galerkin projection error as error indicator.

For the greedy algorithm, we need a finite training set $\Xi_{\text{train}} \subset \mathcal{D}$, a desired tolerance tol_{RB}, a maximal number $N_{\max}$ of basis functions for the reduced space and an initial parameter μ_1^*. Starting with the initial sample set $S_1 = \{\mu_1^*\}$ and the initial reduced basis space $X_1 = \text{span}\{u(\mu_1^*)\}$, the following iterative algorithm is performed ($u_N(\mu)$ is the Galerkin projection of $u^{\mathcal{N}}(\mu)$ onto X_N):

Projection Greedy Algorithm:

$$\begin{aligned}
&\text{while} \quad (\varepsilon_{\text{RB}} > \text{tol}_{\text{RB}} \quad \text{and} \quad N < N_{\max})\\
&\qquad \mu_N^* = \underset{\mu \in \Xi_{\text{train}}}{\arg\max} \|u^{\mathcal{N}}(\mu) - u_N(\mu)\|_X\\
&\qquad \varepsilon_{\text{RB}} = \|u^{\mathcal{N}}(\mu_N^*) - u_N(\mu_N^*)\|_X\\
&\qquad S_N = S_{N-1} \cup \{\mu_N^*\}\\
&\qquad X_N = X_{N-1} + \text{span}\{u^{\mathcal{N}}(\mu_N^*)\}.
\end{aligned}$$

Of course, the projection based greedy is merely used for numerical experiments (e.g. for comparisons in order to investigate the quantitative properties of the error estimator). If the greedy process terminates, it returns the reduced basis space X_N with dimension $N_X \ll \mathcal{N}$ and the set of chosen parameters $S_{N_X} = \{\mu_1^*, ..., \mu_{N_X}^*\}$. This is a standard approach in RBM.

For the dual reduced basis, the situation is less standard so that we compare two variants for the basis generation:
(A) If we assume that the chosen snapshots for the primal variable u are also a good choice for the dual variable λ and set $W_{N_W} = \text{span}\{\lambda^{\mathcal{N}}(\mu_n^*), 1 \le n \le N_X\}$. In this case, we have $\dim(W_{N_W}) := N_W = N_X$.
(B) A greedy algorithm similar to the one described above is performed for the dual variable λ until a given tolerance is satisfied. In this case, we may have $N_W \ne N_X$.

After determining a primal and dual basis, we arrive at a low dimensional saddle point problem: Setting $K_N := \{v_N \in X_N : b(v_N, q_N) \le g(q_N), q_N \in M_N\}$, $M_N := \{\sum_{i=1}^N \alpha_i \lambda^{\mathcal{N}}(\mu_i) : \alpha_i \ge 0\}$ and choosing a suitably stable test space Y_N, the reduced saddle-point problem corresponding to (3) reads:
Find $(u_N, \lambda_N) \in X_N \times M_N$ such that

$$\begin{aligned}
a(u_N, v_N; \mu) + b(v_N, \lambda_N) &= f(v_N; \mu) &\quad \forall\, v_N \in Y_N,\\
b(u_N, q_N - \lambda_N) &\le g(q_N - \lambda_N; \mu) &\quad \forall\, q_N \in M_N.
\end{aligned}$$

4. RB ERROR BOUND

As already seen in Section 2, the space-time setting yields a variational inequality which looks similar to an elliptic one with two main differences: (i) Trial and test spaces differ, $X \subsetneq Y$; (ii) The bilinear form is noncoercive.

Still, the idea in [Glas and Urban (2014)] is to adjust the elliptic error bound from [Haasdonk et al. (2012a)]. It turned out that the main structure can be preserved with some crucial modifications. We briefly recall the main ingredients and refer for details to [Glas and Urban (2014)]. In this section, we omit the μ-dependence due to brevity.

4.1 Residuals and Projectors

The following framework is well-known from [Haasdonk et al. (2012a)] and references therein. Denote the *error* of the primal variable by $e_N := u^{\mathcal{N}} - u_N$ and the *residual* $r_N \in X'$ of the *equation* (3a) is defined by

$$r_N(v) := f(v) - a(u_N, v) - b(v, \lambda_N) = a(e_N, v) + b(v, \delta_N),$$

with the *dual error* $\delta_N := \lambda^{\mathcal{N}} - \lambda_N$. The *inequality residual* of (3b) is $s_N \in W'$

$$s_N(q) := b(u_N, q) - g(q), \qquad q \in W.$$

Next, the Riesz representators $\hat{r}_N \in X, \hat{s}_N \in W, \sigma \in W$ of residuals and the inequality functional, respectively, read

$$\begin{aligned}
(v, \hat{r}_N)_X &= r_N(v),\ v \in X,\\
(q, \hat{s}_N)_W &= s_N(q),\ q \in W,\\
(\sigma, q)_W &= b(u, q) - g(q),\ q \in W.
\end{aligned}$$

As one would incorrectly penalize the inequality residual if $s_N(q) < 0$, an additional projection $\pi : W \to M$ is introduced, which is assumed to be orthogonal with respect to some scalar product $\langle \cdot, \cdot \rangle_\pi$ on W. Furthermore, an induced norm is defined $\|\eta\|_\pi := \sqrt{\langle \eta, \eta \rangle_\pi}$, which is assumed to be equivalent to the norm $\|\cdot\|_W$. Finally, π is assumed to have the following properties:

$$\begin{aligned}
(q - \pi(q), \eta)_W &\le 0, &\qquad q \in W, \eta \in M,\\
\pi(\sigma) &= 0,\\
\langle q, \sigma \rangle_\pi &\le 0, &\qquad q \in M.
\end{aligned}$$

4.2 Space-Time-Type Error Bound

Even though $b(\cdot, \cdot)$ is inf-sup-stable on $Y \times W$, we cannot expect a stronger inf-sup stability on $X \times W$ since $X \subsetneq Y$. One possible way-out is the concept of what we call X-*compatibility*, defined in the following:

Definition 4. The convex set K is called X-*compatible* if there exists a linear mapping $D : M \to X$ such that
(1) $b(Dp, q) = (p, q)_W$ for $p, q \in M$;
(2) There exists a $C_D < \infty$, such that $\|Dp\|_X \le C_D \|p\|_W$ for all $p \in M$.

As we will see below, this is a natural assumption, which can easily be realized. We proceed by recalling the primal-dual error relation.

Lemma 5. Let K be X-compatible and let $a : X \times Y \to \mathbb{R}$ be symmetrically bounded. Then, $\|\delta_N\|_W \le C_D(\|r_N\|_{X'} + \gamma_s [\![e_N]\!]_X)$.

Proof. See [Glas and Urban (2014), Lemma 3.11].

The last ingredient, that we need for deriving the error estimation, is the Nečas condition, which is well-known from well-posedness results of operator equations of Petrov-Galerkin type.

Definition 6. We say that the bilinear form $a : X \times Y \to \mathbb{R}$ satisfies a *Nečas condition* on $U \subseteq Y$, if there exists a $\beta_a > 0$ such that

$$\sup_{w \in U} \frac{a(v,w)}{\|w\|_Y} \ge \beta_a \|v\|_X \ \forall v \in X \cap U,$$
$$\sup_{v \in X \cap U} a(v,w) > 0 \ \forall 0 \ne w \in U.$$

Finally, we can deduce the error estimator.

Theorem 7. Let $a : X \times Y \to \mathbb{R}$ be bounded, symmetrically bounded, weakly coercive and satisfy a Nečas condition on $\mathrm{Ker}(B)$ for $X \hookrightarrow Y$ dense. Let $b : Y \times W \to \mathbb{R}$ be bounded and inf-sup stable. If K is X-compatible, the following error/residual estimates hold

$$[\![e_N]\!]_X \le \Delta_N^u := c_1 + (c_1^2 + c_2)^{1/2},$$
$$\|\delta_N\|_W \le \Delta_N^\lambda := C_D(\|r_N\|_{X'} + \gamma_s \Delta_N^u),$$

where

$$c_1 := \frac{1}{2\alpha_{\mathrm{w}}}([\![r_N]\!]_{X'} + \gamma_s C_D \|\pi(\hat{s}_N)\|_W),$$
$$c_2 := \frac{1}{\alpha_{\mathrm{w}}}(C_D \|r_N\|_{X'} \|\pi(\hat{s}_N)\|_W + (p_N, \pi(\hat{s}_N))_W).$$

Proof. See [Glas and Urban (2014), Theorem 3.12].

Remark 8. (a) It should be noted that the error bounds Δ_N^u and Δ_N^λ both tend to zero as $N \to \mathcal{N}$.
(b) The above estimate is independent of the final time T.
(c) Every space-time bilinear form, which satisfies boundedness and a Gårding inequality, automatically fulfills a Nečas condition, see [Schwab and Stevenson (2009)].

5. NUMERICAL RESULTS

We report results of two experiments concerning the space-time error estimator. In a first experiment we focus on the effectivity of the error estimator and how the choice of primal and dual basis functions as well as the shape of the obstacle influence the effectivity. In the second one, we show that we can significantly reduce the runtime using the RBM.

5.1 Model Problem Setting

We consider a parameterized heat inequality model. The time interval is $[0, 0.1]$ ($T := 0.1$) and the domain $\Omega := [0,1]$. The domain Ω is split into two subdomains $\Omega_1 := [0, \frac{1}{2})$ and $\Omega_2 := [\frac{1}{2}, 1]$. The parameter $\mu \in \mathbb{R}^2$ enters the model as a piecewise constant coefficient function in the bilinear form $a(\cdot,\cdot;\mu)$ as heat conductivities μ_1, μ_2 on

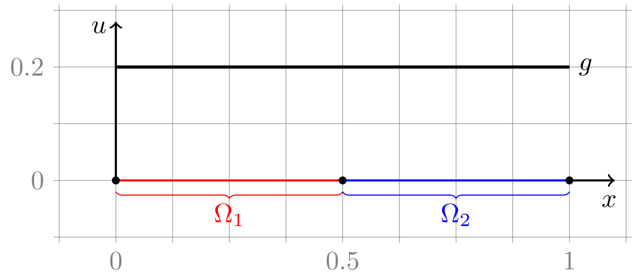

Fig. 1. Initial setting for the example at $t = 0$

Ω_1, Ω_2. We use constant unity Neumann initial conditions at $t = 0$ and homogeneous Dirichlet conditions at $t = T$. Homogeneous boundary conditions are enforced and the obstacle g is a constant. Figure 1 shows the initial setting for $t = 0$.

For the variational formulation we set $V = H_0^1(\Omega)$ (thus $V' = H^{-1}(\Omega)$) and $H = L_2(\Omega)$. We choose $\tilde{W} = V'$ and construct the space-time spaces as in Section 2. The linear and bilinear forms read

$$a(u,v;\mu) = \int_0^T \int_\Omega \langle u_t, v\rangle_{V' \times V} \, dx\, dt$$
$$+ \int_0^T \int_0^{1/2} \mu_1 \nabla u \nabla v \, dx\, dt + \int_0^T \int_{1/2}^1 \mu_2 \nabla u \nabla v \, dx\, dt,$$

$u \in X, v \in Y$, where $\mu := \mu_1 \chi_{[0,\frac{1}{2})} + \mu_2 \chi_{[\frac{1}{2},1]}$,

$$f(v;\mu) = f(v) = \int_0^T \int_\Omega v \, dx\, dt, \ v \in Y,$$
$$b(v,p) = \int_0^T \langle v, p\rangle_{Y \times W} \, dt, \ v \in Y, p \in W$$
$$g(p;\mu) = g(p) = const.$$

For this example, we set the obstacle to be a constant temperature. We obtained similar results also for stationary and (in space) sufficiently smooth obstacles as well.

For the detailed ("truth") discretization we set as in [Urban and Patera (2014)] $X_\delta := S_{\Delta t} \otimes V_h$ and $Y_\delta := Q_{\Delta t} \otimes V_h$ with $\delta = (\Delta t, h)$, where $S_{\Delta t}$ and V_h are spanned by piecewise linear finite elements and $Q_{\Delta t}$ by piecewise constant finite elements. We choose $W = Y'$ and use a dual finite element basis. Therefore, the matrix corresponding to the bilinear form $b(\cdot,\cdot)$ is a multiple of the identity, see also [Haasdonk et al. (2012a)].

For the detailed discretization, we solve a linear complementary problem for u and obtain the dual variable by solving (3a) for λ. However, the online phase amounts solving an ill-conditioned asymmetric variational inequality. Therefore, we use an alternating algorithm from [Hu (2005)], where the problem is rewritten in an alternating form (similar to an augmented Lagrangian) in terms of a convex quadratic programming with simple constrains and a well-conditioned system of nonlinear equations. The arising sub-problems are linear complementary problems and Newton iterations.

Remark 9. (Projection π and operator D). (a) As in this case the cone is stationary, the projection π can be chosen as in [Haasdonk et al. (2012a), §5.2] with $\tilde{W} = V'$, and the extension in [Glas and Urban (2014), Lemma 4.10 ff.].

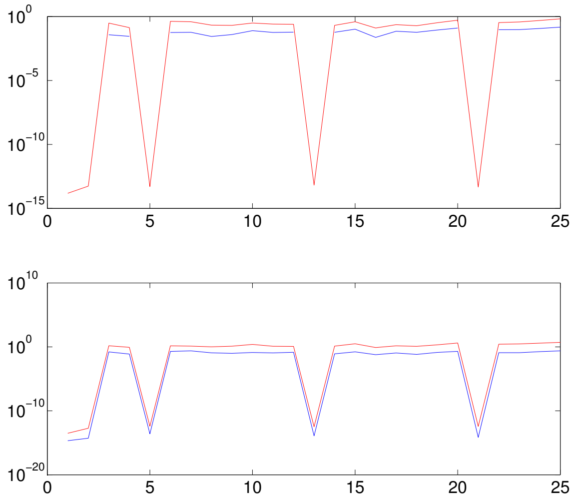

Fig. 2. Error estimator (red line) vs. true error in $[\![\cdot]\!]_X$ (blue line) for all $\mu \in \Xi_{\text{train}}$. Top: primal variable, bottom: dual variable; version (A).

(b) The operator $D : M \to X$ is chosen to be the Riesz operator.

5.2 Rigor and Efficiency of the Error Bound

For the spatial discretization on Ω we use $h := \frac{1}{10}$ and for the time interval we choose $\Delta t := \frac{1}{50}$. This results in a space-time size for the detailed model of $\mathcal{N} = 11 \times 51 = 561$.

The parameter range is chosen as $\mathcal{D} := [1, 5]^2$ and we take a 5×5 grid of snapshots as Ξ_{train}. We set the reduced basis tolerance to $\text{tol}_{\text{RB}} := 10^{-2}$ and determine the reduced basis for the primal variable by the projection based greedy algorithm in Section 3. We obtain $N_X = 5$ and follow option (A) for the dual space, i.e., $N_X = N_W = 5$ and W_{N_W} is formed by the same samples used for the snapshots of the primal basis. In this and the following examples, we set $Y_N := X_N$. The results are shown in Figure 2, where we compare the error estimator with the true error. The plots indicate effectivity of primal and dual error estimators as the curves closely follow the ones for the true error.

For a quantitative comparison, we also report on the minimal and maximal values of the primal effectivity $\text{eff}_u = \Delta_N^u / [\![u - u_N]\!]_X$ and the dual one $\text{eff}_\lambda = \Delta_N^\lambda / \|\lambda - \lambda_N\|_W$ in Table 1. It turns out that the primal effectivities are equally distributed in the interval $[\min_{\text{eff}_u}, \max_{\text{eff}_u}]$. The situation is somewhat different for the dual variable λ, where a closer look shows that the effectivity for 4 out of the $25 = \#\Xi_{\text{train}}$ values exceeds $\frac{1}{2}\max_{\text{eff}_\lambda}$, whereas the others remain small. This also explains the huge difference between $\min_{\text{eff}_\lambda}$ and $\max_{\text{eff}_\lambda}$. On the other hand, a maximal effectivity of about 50 still seems acceptable keeping the nonlinear behavior of the problem due to the obstacle in mind.

5.3 Does the Choice of the Basis Influence Effectivity?

Now we consider the version (B) for the determination of the dual reduced basis, namely performing a separate

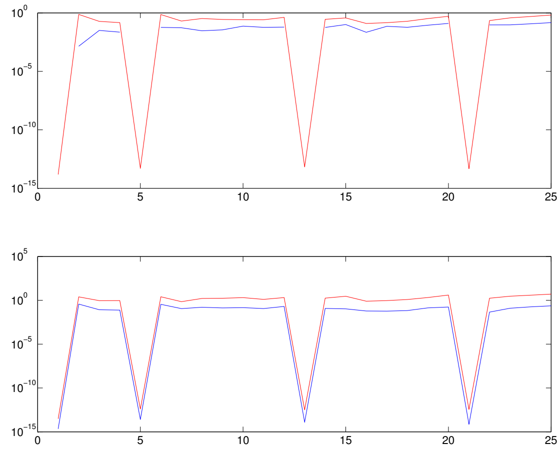

Fig. 3. Error estimator (red line) vs. true error in $[\![\cdot]\!]_X$ (blue line) for $\mu \in \Xi_{\text{train}}$. Top: primal variable, bottom: dual variable; version (B).

Table 1. Minimal and maximal effectivity for all $\mu \in \Xi_{\text{train}}$

Variable	$\min_{\text{eff}}$	$\max_{\text{eff}}$
u	3.2613	8.2256
λ	5.4592	51.0879

greedy algorithm for the dual variable. We use the same primal reduced basis with $N_X = 5$. The greedy for the dual variable is done with the same Ξ_{train} and the same tolerance of 10^{-2} and we obtain a dual basis of cardinality $N_W = 11$.

The results are presented in Figure 3. One might think that the results concerning version (A) in Figure 2 and version (B) in Figure 3 are almost the same. Even though this is true for most of the chosen $\mu \in \Xi_{\text{train}}$, we see a significant difference for the parameter with number 2. This sample is special since μ_2 is chosen by the greedy algorithm for the primal basis but *not* for the dual basis. This causes the fact that the error estimator overestimates the true error by a factor of 526, which is clearly not sufficient. Another reason is that the residual is a part of the error estimator, which involves the error of the dual variable.

Moreover, note that we might loose reproductivity of snapshots in the sense that we might get $u_N(\mu) \neq u^{\mathcal{N}}(\mu)$ even for $\mu \in S_{N_X}$, i.e., parameters that are in the primal sample set (which means that the corresponding snapshots are in X_N). This is due to the use of the alternating algorithm for the case $\mu \notin S_{N_W}$.

5.4 How Does the Obstacle Change Effectivity?

Now, we change the height of the (stationary, constant) obstacle from $g = 0.2$ to $g = 0.35$. All other data is kept. The results are shown in Figure 4.

The greedy algorithm for the primal variable results in a reduced space of dimension $N_X = 8$, which seems reasonable. Motivated by the previous experiments, we choose version (A) for constructing the dual basis, i.e. based upon the same sample parameters. However, a closer look at the dual variable shows that for at least two basis

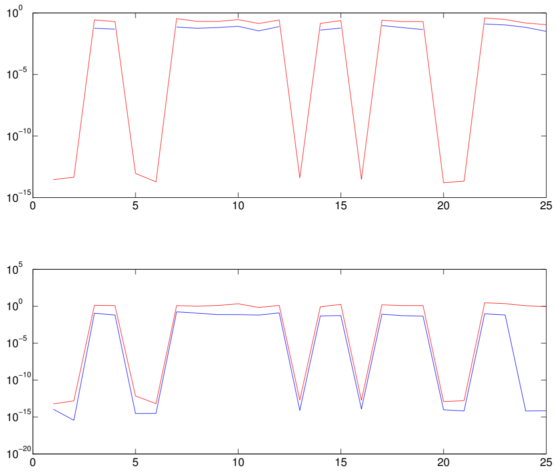

Fig. 4. Error estimator (red line) vs. true error in $[\![\cdot]\!]_X$ (blue line) for $\mu \in \Xi_{\text{train}}$ with $g = 0.35$. Top: primal variable, bottom: dual variable; version (A).

functions the error is by far overestimated. The reason for this behavior is that for these two cases, the solution does not touch the obstacle. Therefore, the exact value of the dual variable is zero, which is not reflected well enough by the estimator.

This is the effect mentioned in the introduction. If the solution happens not to touch an obstacle (which has to be expected in particular for instationary obstacles), the quality of error estimators are expected to be poor. This effect is to be expected to be present also for elliptic problems.

5.5 RB Efficiency

Finally, we investigate the quantitative speedup provided by the RBM. In order to do so, we increase the size of the detailed problem by setting $h = \frac{1}{20}$, $\Delta t = \frac{1}{100}$ resulting in a "truth" space-time size of $\mathcal{N} = 21 \times 101 = 2121$. The dual reduced basis is determined by version (A).

We performed CPU-time comparisons for a whole variety of parameters. In Table 2, we report timings for three particular examples, namely $\mu \in \{(4,3),(4,1),(2.5,3.5)\}$. The first one is a sample chosen by the greedy, i.e., in S_N, the second is in Ξ_{train}, but not in S_N and the third one is not in the training set.

Table 2. CPU time in seconds

μ	$\mu \in$	Offline	Online	Error Estimator
$(4,3)$	S_N	66.62	0.0015	0.7132
$(4,1)$	$\Xi_{\text{train}} \setminus S_N$	66.79	1.6822	0.7472
$(2.5,3.5)$	$\mathcal{D} \setminus \Xi_{\text{train}}$	66.25	2.3233	0.7163

The offline CPU time is about 67 seconds in all three cases. The online timings depend on the choice of μ, in particular for a chosen sample, which indicates that the online algorithm detects such "simple" cases. The timings for the other two cases are almost the same, which seems reasonable because these are non-sample parameters. We get a speedup of at least a factor of 30.

6. CONCLUSION

In this paper, we present numerical investigations corresponding to a novel error estimator for parabolic (i.e., noncoercive) variational inequalities based upon the space-time variational formulation.

The experiments indicate rigor and effectivity of the error bound and show that the bound in general reflects the behavior of the true error. Depending on the particular construction of the (dual) basis and the shape of the obstacle, we observe deficiencies in specific parameter constellations. In all such cases, we give a corresponding explanation which we think is also valuable for the understanding of such noncoercive variational inequalities in general, in particular also for other kind of problems (not only for the considered parameterized heat inequality).

Moreover, our experiments indicate a speedup of at least a factor of 30 by the RBM. This is promising, in particular in view of the fact that the online timings are not yet optimal due to the use an alternating algorithm in the online phase. Future work will be devoted to analysis for suitable stable test spaces, a speedup of the online phase as well as to a refinement of the error estimator, in particular on efficient offline/online-decompositions, which will speedup the basis generation enormously.

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