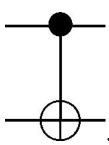
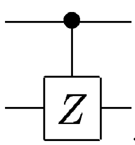
Übungsblatt 1 - QIV I

The Pauli matrices are $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, and $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. The identity matrix is $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. $\otimes$ denotes a tensor product. The controlled-NOT gate performs the transform $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ and has the circuit symbol . The controlled-Z gate performs the transform $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ and has the circuit symbol 

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Problems: Quantum Mechanics

1. **Complex numbers.** Give numerical values for the following expressions: (a) $\sqrt{i}$, (b) $\exp(-i\pi/4)$, (c) $\exp(i\pi)$.

2. **Eigenvalues and eigenvectors.** What are the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}?$$

3. **Commutators.** What is $[X, Y]$? $[X \otimes I, Y \otimes Z]$?

4. **Matrix exponentials.** Express $\exp[i\pi X/2]$ as 2×2 matrix with numerical entries.

5. **Simple measurement.** A qubit is in the state

$$\frac{|0\rangle + |1\rangle}{\sqrt{2}}.$$

When this is measured (in the usual computational basis, $|0\rangle$ and $|1\rangle$), what is the probability of obtaining 1?

6. **Post-measurement states.** A two-qubit system is in the state

$$\frac{3|00\rangle + 4|11\rangle}{5}\,.$$

The first qubit is measured. What is the probability that the second qubit will end up in the state $|0\rangle$?

7. **Measurement observables.** What is the expectation $\langle\psi|O|\psi\rangle$ of the observable $O = X\cos\theta + I\sin\theta$ when the state is $|\psi\rangle = |0\rangle$?

8. **Schrödinger's equation.** Let H be a time-independent Hamiltonian. The solution to Schrödinger's equation

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle$$

can be written as

$$|\psi(t)\rangle = U|\psi(0)\rangle\,.$$

Give an expression for U in terms of H.

9. **Density matrix.** The state of a qubit (two level quantum system) can be written as $|\psi\rangle = a|0\rangle + b|1\rangle$, where $|a|^2 + |b|^2 = 1$. The density operator representation of this state is, symbolically, $\rho = |\psi\rangle\langle\psi|$. Give this as a 2×2 matrix.

10. **Subsystem density matrices.** The density matrix

$$\rho = \frac{1}{4}\begin{bmatrix} 1 & 1 & i & i \\ 1 & 1 & i & i \\ -i & -i & 1 & 1 \\ -i & -i & 1 & 1 \end{bmatrix}$$

describes the joint state of two qubits A and B, and $\rho = \rho_A \otimes \rho_B$. Give the 2×2 density matrices ρ_A and ρ_B.

11. **Tensor products of spinors.** A qubit $a|0\rangle + b|1\rangle$ can be written as a two-component vector,

$$\begin{bmatrix} a \\ b \end{bmatrix}.$$

Suppose we have two qubits, in the states $|\psi_1\rangle = a_1|0\rangle + b_1|1\rangle$ and $|\psi_2\rangle = a_2|0\rangle + b_2|1\rangle$. Write out the state of the joint system, $|\psi_1\rangle \otimes |\psi_2\rangle$ as a four component vector.

12. **Tensor products of Pauli matrices.** Give $X \otimes Y$ as a 4×4 matrix.

13. **Greenberger-Horne-Zeilinger.** Let $|\psi\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$ be the state of three qubits, and let X, Y, and Z be the Pauli matrices. Give numerical values for the expectation value $\langle\psi|O|\psi\rangle$, where: (a) $O = X \otimes X \otimes X$ (sometimes this is also written as $X_1X_2X_3$, or $\sigma_x^1\sigma_x^2\sigma_x^3$) (b) $O = X \otimes Y \otimes Y$, (c) $O = X \otimes X \otimes Y$.

14. **Rotation operators.** Let $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$ be the state of a qubit. Give the state

$$|\psi'\rangle = e^{-i\alpha Z/2}|\psi\rangle\,.$$

15. **Purification.** Exhibit a pure state $|\psi\rangle_{AB}$ for two qubits such that the reduced density matrix for the first qubit is $\rho_A = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|$.

16. **Hadamard gates.** A Hadamard gate H performs the unitary transform

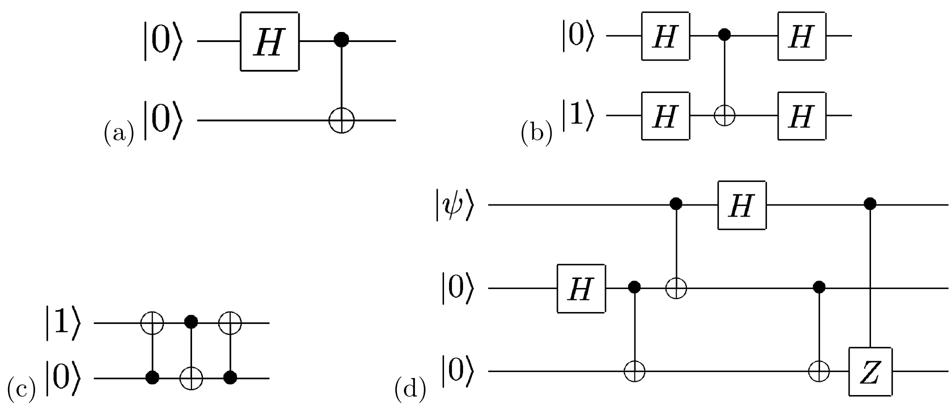
$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

on a single qubit. Draw a quantum circuit using H to create the state

$$\frac{1}{\sqrt{8}}\sum_{x\in\{0,1\}^3}|x\rangle = \frac{|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle}{\sqrt{8}}$$

from three qubits, which are initially each in the state $|0\rangle$.

17. **Simple quantum circuits.** Give the output states of the following quantum circuits:



18. **SU(2) rotations.** Simplify $e^{i(n_x X + n_y Y + n_z Z)}$ and express as a weighted sum of I, X, Y, and Z, using $n = \sqrt{n_x^2 + n_y^2 + n_z^2}$ and $\hat{n} = (n_x, n_y, n_z)/n$.

21. **Repetition code.** Consider the binary symmetric channel of the problem above, and show that by repeating each symbol three times, the probability of error can be reduced from $O(p)$ to $O(p^2)$. Give the lowest probability of error achievable using this code, for this channel.

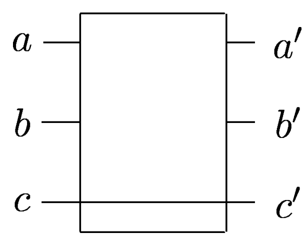
22. **Data compression.** A source emits three symbols, 0, 1, and 2 with probabilities 1/4, 1/4, and 1/2. On average, how few bits per symbol are required to faithfully represent messages from this source?

Problems: Computer Science

23. **Run times: search problem.** Here's a game: hidden behind one of four doors is a prize. The location is random and different each time you play. On average, how many doors do you have to open before you know where the prize is located?

24. **Fredkin and Boolean logic gates.** The Fredkin gate has the following truth table and circuit symbol:

Inputs			Outputs		
a	b	c	a'	b'	c'
0	0	0	0	0	0
0	0	1	0	0	1
0	1	0	0	1	0
0	1	1	1	0	1
1	0	0	1	0	0
1	0	1	0	1	1
1	1	0	1	1	0
1	1	1	1	1	1



Construct AND, NOT, and OR gates using circuits with just Fredkin gates.

25. **Universality.** The NAND gate takes two bits x and y and outputs $1 - xy$. Similarly, the XOR gate outputs $x \oplus y$, where $\oplus$ denotes addition modulo two. Construct an XOR gate using just NAND gates. Also show that a NAND gate *cannot* be constructed from just XOR gates.

26. **Boosting and the Chernoff bound.** An important property of random algorithms is that if their probability of success and failure are bounded away from 1/2, then repetition of the algorithm can quickly increase the certainty of the result. Specifically, suppose $X_1, \ldots, X_n$ are independent and identically distributed random variables, each taking the value 1 with probability $1/2 + \epsilon$, and the value 0 with probability $1/2 - \epsilon$. Prove that

$$\text{prob}\left(\sum_{i=1}^{n} X_i \leq n/2\right) \leq e^{-2\epsilon^2 n}\,.$$