Übungsblatt

Aufgabe 1: Calculate the following in terms of I_2, X, Y, Z

(i) XZ
(ii) ZX
(iii) $U_{CNOT}(X \otimes I_2)U_{CNOT}$
(iv) $U_{CNOT}(I_2 \otimes X)U_{CNOT}$
(v) $U_{CNOT}(Z \otimes I_2)U_{CNOT}$
(vi) $U_{CNOT}(I_2 \otimes Z)U_{CNOT}$
(vii) $U_{CNOT}(X \otimes X)U_{CNOT}$
(viii) $U_{CNOT}(Z \otimes Z)U_{CNOT}$
(ix) $U_{CNOT}U_{CNOT}$

where

$$
\begin{aligned}
I_2 &:= |0\rangle\langle 0| + |1\rangle\langle 1| \\
X &:= |0\rangle\langle 1| + |1\rangle\langle 0| \\
Y &:= |0\rangle\langle 1| - |1\rangle\langle 0| \\
Z &:= |0\rangle\langle 0| - |1\rangle\langle 1| \\
U_{CNOT} &:= |0\rangle\langle 0| \otimes I_2 + |1\rangle\langle 1| \otimes X\,.
\end{aligned}
$$

Aufgabe 2

(i) Let $A := |0\rangle\langle 0| - |1\rangle\langle 1|$ in the Hilbert space $\mathbf{C}^2$. Calculate

$$U_H A U_H |0\rangle, \qquad U_H A U_H |1\rangle$$

where U_H is the *Walsh-Hadamard transform.* The unitary transform U_H is defined by

$$U_H|k\rangle = \frac{1}{\sqrt{2}}(|0\rangle + (-1)^k|1\rangle), \qquad k \in \{0,1\}.$$

(ii) Calculate

$$(U_H \otimes U_H)U_{CNOT}(U_H \otimes U_H)|j,k\rangle$$

where $|j,k\rangle \equiv |j\rangle \otimes |k\rangle$ with $j,k \in \{0,1\}$, and the answer is in the form of a ket $|m,n\rangle$ with $m,n \in \{0,1\}$. The unitary transform

$$U_{CNOT} := |0\rangle\langle 0| \otimes I_2 + |1\rangle\langle 1| \otimes U_{NOT}$$

is the *controlled NOT* operation and the unitary transform

$$U_{NOT} = |0\rangle\langle 1| + |1\rangle\langle 0|$$

is the *NOT* operation.

Aufgabe 3

Problem 2. Consider the linear operator

$$H := i\hbar\omega(|0\rangle\langle 1| - |1\rangle\langle 0|)$$

operating in the Hilbert space $\mathbf{C}^2$, where $\{ |0\rangle, |1\rangle \}$ is an orthonormal basis in $\mathbf{C}^2$ and ω is a real parameter (frequency).

(i) Is H self-adjoint?

(ii) Find the eigenvalues and corresponding normalized eigenvectors of H.

(iii) Find the unitary matrix

$$U(t) := \exp(-iHt/\hbar).$$

Find the values of t such that $U(t)$ performs the NOT operation

$$U(t)|0\rangle \to |1\rangle, \qquad U(t)|1\rangle \to |0\rangle .$$

(iv) Calculate $U(t = \pi/4\omega)$ and $(U(t = \pi/4\omega))^2$.

Aufgabe 4

Let σ_x, σ_y, σ_z be the *Pauli spin matrices*

$$\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(i) Find

$$R_{1x}(\alpha) := \exp(-i\alpha(\sigma_x \otimes I_2)), \qquad R_{1y}(\alpha) := \exp(-i\alpha(\sigma_y \otimes I_2))$$

where $\alpha \in \mathbf{R}$ and I_2 denotes the 2×2 unit matrix.

(ii) Consider the special case $R_{1x}(\alpha = \pi/2)$ and $R_{1y}(\alpha = \pi/4)$. Calculate $R_{1x}(\pi/2)R_{1y}(\pi/4)$. Discuss.

Aufgabe 5

4. Consider the state in the Hilbert space $\mathcal{H} = \mathbf{C}^{16}$

$$|\psi_0\rangle = |0101\rangle \equiv |0\rangle \otimes |1\rangle \otimes |0\rangle \otimes |1\rangle$$

where $\{ |0\rangle, |1\rangle \}$ is the standard basis in $\mathbf{C}^2$. Let

$$|\psi_1\rangle = B|\psi_0\rangle = \frac{1}{\sqrt{2}}(|0101\rangle + |0110\rangle)$$

$$|\psi_2\rangle = U|\psi_1\rangle = \frac{1}{\sqrt{2}}(|0101\rangle + |1010\rangle)$$

$$|\psi_3\rangle = S|\psi_2\rangle = \frac{1}{\sqrt{2}}(|0101\rangle - |1010\rangle)$$

$$|\psi_4\rangle = U^*|\psi_3\rangle = \frac{1}{\sqrt{2}}(|0101\rangle - |0110\rangle)$$

$$|\psi_5\rangle = B^*|\psi_4\rangle = -|0110\rangle .$$

Find the 16×16 unitary matrices B, U, S which perform these transformations.

Aufgabe 6

The *Fredkin gate* is the unitary operator acting as

$$U_F|c, x, y\rangle = |c, cx + \bar{c}y, \bar{c}x + cy\rangle$$

in the Hilbert space $\mathbf{C}^8$, where $c, x, y \in \{ 0, 1 \}$.
(i) Consider the cases $c = 0$ and $c = 1$.
(ii) Find the matrix representation for the standard basis.

Aufgabe 7

Consider the 8×8 matrix

$$U(\alpha) = \frac{e^{i\alpha}}{\sqrt{2}}(I_2 \otimes I_2 \otimes I_2 + i\sigma_x \otimes \sigma_x \otimes \sigma_x)$$

where $\alpha \in \mathbf{R}$.
(i) Show that U is unitary.
(ii) Let

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Consider the state

$$|\psi\rangle = |\downarrow\rangle \otimes |\downarrow\rangle \otimes |\downarrow\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Calculate $U|\psi\rangle$.
(iii) Consider $U(\alpha = 0)$ and the unitary 8×8 diagonal matrix

$$V = \text{diag}(e^{i3\phi/2}, 1, 1, 1, 1, 1, 1, e^{-i3\phi/2}).$$

Calculate $VU(\alpha = 0)|\psi\rangle$.
(iv) Calculate $U(\alpha = 0)VU(\alpha = 0)|\psi\rangle$.
(v) Let

$$|\xi_1\rangle = |\downarrow\rangle \otimes |\downarrow\rangle \otimes |\downarrow\rangle, \qquad |\xi_2\rangle = |\uparrow\rangle \otimes |\uparrow\rangle \otimes |\uparrow\rangle.$$

Calculate the probabilties

$$|\langle\xi_1|U(\alpha = 0)VU(\alpha = 0)|\psi\rangle|^2, \qquad |\langle\xi_2|U(\alpha = 0)VU(\alpha = 0)|\psi\rangle|^2.$$

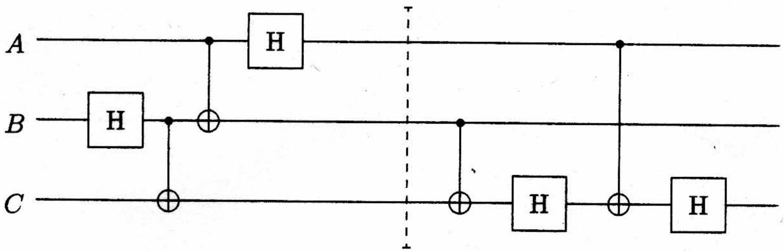
Draw $|\langle\xi_1|U(\alpha = 0)VU(\alpha = 0)|\psi\rangle|^2$ and $|\langle\xi_2|U(\alpha = 0)VU(\alpha = 0)|\psi\rangle|^2$ as functions of ϕ.

Aufgabe 8

In quantum teleportation we start with the following state in the Hilbert space $\mathbf{C}^8$

$$|\psi\rangle \otimes |0\rangle \otimes |0\rangle \equiv (a|0\rangle + b|1\rangle) \otimes |0\rangle \otimes |0\rangle \equiv |\psi 00\rangle$$

where $|a|^2 + |b|^2 = 1$. The quantum circuit for teleportation is given by



where A is the input $|\psi\rangle$, B the input $|0\rangle$ and C the input $|0\rangle$. Study what happens when we feed t Aufgabe 8 e $|\psi 00\rangle$ into the quantum circuit. From the circuit we have the following eight 8×8 unitary matrices (left to right)

$$U_1 = I_2 \otimes U_H \otimes I_2, \quad U_2 = I_2 \otimes U_{XOR},$$
$$U_3 = U_{XOR} \otimes I_2, \quad U_4 = U_H \otimes I_2 \otimes I_2,$$
$$U_5 = I_2 \otimes U_{XOR}, \quad U_6 = I_2 \otimes I_2 \otimes U_H,$$
$$U_7 = I_4 \oplus U_{NOT} \oplus U_{NOT}, \quad U_8 = I_2 \otimes I_2 \otimes U_H$$

where $\oplus$ denotes the direct sum of matrices, U_H denotes the Hadamard gate, U_{XOR} denotes the XOR-gate and

$$U_{NOT} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(i) Find $U_8 U_7 U_6 U_5 U_4 U_3 U_2 U_1 |\psi 00\rangle$.