Problem 1. Let $A := (a_{ij})_{ij}$ be an $m \times n$ matrix and B be an $r \times s$ matrix. The *Kronecker product* of A and B is defined as the $(m \cdot r) \times (n \cdot s)$ matrix

$$A \otimes B := \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{pmatrix}$$

(i) Let

$$|\phi_1\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |\phi_2\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Thus $\{ |\phi_1\rangle, |\phi_2\rangle \}$ forms a basis in $\mathbf{C}^2$ (the standard basis). Calculate

$$|\phi_1\rangle \otimes |\phi_1\rangle, \quad |\phi_1\rangle \otimes |\phi_2\rangle, \quad |\phi_2\rangle \otimes |\phi_1\rangle, \quad |\phi_2\rangle \otimes |\phi_2\rangle$$

and interpret the result.

(ii) Consider the Pauli matrices

$$\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Find $\sigma_x \otimes \sigma_z$ and $\sigma_z \otimes \sigma_x$ and discuss.

Problem 2. Given the orthonormal basis

$$|\psi_1\rangle = \begin{pmatrix} e^{i\phi} \cos\theta \\ \sin\theta \end{pmatrix}, \qquad |\psi_2\rangle = \begin{pmatrix} -\sin\theta \\ e^{-i\phi} \cos\theta \end{pmatrix}$$

in the Hilbert space $\mathbf{C}^2$. Use this basis to find a basis in $\mathbf{C}^4$.

Problem 3. A system of n-qubits represents a finite-dimensional Hilbert space over the complex numbers of dimension 2^n. A state $|\psi\rangle$ of the system is a superposition of the basic states

$$|\psi\rangle = \sum_{j_1, j_2, \dots, j_n = 0}^{1} c_{j_1 j_2 \dots j_n} |j_1\rangle \otimes |j_2\rangle \otimes \dots \otimes |j_n\rangle.$$

In a short cut notation this state is written as

$$|\psi\rangle = \sum_{j_1, j_2, \dots, j_n = 0}^{1} c_{j_1 j_2 \dots j_n} |j_1 j_2 \dots j_n\rangle.$$

Consider as a special case the state in the Hilbert space $\mathcal{H} = \mathbf{C}^4$ $(n = 2)$

$$|\psi\rangle = \frac{1}{2}(|0\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle) \equiv \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle).$$

Can this state be written as a product state?

Problem 4 Consider the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Consider the *GHZ-state*

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right).$$

(i) Show that $|\psi\rangle$ is an eigenvector of the operator $\sigma_y \otimes \sigma_y \otimes \sigma_x$. What is the eigenvalue?
(ii) Is

$$|\phi\rangle = \frac{1}{\sqrt{2}} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{i\phi} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

an eigenvector of $\sigma_y \otimes \sigma_y \otimes \sigma_x$?

Problem 5 Consider the three-qubit *GHZ-state*

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|0\rangle \otimes |0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle \otimes |1\rangle)$$

with the standard basis

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Let σ_x, σ_y be the Pauli spin matrices.
(i) Calculate

$$\langle\psi|(\sigma_y \otimes \sigma_y \otimes \sigma_y)|\psi\rangle, \quad \langle\psi|(\sigma_x \otimes \sigma_x \otimes \sigma_y)|\psi\rangle,$$
$$\langle\psi|(\sigma_x \otimes \sigma_y \otimes \sigma_x)|\psi\rangle, \quad \langle\psi|(\sigma_y \otimes \sigma_x \otimes \sigma_x)|\psi\rangle.$$

(ii) Calculate

$$\langle\psi|(\sigma_x \otimes \sigma_y \otimes \sigma_y)|\psi\rangle, \quad \langle\psi|(\sigma_y \otimes \sigma_x \otimes \sigma_y)|\psi\rangle,$$
$$\langle\psi|(\sigma_y \otimes \sigma_y \otimes \sigma_x)|\psi\rangle, \quad \langle\psi|(\sigma_x \otimes \sigma_x \otimes \sigma_x)|\psi\rangle.$$

Problem 6 Consider the states

$$|\psi\rangle = \frac{1}{\sqrt{3}}|00\rangle + \sqrt{\frac{2}{3}}|11\rangle \equiv \frac{1}{\sqrt{3}}|0\rangle \otimes |0\rangle + \sqrt{\frac{2}{3}}|1\rangle \otimes |1\rangle$$

and

$$|\phi\rangle = |11\rangle \equiv |1\rangle \otimes |1\rangle .$$

Find $p := |\langle\phi|\psi\rangle|^2$, i.e., the probability of finding $|\psi\rangle$ in the state $|\phi\rangle$.

Problem 7 Consider the states

$$|0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

in the Hilbert space $\mathbf{C}^2$ and the Bell state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle)$$

in the Hilbert space $\mathbf{C}^4$. Let $(\alpha, \beta \in \mathbf{R})$

$$|\alpha\rangle := \cos\alpha|0\rangle + \sin\alpha|1\rangle, \qquad |\beta\rangle := \cos\beta|0\rangle + \sin\beta|1\rangle$$

be states in $\mathbf{C}^2$. Find the probability

$$p(\alpha, \beta) := |(\langle\alpha| \otimes \langle\beta|)|\psi\rangle|^2 .$$

Discuss p as a function of α and β.

Problem 8 Consider the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \equiv \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle)$$

and $\langle 0| \otimes I_2$, where I_2 is the 2×2 unit matrix. Find

$$(\langle 0| \otimes I_2)|\psi\rangle .$$

Discuss.