Übungsaufgaben

1) **(Criterion to decide if a state is mixed or pure)** Let ρ be a density operator. Show that $\mathrm{tr}(\rho^2) \leq 1$, with equality if and only if ρ is a pure state.

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2) (i) The *Hilbert-Schmidt distance* between any two density operators ρ_1 and ρ_2 is given by the Frobenius-Hilbert-Schmidt norm of their differences

$$D_{HS}(\rho_1, \rho_2) := \sqrt{\mathrm{tr}((\rho_1 - \rho_2)^2)}\,.$$

Let

$$\rho_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \rho_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Calculate $D_{HS}(\rho_1, \rho_2)$.

(ii) The *Bures distance* in the space of mixed quantum states described by the density matrices ρ_1 and ρ_2 is defined as

$$D_B(\rho_1, \rho_2) := \sqrt{2(1 - \mathrm{tr}((\rho_1^{1/2} \rho_2 \rho_1^{1/2})^{1/2}))}\,.$$

Let

$$\rho_1 = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}, \qquad \rho_2 = \begin{pmatrix} 3/4 & 0 \\ 0 & 1/4 \end{pmatrix}.$$

Calculate $D_B(\rho_1, \rho_2)$.

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3) Given the *Schrödinger equation*

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = \hat{H} |\psi\rangle\,.$$

Find the time-evolution of the density matrix

$$\rho(t) := \sum_{j=1}^{n} |\psi^{(j)}(t)\rangle\langle\psi^{(j)}(t)|\,.$$

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4) Consider the 2×2 matrix

$$A = \begin{pmatrix} -1 & 3 \\ 3 & 2 \end{pmatrix}.$$

Calculate the trace of A with respect to the standard basis. Calculate the trace of A with respect to the Hadamard basis

$$\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}.$$

5) Consider the 4×4 matrix (*density matrix*)

$$|\mathbf{u}\rangle\langle\mathbf{u}| = \begin{pmatrix} u_1\bar{u}_1 & u_1\bar{u}_2 & u_1\bar{u}_3 & u_1\bar{u}_4 \\ u_2\bar{u}_1 & u_2\bar{u}_2 & u_2\bar{u}_3 & u_2\bar{u}_4 \\ u_3\bar{u}_1 & u_3\bar{u}_2 & u_3\bar{u}_3 & u_3\bar{u}_4 \\ u_4\bar{u}_1 & u_4\bar{u}_2 & u_4\bar{u}_3 & u_4\bar{u}_4 \end{pmatrix}$$

in the product Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B \equiv \mathbf{C}^4$, where $\mathcal{H}_A = \mathcal{H}_B = \mathbf{C}^2$.
(i) Calculate

$$\mathrm{tr}_A(|\mathbf{u}\rangle\langle\mathbf{u}|)$$

where the basis is given by

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes I_2, \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes I_2$$

and I_2 denotes the 2×2 unit matrix.
(ii) Find

$$\mathrm{tr}_B(|\mathbf{u}\rangle\langle\mathbf{u}|)$$

where the basis is given by

$$I_2 \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad I_2 \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$