# Übungsblatt x

**Aufgabe 1** Consider the Hilbert space $\mathbf{C}^2 \otimes \mathbf{C}^2$ and the unitary $2 \times 2$ matrix

$$U(\theta, \phi) := \begin{pmatrix} \cos\frac{\theta}{2} & e^{-i\phi}\sin\frac{\theta}{2} \\ -e^{i\phi}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}.$$

Which of the following states are entangled?

$$\text{(i)} \quad \Big(U(\theta_1, \phi_1) \otimes U(\theta_2, \phi_2)\Big)(1, 0, 0, 0)^T$$

$$\text{(ii)} \quad \Big(U(\theta_1, \phi_1) \otimes U(\theta_2, \phi_2)\Big)(0, 0, 0, 1)^T$$

$$\text{(iii)} \quad \Big(U(\theta_1, \phi_1) \otimes U(\theta_2, \phi_2)\Big)\frac{1}{\sqrt{2}}(1, 0, 0, 1)^T$$

where $\otimes$ denotes the Kronecker product and $^T$ denotes the transpose.

**Aufgabe 2** We consider the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, where

$$\mathcal{H}_A = \mathcal{H}_B = \mathbf{C}^2.$$

(i) Consider the state

$$|\psi\rangle := \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}.$$

Calculate

$$\rho_A := \mathrm{tr}_{\mathcal{H}_B}(|\psi\rangle\langle\psi|), \qquad \rho_B := \mathrm{tr}_{\mathcal{H}_A}(|\psi\rangle\langle\psi|)$$

and

$$-\mathrm{tr}(\rho_A \log_2 \rho_A), \qquad -\mathrm{tr}(\rho_B \log_2 \rho_B)$$

where $-\mathrm{tr}(\rho_A \log_2 \rho_A)$ denotes the *von Neumann entropy*.

(ii) Consider the state

$$|\psi\rangle := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

Calculate

$$\rho_A := \mathrm{tr}_{\mathcal{H}_B}(|\psi\rangle\langle\psi|)$$

and

$$-\mathrm{tr}(\rho_A \log_2 \rho_A).$$

(iii) Consider the state

$$|\psi\rangle := \frac{1}{2}(U_1 \otimes U_2)\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

where $U_1$ and $U_2$ are unitary matrices acting on $\mathbf{C}^2$. Calculate

$$\rho_A := \mathrm{tr}_{\mathcal{H}_B}(|\psi\rangle\langle\psi|), \qquad -\mathrm{tr}(\rho_A \log_2 \rho_A).$$

(iv) Consider the state

$$|\psi\rangle := \frac{1}{\sqrt{2}}(U_1 \otimes U_2)\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

where $U_1$ and $U_2$ are unitary matrices acting on $\mathbf{C}^2$. Calculate

$$\rho_A := \mathrm{tr}_{\mathcal{H}_B}(|\psi\rangle\langle\psi|)$$

and

$$-\mathrm{tr}(\rho_A \log_2 \rho_A).$$

## Aufgabe 3

Consider the *Greenberger-Horne-Zeilinger state*

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle \otimes |1\rangle)$$

and the *W-state*

$$|W\rangle = \frac{1}{\sqrt{3}}(|0\rangle \otimes |0\rangle \otimes |1\rangle + |0\rangle \otimes |1\rangle \otimes |0\rangle + |1\rangle \otimes |0\rangle \otimes |0\rangle).$$

(i) Calculate

$$(\sqrt{2}\langle 0| \otimes I_2 \otimes I_2)|GHZ\rangle, \quad (\sqrt{2}I_2 \otimes \langle 0| \otimes I_2)|GHZ\rangle$$

$$(\sqrt{2}I_2 \otimes I_2 \otimes \langle 0|)|GHZ\rangle, \quad (\sqrt{2}\langle 1| \otimes I_2 \otimes I_2)|GHZ\rangle$$

$$(\sqrt{2}I_2 \otimes \langle 1| \otimes I_2)|GHZ\rangle, \quad (\sqrt{2}I_2 \otimes I_2 \otimes \langle 1|)|GHZ\rangle.$$

and discuss.

(iii) Calculate

$$\left(\frac{\sqrt{3}}{\sqrt{2}}\langle 0| \otimes I_2 \otimes I_2\right)|W\rangle, \quad \left(\frac{\sqrt{3}}{\sqrt{2}}I_2 \otimes \langle 0| \otimes I_2\right)|W\rangle,$$

$$\left(\frac{\sqrt{3}}{\sqrt{2}}I_2 \otimes I_2 \otimes \langle 0|\right)|W\rangle, \quad (\sqrt{3}\langle 1| \otimes I_2 \otimes I_2)|W\rangle,$$

$$(\sqrt{3}I_2 \otimes \langle 1| \otimes I_2)|W\rangle, \quad (\sqrt{3}I_2 \otimes I_2 \otimes \langle 1|)|W\rangle$$

and discuss.

## Aufgabe 4

Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be finite-dimensional Hilbert spaces. Let $\mathcal{H}$ be the Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, i.e., $\mathcal{H}$ is the tensor product of the two Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. Let $|\psi\rangle$ be a normalized vector (pure state) in $\mathcal{H}$. Let $X$ be an observable (described as a hermitian matrix $\hat{X}$) in $\mathcal{H}$. Then $\langle\psi|\hat{X}|\psi\rangle$ defines the expectation values. The following three conditions are equivalent when applied to pure states.

1. *Factorisability*: $|\psi\rangle = |\alpha\rangle \otimes |\beta\rangle$, where $|\alpha\rangle \in \mathcal{H}_A$ and $|\beta\rangle \in \mathcal{H}_B$ with $|\alpha\rangle$ and $|\beta\rangle$ normalized.

2. The generalized *Bell inequality*: Let $\hat{A}_1$, $\hat{A}_2$ be hermitian operators (matrices) in $\mathcal{H}_A$ with

$$\hat{A}_1^2 = I_A, \qquad \hat{A}_2^2 = I_A$$

where $I_A$ is the identity operator in $\mathcal{H}_A$. Let $\hat{B}_1$, $\hat{B}_2$ be hermitian operators (matrices) in $\mathcal{H}_B$ with

$$\hat{B}_1^2 = I_B, \qquad \hat{B}_2^2 = I_B$$

where $I_B$ is the identity operator in $\mathcal{H}_B$. Thus the eigenvalues of $\hat{A}_1$, $\hat{A}_2$, $\hat{B}_1$ and $\hat{B}_2$ can only be $\pm 1$. The generalized Bell inequality is

$$|\langle\psi|\hat{A}_1 \otimes \hat{B}_1|\psi\rangle + \langle\psi|\hat{A}_1 \otimes \hat{B}_2|\psi\rangle + \langle\psi|\hat{A}_2 \otimes \hat{B}_1|\psi\rangle - \langle\psi|\hat{A}_2 \otimes \hat{B}_2|\psi\rangle| \leq 2.$$

3. *Statistical independence:* For all hermitian operators $\hat{A}$ on $\mathcal{H}_A$ and $\hat{B}$ on $\mathcal{H}_B$ with the conditions given above

$$\langle\psi|\hat{A} \otimes \hat{B}|\psi\rangle = \langle\psi|\hat{A} \otimes I_B|\psi\rangle\langle\psi|I_A \otimes \hat{B}|\psi\rangle.$$

(i) Show that condition 3 follows from condition 1.
(ii) Show that condition 2 follows from condition 3.

Aufgabe 5 Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be finite-dimensional Hilbert spaces. Let $\mathcal{H}$ be the Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, i.e., $\mathcal{H}$ is the tensor product of the two Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$.

Let $\hat{A}_1$, $\hat{A}_2$ be hermitian operators (matrices) in $\mathcal{H}_A$ with

$$\hat{A}_1^2 = I_A, \qquad \hat{A}_2^2 = I_A .$$

Let $\hat{B}_1$, $\hat{B}_2$ be hermitian operators (matrices) in $\mathcal{H}_B$ with

$$\hat{B}_1^2 = I_B, \qquad \hat{B}_2^2 = I_B .$$

The generalized *Bell inequality* is given by

$$|\langle\psi|\hat{A}_1 \otimes \hat{B}_1|\psi\rangle + \langle\psi|\hat{A}_1 \otimes \hat{B}_2|\psi\rangle + \langle\psi|\hat{A}_2 \otimes \hat{B}_1|\psi\rangle - \langle\psi|\hat{A}_2 \otimes \hat{B}_2|\psi\rangle| \leq 2 .$$

Let $\mathcal{H}_A = \mathcal{H}_B = \mathbf{C}^2$. Let $\{\, |0\rangle\, , |1\rangle \,\}$ be the standard basis in $\mathbf{C}^2$. Consider the entangled state in $\mathcal{H} = \mathbf{C}^4$ (*EPR-state*)

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle) .$$

Show that this state and the operators

$$\hat{A}_1 := \sigma_x, \qquad \hat{A}_2 := \sigma_y$$

$$\hat{B}_1 := \frac{1}{\sqrt{2}}(\sigma_x + \sigma_y), \qquad \hat{B}_2 := \frac{1}{\sqrt{2}}(\sigma_x - \sigma_y)$$

violate the Bell inequality.