Das Strahlungsfeld
des Freie-Elektronen-Lasers
im Quantenlimit

Diplomarbeit an der Universität Ulm

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The Radiation Field of the Free-Electron Laser in the Quantum Limit

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1 INTRODUCTION

1.1 Motivation

Free-electron lasers (in the following: FELs) are a class of coherent light sources in focus of today’s research. Many areas of science benefit from their unique properties, such as coherence, widely ranged tunability, and high brilliance, i.e. high intensity in a narrow frequency spectrum and a highly focused radiation cone. The fields of application are semiconductor physics, material science, femtochemistry, biophysics, medical surgeries and much more [1]. In contrast to conventional lasers, there are no restrictions given by the atomic energy structure of bound electrons in an active medium, and this is why these type of light sources are called free-electron lasers.

Especially the tunability of an FEL’s wavelength is of great interest and makes it possible to operate this light source in a vast range of the electromagnetic spectrum, even in the X-ray region. So today, a new generation of FELs is under construction such as the European XFEL in Hamburg that produces coherent light in exactly this region. Even though the development of FELs reaches back to the 60s and 70s [2, 3], it is still a hot topic.

Currently, there are many types of different operation modes. One can use FELs by seeding them with an initial laser field, which is then amplified. In addition to that, the so called SASE-mode can create a laser field from the vacuum, where SASE stands for self-amplified spontaneous emission.

In [4, 5] it is argued that quantum effects are of no importance for an accurate description of FELs and hence the usual approach is a classical one. Of course this can be fairly complicated, when multi-particle and three-dimensional effects are included. The predictions from classical theory match the experimental results extremely well. Thus it has been assumed to be sufficient to understand FELs’ principles of operation. We will briefly recapitulate the main ideas of classical theory and its most important properties in section 1.2 of this chapter.

In this description, we neglect quantum effects such as the recoil of the electron due to the scattering with a light wave. Of course, there have been many approaches to include these quantum effects, see for example [6, 7, 8, 9], as we will summarize in section 1.3.1. At the Helmholtz-Zentrum Dresden-Rossendorf the question has been raised whether it
is possible to operate FELs in a regime, where quantum effects are of importance and cannot be neglected. At this facility, there is great interest in the definition of a quantum regime and the observable effects on the light spectrum. It was predicted by [7, 10] and others, that the intrinsic linewidth would be smaller in this regime.

A general quantum description of FELs in a special frame has been known for a long time [11] and we will derive it in an intuitive way in section 1.3 of this chapter. Unfortunately, this leads to a time-dependent Hamiltonian for which the time evolution cannot be solved analytically. Hence, previous approaches [12, 13] used perturbation theory to face this problem. In contrast to conventional perturbation theory we will derive an alternative model of the FEL in the quantum regime in chapter 2. During this process we find a definition of this quantum regime, which is one of the main results of this thesis.

Our model will be an effective two-level system, where the interaction takes place between two intervals in momentum space. As quantum physics limits the electron to discrete energy levels of an atom in conventional laser physics, the interaction in the quantum regime of an FEL is restricted to resonances in the electrons’ momenta, i.e. the kinetic energy. So in a matter of speaking we have a quantization of the continuous classical theory where we have no restriction of the interaction. This is the second main result of this thesis.

The model we will derive can be solved analytically so the time evolution of the laser field and the steady state properties can be investigated easily in chapter 3. We will find, that some features of FELs are different in this quantum regime.

To better understand the validity of the model developed in this thesis, we will view it in the context of the results of perturbation theory and look at the connection to the classical description in chapter 4. As we will see, there is a deep link between our model and perturbation theory.

1.2 The Classical Free-Electron Laser

The qualities mentioned previously are the reason for the interest in FELs as alternative light sources. In this section we will discuss some of these requested properties, when we examine the basic principles of an FEL.

1.2.1 A Forth Generation Synchrotron Radiation Device

The fundamental configuration of an FEL is simple but intriguing. It consists of an electron beam moving with relativistic velocity through a periodic magnetic field, as
1.2 The Classical Free-Electron Laser

Figure 1.1: Principle of an FEL: The electron bunch wiggles due to the alternating magnetic field of the wiggler and emits radiation. The blue line shows the path of an electron through the wiggler.

shown in Figure 1.1. In addition to the electron beam and the magnetic field, a cavity may be added, depending on the operation mode of the FEL.

To produce an alternating magnetic field, alignments of bar magnets, called wigglers, can be used, as shown in that figure. Even though linear and helical wigglers are the most common form [4], the field can be produced any other way. For example, another option is to use so called optical wigglers, i.e. oncoming electromagnetic waves, generated by a conventional laser, which is necessary for a realization of an FEL working in the quantum regime [14]. We will come back to this type of wigglers later when we discuss the Bambini-Renieri frame in section 1.2.2.

Focus of the emitted radiation

An electron passing a periodic magnetic field is deflected by the Lorentz force perpendicular to the field, as sketched in Figure 1.1. The velocity picked up through this acceleration is small compared to the one in forward direction, if the electron moves at relativistic speed. So in good approximation, the acceleration $\ddot{v}$ of a charge $e$ with the relativistic velocity $v$ can be assumed as perpendicular to its main velocity.

This acceleration of a charge causes the emission of radiation, as known from classical electrodynamics. In this approximation we assume the electron to radiate like a dipole moving with velocity $v$ and oscillating perpendicular to this direction. Hence, the angular...
Figure 1.2: Polar plots of the angular power distribution of a charge accelerated perpendicular to the velocity in units of $e^2 \dot{v}^2 / (4 \pi c^3)$ for different velocities $v$. For increasing velocities, the radiation becomes focussed in forward direction.

The power distribution is, as described in [15],

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^3} \left(1 - \frac{v}{c} \cos \theta\right)^3 \left(1 - \frac{\sin^2 \theta}{\gamma^2 \left(1 - \frac{v}{c} \cos \theta\right)^2}\right)$$

with the relativistic factor $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$ and the speed of light $c$. Here, $\theta$ is the angle between the observer and the velocity of the dipole. This power distribution is plotted in Figure 1.2. As we can see, the power becomes more focused in forward direction as $v$ approaches $c$. In FELs, the electrons propagate almost at the speed of light, and hence, the emitted radiation is highly focused and backward emission is almost completely suppressed. In this limit, the power is just emitted between angles

$$\theta_{\text{max}} \approx \frac{1}{\gamma}$$

into the forward direction [9]. Therefore, the reason for the highly focused beam of an FEL is the high electron velocity.
1.2 The Classical Free-Electron Laser

**Tunable wavelength**

Another advantage of FELs is their tunability. It can be explained by the following argument [4, 9]:

We again assume the electron to radiate like a dipole perpendicular to its velocity \( v \approx c \). If the wiggler has \( N_W \) alternating magnets on a length of \( L_W \), and thus a period of \( \lambda_W = L_W/N_W \) in the laboratory frame, the dipole sees in its rest frame due to Lorentz contraction the wavelength

\[
\lambda'_W = \frac{L_W'}{N_W} = \frac{1}{\gamma} \frac{L_W}{N_W} = \frac{\lambda_W}{\gamma}.
\]

We assume that the dipole is stimulated at this wavelength. We will discuss this assumption in section 1.2.2 in more detail. The emitted wavelength \( \lambda'_L \) of an oscillating dipole at rest is equal to the stimulating wavelength. When transforming back to the laboratory system, we have to consider the Doppler shift [9] and obtain

\[
\lambda_L = \sqrt{1 - \frac{v^2}{c^2}} \lambda'_L = \sqrt{1 - \frac{v^2}{c^2}} \lambda'_W \approx \frac{1}{2\gamma} \lambda'_W = \frac{\lambda_W}{2\gamma^2}, \tag{1.1}
\]

where we used \( 1 + v/c \approx 2 \). Hence, the wavelength \( \lambda_L \) can be tuned by varying the velocity, i.e. \( \gamma \). The faster the electrons enter the experimental set-up, the smaller becomes the emitted wavelength. Another way to modify \( \lambda_L \) is to change \( \lambda_W \) and by that the periodicity of the wiggler. In this simple model, the tunability of FELs can be understood quite easily.

Of course we made several crude approximations in the derivation given above. First of all, the recoil effects of the Compton scattering process are neglected. We will drop this assumption in the quantum case. In addition to that, the electron oscillates more in a figure-of-eight pattern rather than a dipole pattern [4]. Furthermore, the magnetic field has been assumed constant perpendicular to the main direction of electron movement. Taking these additional effects into account, the emitted wavelength is approximately, as shown by [4],

\[
\tilde{\lambda}_L \approx \frac{\lambda_W}{2\gamma^2} \left[ 1 + \left( \frac{eB_W\lambda_W}{2\pi mc} \right)^2 \right].
\]

Here, \( B_W \) denotes the rms average magnetic induction of the wiggler. We see, that our ad hoc assumption is an approximation of a more sophisticated treatment. Looking at this expression, there is one more option to change the emitted wavelength: Tuning the
1 Introduction

strength of the wiggler field $B_W$. In the actual experiment this can be easier realized than changing the other two possible parameters. If the initial velocity of the electrons is varied, the electron beam has to be elaborately realigned with the setup. To change $\lambda_W$, one has to rearrange the magnets of the wiggler. In contrast to that, tuning the wiggler field is simple by moving alignments of magnets apart.

A coherent light source?

So far, we just discussed the interaction of one electron with the wiggler field. In this process, the emitted radiation has no coherence properties, since it is, in a way, spontaneous synchrotron radiation, as in storage rings or synchrotrons. But how does stimulated emission occur? Is even the term free-electron laser justified?

In the quantum theory of FELs developed in this thesis, we give some more explanations to this questions, but for now, we want to concentrate on a classical theory.

Figure 1.3: Coherent superposition of the radiation emitted by each microbunch in the ponderomotive potential.

As mentioned above, FELs provide coherent light. This can only be explained by multiparticle effects. The Spin-Spin and Coulomb interaction between relativistic electrons in a wiggler can be neglected [4]. As we will show in section 1.2.2 the electrons move in a periodic ponderomotive potential, which is created by an existent laser light wave. So, the interaction between the electrons is mediated by a self consistent laser field. Under the influence of this potential the electrons get rearranged in periodic structures called microbunches. These microbunches move with the velocity of the potential and are separated by the potential’s wavelength $\lambda_P$. This distance between two microbunches is, as we will show in section 1.2.2 with Eq. (1.2),

$$\lambda_P = \frac{2\pi}{k_L + k_W} = \frac{1}{1/\lambda_L + 1/\lambda_W} = \frac{\lambda_L}{1 + \lambda_L/\lambda_W} \approx \frac{\lambda_L}{1 + \frac{1}{2\gamma^2}} \approx \lambda_L,$$
where we used Eq. (1.1) derived above and \( 1 \ll \gamma \). The periodicity of the ponderomotive potential will be explained later. In each bucket of the potential, all electrons wiggle in phase and emit light of wavelength \( \lambda_L \). Since the distance between two buckets corresponds to this laser wavelength \( \lambda_L \), there is a coherent superposition of the radiation produced by every microbunch. This superpostion of the radiation emitted by different microbunches is illustrated in Figure 1.3.

This process is called, in analogy to a conventional laser, *stimulated emission*. Even though it can be explained just by classical considerations, an FEL produces coherent radiation. Hence, it can be considered as a forth generation synchrotron radiation device. In contrast to that, a quantum mechanical description or semi-classical theory of stimulated emission is necessary to explain the coherence properties of conventional lasers.

### 1.2.2 The Bambini-Renieri Frame

To build up a classical theory of FELs, we have to solve the equations of motion for relativistic electrons under the influence of a periodic alternating wiggler field. Once this motion is known, we can calculate the emitted radiation. Usually, so called *pendulum equations* describe the dynamics of this type of system. We will derive them in Eq. (1.4). Many descriptions of FELs start with a one-dimensional theory and then generalize it to higher dimensions, where additional effects are included. But the results gained from one-dimensional theories are surprisingly good and match the basics of the experimental situation in an accurate way \[4, 5\]. Therefore, the model developed in this thesis will also pursue an one-dimensional approach.

Usually, a small initial modulation in the charge distribution is assumed to describe the strong signal gain of the intensity and the process of microbunching. In \[4, 5\] and many other approaches, the theory is developed in the laboratory frame, where internal electron bunch coordinates are introduced. In contrast to that, we will use the ideas of Bambini, Renieri, and Stenhom \[16, 17\] and introduce a special reference frame. In this frame, the Hamilton function will take a rather simple form.

**The ponderomotive potential**

We will follow now the derivation of \[9\] since it is an intuitive and clear approach. The same form of the Hamiltonian can be found through *canonical transformations*, as shown in \[12\] for various types of wigglers.
1 Introduction

Let us assume one relativistic electron $e$ of mass $m_0$ with momentum $\vec{p}$ to interact with a laser field $\vec{A}_L$ moving parallel to the momentum, i.e. $\vec{k}_L \cdot \vec{p} = |\vec{k}_L||\vec{p}|$, and a wiggler field $\vec{A}_W$ in opposite direction, i.e. $\vec{k}_W \cdot \vec{p} = -|\vec{k}_W||\vec{p}|$. The Hamiltonian reads then

$$\mathcal{H} = \sqrt{\left(\vec{p} - \frac{e}{c} \left(\vec{A}_L + \vec{A}_W\right)\right)^2 c^2 + m_0^2 c^4},$$

where we used the principle of minimal coupling to introduce interaction between the charge and the vector potentials [18]. Let the electron move initially just in $z$-direction, which is parallel to the wiggler axis. We now neglect a spatial variation of the wiggler field perpendicular to the wiggler axis, i.e. $\vec{A}_W = \vec{A}_W(z, t)$, and assume

$$\vec{A}_W = \vec{e} A_W e^{-i(\omega_W t + k_W z)} + \text{c.c.}$$

with the polarization $\vec{e}$ of the vector potential and its amplitude $A_W$. The spatial variation is $k_W = 2\pi/\lambda_W$ in $z$-direction. In the laboratory frame, we have a static alternating wiggler field and hence there is no time dependence of the vector potential, i.e. $\omega_W = 0$, but when Lorentz-transformed to a different frame, a time dependence occurs. That is why it is included here. Since the ansatz of an electromagnetic wave for a static magnetic field is made, it might be called a quasi-wave.

The initial laser field is assumed to be

$$\vec{A}_L = \vec{e} A_L e^{-i(\omega_L t - k_L z)} + \text{c.c.}$$

and by that a wave moving into the opposite direction compared to the wiggler field. In this approximation, we have implied a one-mode behavior.

Since there is no $x$ and $y$-dependence of $\mathcal{H}$, the equations of motion give

$$\frac{\partial}{\partial t} p_{x_j} = -\frac{\partial}{\partial x_j} \mathcal{H} = 0$$

for $x_j \in \{x, y\}$. If there is just an initial momentum in $z$-direction, we can thus use a one dimensional theory and write $\vec{p} = p \vec{e}_z$.

We now choose the polarization of the vector potentials to be circular, i.e. $\vec{e} = (e_x + i e_y)/\sqrt{2}$, which corresponds to a helical wiggler configuration. This brings us to

$$e^2 \left[\vec{p} - \frac{e}{c} \left(\vec{A}_L + \vec{A}_W\right)\right]^2 = p^2 c^2 - 2 e c p \vec{e}_z \cdot \left(\vec{A}_L + \vec{A}_W\right) + e^2 \left(\vec{A}_L + \vec{A}_W\right)^2 = 0$$

$$= p^2 c^2 + e^2 \left(\vec{A}_L^2 + \vec{A}_W^2\right) + 2 e^2 \vec{A}_L \cdot \vec{A}_W.$$
1.2 The Classical Free-Electron Laser

With the orthogonality \( \vec{e} \cdot \vec{e} = 0 \) and normalization \( \vec{e} \cdot \vec{e}^* = 1 \) of the circular polarization vector we get \( \vec{A}_j^2 = 2 |\mathbf{A}_j|^2 \) with \( j \in \{L, W\} \) and

\[
\vec{A}_L \cdot \vec{A}_W = \mathbf{A}_W^* \mathbf{A}_L \exp[-i \{ (\omega_L - \omega_W) t - (k_L + k_W) z \}] + c.c.
\]

for the cross term. We now introduce a shifted electron mass

\[
m'^2 = m_0^2 + 2 e^2 c^4 \left( |\mathbf{A}_W|^2 + |\mathbf{A}_L|^2 \right)
\]

and get the Hamiltonian

\[
\mathcal{H} = \sqrt{m^2 c^4 + p'^2 c^2 + \left( 2 e^2 \mathbf{A}_W^* \mathbf{A}_L \right) e^{-i \left[ (\omega_L - \omega_W) t - (k_L + k_W) z \right]} + c.c.}, \tag{1.2}
\]

which is the Hamiltonian of a relativistic particle interacting with a periodic potential, called ponderomotive potential. It is spatially dependent with wave vector \( k_P = k_L + k_W \) and time dependent with frequency \( \omega_P = \omega_L - \omega_W \). Its amplitude depends both on the strength of the laser and wiggler field. So the motion of one electron is determined by the vector potential of the laser field, and since this laser field is produced by the emission of all electrons, there is an indirect interaction between them, even though we just have a one-particle theory. If we change the reference frame, we can simplify this Hamiltonian further and get a non-relativistic motion, as shown in the following.

### Choosing a different frame

We now move to a reference frame where the momentum of the electron is small enough so that it can be treated non-relativistically. From now on, variables with a prime denote the new system. If we transform the Hamiltonian into this frame, we can expand the square root appearing in Eq. (1.2) into a Taylor series, since, as argued in [9],

\[
p'^2 c^2 \ll m'^2 c^4 \quad \text{and} \quad e^2 \mathbf{A}_W' \mathbf{A}_L' \ll m'^2 c^4.
\]

With that, we get

\[
\mathcal{H}' \approx m' c^2 + \frac{p'^2}{2m'} + \left( \frac{e^2 \mathbf{A}_W^* ' \mathbf{A}_L'}{m' c^2} e^{-i \left[ (\omega_L' - \omega_W') t' - (k_L' + k_W') z' \right]} + c.c. \right)
\]

in the new frame. This Hamiltonian can be interpreted as follows: A non-relativistic electron with kinetic energy \( p'^2 / 2m' \) moves in the periodic ponderomotive potential.

So far, we have not specified the reference frame yet. The only assumption made was that the particle moves with non-relativistic velocity in it. To find a most convenient frame, we will use the ideas from [16, 17], but make a slight different definition of this frame. Nevertheless, we call it the Bambini-Renieri frame.
1 Introduction

In order to get a time independent ponderomotive potential, we change to this Bambini-
Renieri frame, where \( \omega'_L = \omega'_W \) and thus \( \omega'_p = 0 \). With the dispersion relations as in [9] we get \( ck'_L = v_{\text{frame}}k'_W \), if \( v_{\text{frame}} \) is the frame’s velocity relative to the laboratory frame. With that, the Hamiltonian reads

\[
H' = m'c^2 + \frac{p'^2}{2m'} + \left( \frac{e^2 \mathcal{A}'_W \mathcal{A}'_L}{m'c^2} \right) e^{i2kz'} + \text{c.c.}, \tag{1.3}
\]

where we defined \( 2k \equiv (1 + c/v_{\text{frame}})k'_L \). Because the ponderomotive potential is now static, the velocity of the Bambini-Renieri frame corresponds to the velocity of the ponderomotive potential in the laboratory frame.

The usual definition of the Bambini-Renieri frame is \( k = k'_L = k'_W \). This means that we still would have a small time dependent phase with frequency \( \omega_p = k(c - v_{\text{frame}}) \). But we rather choose a more complicated definition of the wave vector \( k \) than keeping a time dependence. Nevertheless, both frames are equivalent for \( v_{\text{frame}} \) approaching \( c \). This is the Weizsäcker-Williams approximation explained below.

\[\lambda_L = \frac{2\pi}{k'_L} \quad \lambda_W = \frac{2\pi}{k_W} \]

(a) Situation in the laboratory frame: An electron interacts with a static magnetic field and a laser light wave.

\[\lambda'_L = \lambda \quad \lambda'_W = \lambda \]

(b) Situation in the Bambini-Renieri frame: An electron interacts with a laser light wave and a oncoming quasi wave from the wiggler field, whose wavelengths coincide.

Figure 1.4: Illustration of the Bambini-Renieri frame in the Weizsäcker-Williams approximation.

In the laboratory frame, the field of a static magnetic wiggler is time-independent. In the Bambini-Renieri frame, where the electrons move with non-relativistic speed, the Lorentz-transformed static magnetic wiggler field becomes an oncoming time-dependent electromagnetic wave. This is demonstrated in Figure 1.4 where due to Lorentz contraction the wavelength of the laser and the wiggler field coincide.
1.2 The Classical Free-Electron Laser

For the derivation of this Hamiltonian it was essential that we used an helical wiggler, i.e. a circular polarized oncoming electromagnetic quasi wave. In [12] it was shown with the help of canonical transformations, that this form of the Hamiltonian holds true for various types of wigglers, as already mentioned above.

The Weizsäcker-Williams approximation

When we discussed the wavelength of the emitted radiation in section 1.2.1, we assumed that the electron is stimulated with the frequency of the wiggler. We now take a closer look at this assumption.

In the Bambini-Renieri frame, simple Thomson scattering takes place. In [19], Weizsäcker investigated the radiation of relativistic electrons when scattered at a static Coulomb potential. He showed, that for relativistic electrons the static electric field can be approximated by an oncoming electromagnetic wave.

We now make the same approximation, where we have $\omega'_{L} = \omega'_{W}$ and $k = k'_{L} = k'_{W}$. This is true for $c \approx v_{\text{frame}}$. As the frame approaches the speed of light, the static field becomes an oncoming electromagnetic wave with $\omega'_{W} \approx ck'_{W}$. For optical wigglers, this relation is exact since then we actually do have approaching electromagnetic waves.

For electrons at relativistic velocities, which is the case for FELs, the approximation of an approaching quasi-wave is an acceptable ansatz.

1.2.3 Radiation Field of the Classical FEL

We now take a closer look at the system of electrons interacting with the ponderomotive potential in the Bambini-Renieri frame. We can use, while dropping the primes, the Hamiltonian from Eq. (1.3) to set up Hamilton’s equations of motion. This leads, for real amplitudes of the vector potential, to the coupled differential equations [17]

$$\frac{d}{dt} W = \frac{d}{dt} \left( \frac{2kp}{m} \right) = \frac{2k}{m} \frac{\partial p}{\partial t} = -\frac{2k}{m} \frac{\partial \mathcal{H}}{\partial z} = \frac{2e^2 A_w A_L}{mc^2} \frac{4k^2}{m} \sin 2kz \equiv \varpi^2 \sin \psi \quad (1.4a)$$

$$\frac{d}{dt} \psi = \frac{d}{dt} (\pi - 2kz) = -2k \frac{\partial z}{\partial t} = -2k \frac{\partial \mathcal{H}}{\partial p} = -\frac{2kp}{m} \equiv -W, \quad (1.4b)$$

where we introduced the ponderomotive phase $\psi = \pi - 2kz$ and the scaled momentum $W = 2kp/m$ as well as $\varpi^2 = 8e^2 A_w A_L k^2/(mc)^2$. When we look at the second derivative of $\psi$ in time, we can bring Eq. (1.4) to the form

$$\ddot{\psi} + \varpi^2 \sin \psi = 0$$

and see the analogy to a pendulum. Because of that, these equations are called pendulum equations.
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Small signal gain

The pendulum equations cannot be solved analytically, as they correspond to an anharmonic oscillator. We just want to give the ideas of [17] how to find an approximate solution for the small signal gain. The authors expand the phase and the momentum in $\varpi^2$ and iteratively solve the equations for different orders of this parameter. After that, they average over the initial phase which is assumed to be uniformly distributed, i.e. the electrons are initially uniformly distributed in space.

![Figure 1.5: Gain of the classical FEL in the small signal regime.](image)

Because of energy conservation, the loss of the electrons’ kinetic energy is converted into an increase of the laser field’s intensity. So the authors calculate the classical gain $G_{cl}$ via the average loss of momentum, i.e.

$$G_{cl} = -\langle W - W_0 \rangle_{av},$$

where $W_0$ denotes the initial momentum distribution and $\langle \cdot \rangle_{av}$ the average over the initial phase. Up to second order in $\varpi^2$, they arrive at

$$G_{cl} = -\varpi^4 \frac{d}{dW_0} \left( \frac{\sin W_0 t/2}{W_0} \right)^2.$$

(1.5)

This is the well known result of the gain profile [4, 5] and plotted in Figure 1.5. If the electrons have the initial momentum $W_0 = 0$, there is no gain of the laser field. We
1.2 The Classical Free-Electron Laser

call this condition classical resonance, where the electrons are in the Bambini-Renieri frame at rest. If they have initially a positive momentum, i.e. they are above classical resonance, they will loose kinetic energy to the laser field and slow down. By that, the laser field is amplified. On the other hand, they are accelerated if they are initially below resonance and the laser field looses energy. We also want to emphasize the fact that these results were deduced in second order perturbation theory.

**Saturation regime**

![Figure 1.6: Schematic behavior of the laser intensity for increasing wiggler lengths according to [5]. In the high gain regime, the intensity starts to grow exponentially. Saturation occurs at some fixed intensity.](image)

For the derivation of the gain in the small signal regime we expanded phase and momentum in orders of \( \varpi^2 = 8e^2 A_W A_L k^2/(mc)^2 \propto A_L \), which we assumed to be constant. This is an approximation, since the intensity of the laser field is amplified, and thus \( A_L \) increases. So the expansion parameter is not constant and the derivation given above is not valid for all intensities, since the expansion can be made just for small \( \varpi^2 \). To solve the pendulum equations for higher intensities, we have to do it self-consistently, and by that include differential equations for the charge distribution and the electric field, too. An approach is given in [5], as well as numerical simulations. We briefly want to discuss the results.

For low intensities Eq. (1.5) is a very good approximation. But as the intensity increases, i.e. with growing length of the wiggler, the negative part of the gain curve
decreases and the positive parts become more important. Because of this, the intensity of the laser grows exponentially with the length of the wiggler, if the electrons are above resonance. At some intensity, the laser saturates, as shown in Figure 1.6. This intensity is always the same, regardless of the initial seeding intensity.

### 1.3 Quantum Theory of the FEL

Already Bambini and Stenholm developed a quantum description of the FEL in the Bambini-Renieri frame [11]. The quantization of the one-dimensional theory is simple and straightforward. We replace the momentum $p$ by the momentum operator $\hat{p}$ and the position $z$ by the position operator $\hat{z}$. These operators fulfill the well known commutation relation $[\hat{p}, \hat{z}] = \hbar/i$.

We also quantize the electromagnetic field by substituting $A_j e^{-i\omega_j t}$ by $\hat{A}_j \hat{a}_j$ and $A_j^* e^{i\omega_j t}$ by $\hat{A}_j \hat{a}_j^\dagger$ for $j \in \{L, W\}$. $\hat{a}_j^\dagger$ is the creation operator of the laser field or wiggler field, respectively, and $\hat{a}_j$ the corresponding annihilation operator. They fulfill the commutation relations $[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{i,j}$. We included the time-dependent phase when substituting the field amplitudes by operators, to account for the time dependence of the creation and annihilation operators. So, the quantized version of the Hamiltonian from Eq. (1.3) reads

$$\hat{H}_e = \frac{mc^2}{\hbar} \equiv \hat{H}_{\text{rest}} + \frac{\hat{p}^2}{2m} + \hbar \tilde{g} \left( \hat{a}_L \hat{a}_W^\dagger e^{i(\omega_L - \omega_W)t + 2k\hat{z}} + \text{h.c.} \right) \equiv \hat{H}_{\text{int}}$$

with the coupling constant $\tilde{g} = e^2 A_W A_L / (\hbar mc^2)$ and h.c. for the hermitian conjugate. Here, the amplitudes of the vector potential in the SI-system are, as in [18] $A_j = \sqrt{\hbar/(2\varepsilon_0 V c k_j)}$.

This Hamiltonian describes the interaction of one electron with the vector potentials. To get a full description of the whole system, we need to add the Hamiltonian of the free field, i.e. $\hat{H}_{\text{field}} = \hbar \omega_L \hat{a}_L^\dagger \hat{a}_L + \hbar \omega_W \hat{a}_W^\dagger \hat{a}_W$. We now move to the interaction picture with the unitary transformation

$$\hat{H} \equiv \exp \left[ \frac{i}{\hbar} \left( \hat{H}_{\text{rest}} + \hat{H}_{\text{kin}} + \hat{H}_{\text{field}} \right) t \right] \hat{H}_{\text{int}} \exp \left[ -\frac{i}{\hbar} \left( \hat{H}_{\text{rest}} + \hat{H}_{\text{kin}} + \hat{H}_{\text{field}} \right) t \right]$$

performed in appendix A.II. With Eq. (A.7) we get the result

$$\hat{H} = \hbar \tilde{g} \left( \hat{a}_L \hat{a}_W^\dagger e^{i 2k\hat{z}} e^{i \phi} (\hat{p}) t + \text{h.c.} \right)$$
where we defined the phase as

\[ \phi_{\pm}(\hat{p}) \equiv \frac{2k}{m} \left( \hat{p} \pm \frac{q}{2} \right). \]  

(1.7)

In this definition, we used the recoil \( q \equiv 2\hbar k \). With the formulation Eq. (A.3) of \( \exp[\pm 2i\hat{z}] \) as ladder operator, we can rewrite the Hamiltonian as

\[ \hat{H} = \hbar \tilde{g} \left( \hat{a}_L \hat{a}_W^\dagger \int_{-\infty}^{\infty} dp \ e^{i\phi_{\pm}(p)t} |p + q\rangle \langle p| + \text{h.c.} \right). \]  

(1.8)

In this representation, the Hamilton operator can be easily interpreted: We are looking at Compton-scattering processes with conservation of momentum and energy, as we will see in the following.

![Diagram](a) The electron scatters a laser photon into the wiggler field.

(b) The electron scatters a wiggler quasi-photon into the laser field.

![Diagram](Figure 1.7: Illustration of the two basic Compton-scattering processes occurring in \( \hat{H} \).

The summand with \( \hat{a}_L \hat{a}_W^\dagger |p + q\rangle \langle p| \) describes absorption of a laser field photon. A laser photon is scattered into the wiggler field, as depicted in Figure 1.7(a). During this scattering process, the electron picks up a recoil \( q = 2\hbar k \) and we see that it is elastic scattering and thus conservation of energy and momentum are ensured.

The hermitian conjugate of that process reads \( \hat{a}_W^\dagger \hat{a}_L |p - q\rangle \langle p|\). We now scatter from the wiggler field into the laser field, as sketched in Figure 1.7(b). Again we have ensured conservation of energy and momentum.

We want to emphasize the fact that \( \hat{H} \) in Eq. (1.8) has a time-dependent phase, and thus is time-dependent. The Schrödinger equation cannot be solved easily by just integrating it, one has to use the time ordering operator [18]. We will discuss different approaches to solve the dynamics of the FEL in section 1.3.1.
1 Introduction

So far, this theory just includes one electron. Of course it can be generalized to a $N$-particle description, where the $\nu$th electron is described by the momentum state $|p_\nu\rangle$, which leads to the Hamiltonian

$$\hat{H}_{\text{multi}} = \sum_{\nu=1}^{N} \hbar g \int_{-\infty}^{\infty} dp_\nu \, \exp\{i \phi(p_\nu)\} |p_\nu + q\rangle \langle p_\nu| + \text{h.c.}. \quad (1.9)$$

Here, as in classical theory, Spin-Spin and Coulomb interaction are neglected. This approach is pursued in \cite{7, 12}. Usually a bunching operator is introduced that comprehends all actions on the momentum states.

For an exact quantum mechanical description, more than two electrons cannot be in the same state because they are fermions and have to obey the Pauli principle, which is valid for indistinguishable particles. This would lead to a quantum energy spread, since all electrons cannot be in one momentum state. But for FELs, as discussed in \cite{20}, the wave packets of electrons do not overlap in phase space and hence they are distinguishable, which makes a bosonic treatment sufficient. However, we will concentrate on a one-particle theory like the Scully-Lamb theory of a conventional laser \cite{21} and keep in mind that neglecting space charge effects, Spin-Spin interaction and the Pauli principle is also adequate in a multi-particle theory.

1.3.1 Previous Approaches

To solve the dynamics of the coupled electron and radiation system, we have to calculate the time evolution operator \cite{18}

$$\hat{U}(t + \tau) \equiv \mathcal{T}\left\{ \exp\left[ -\frac{i}{\hbar} \int_{t}^{t+\tau} dt' \hat{H}(t') \right] \right\} \equiv \sum_{\nu=0}^{\infty} \left(-\frac{i}{\hbar}\right)^{\nu} \int_{t}^{t+\tau} dt_{\nu} \cdots \int_{t}^{t_{2}} dt_{1} \hat{H}(t_{\nu}) \cdots \hat{H}(t_{1})$$

The operator $\mathcal{T}$ is called time ordering operator. The Bambini-Renieri Hamiltonian in the interaction picture is time-dependent, and the time evolution cannot be solved analytically. There are several methods of facing this problem, but most of them involve some kind of perturbation theory, as we will see in the following.

When we developed the classical theory, we neglected the appearance of recoil occurring at the scattering process. Under the assumption that this recoil is sufficiently small, we can expand the time evolution operator $\hat{U}$ into orders of the recoil \cite{6} [12].
What sufficiently small means, becomes clear if we look at the quantum parameter

\[ \alpha \equiv \frac{m}{2 q k \tau} \]  

(1.10)

introduced by Becker and McIver [12] to distinguish between a classical and a quantum regime. If this parameter is much larger than unity, the expansion into small recoils is valid. In zeroth order we regain results from the classical theory, such as the small signal gain. For higher orders, we get corrections to that, and can even calculate a steady state.

A different approach is to expand the time evolution operator in orders of \( g \sqrt{n} \), where \( n \) is the photon number of the laser field. This was done in [9, 13]. It is exact in the recoil, but just valid for small laser fields, i.e. in contrast to the idea above valid for \( \alpha \leq 1 \). To regain the classical results, we have to set \( \hbar \) and with that the recoil \( q \) equal to zero. In second order the classical gain curve can be reproduced, in forth order we see a steady state solution. One of the main results is, that interaction mainly occurs for electrons with initial momenta around \( \pm q/2 \). We will discuss that in 1.3.2. In chapter 2, we use this insight to derive an effective Hamiltonian.

Moreover, the problem can be faced numerically. In [8], a set of differential equations similar to the ones in chapter 2 was numerically solved. For \( g \sqrt{n} \ll qk/m \) it was shown that the interaction takes place mainly between two electron momenta, separated by the recoil \( q \) and that coupling to other momenta is suppressed. This is in agreement with the results from [9] and gives us a motivation of the quantum regime in the next section. We will try to find an analytical explanation for this fact in chapter 2. But in contrast to [8], we will admit interaction between momenta close to their resonances.

Of course these approaches are only a few among others, but we are concentrating on these results, since they give rise to the assumption of an effective two-level behavior.

### 1.3.2 Classical vs. Quantum Regime: A First Glimpse

We have discussed a quantum description of an FEL above. So far, we have just made the Weizsäcker-Williams and one-mode approximation. But even under these assumptions, the time evolution cannot be solved exactly. Still, classical results can be regained from an approximate solution. When we look at the Becker parameter \( \alpha \propto 1/q \) we see that quantum effects occur for non-vanishing recoil \( q \). It becomes clear, that a quantum regime can be defined by this parameter. Operating in this regime, the FEL would display genuine different features, as argued by [6, 7, 12] and others.
1 Introduction

From the considerations in those previous works we do know that the recoil needs to be sufficiently large and cannot be neglected to see these quantum mechanical properties. But is a large recoil the only condition for this quantum regime? We will focus on this question in section 2.4.

In [12], it is argued that the quantum regime is characterized by a two-level behavior, as also predicted by [8, 9]. This insight is of some importance for the proceeding in this thesis, since we will use a two-level type system as the pivot of our ansatz. We motivate it by our new definition of the quantum regime: Every electron can just have two different momenta and jumps between them through the emission and absorption of laser photons. Other transitions in momentum space are suppressed. The jumps are mediated by the recoil, which cannot be neglected in the quantum regime.

In the following chapter, we will find an effective Hamiltonian that describes a two-level behavior and find by that the conditions of this quantum regime.
2 Effective Two-Level Hamiltonian

As discussed in section 1.3.1, a Schrödinger equation with the Hamiltonian (1.8) cannot be solved analytically, but just with some kind of perturbation theory. In this thesis, we use a different approach. Rather than finding an approximate time evolution resulting from a Schrödinger equation with the exact Bambini-Renieri Hamiltonian, we use a modified Hamiltonian that leads to a system for which the time evolution can be solved exactly.

In this chapter, we motivate this effective Hamiltonian in two different ways. The results are very similar, and we gain some insight into the conditions for the quantum regime and the basic physics behind this scattering process.

2.1 Rotating Wave Approximation

We now start with the Hamiltonian in the Bambini-Renieri frame derived in section 1.3. As in [8], we assume the wiggler field to be strong, i.e. $1 \ll \sqrt{n_W} \approx \sqrt{n_W} + 1$, where $n_W$ is the quasi-photon number of the wiggler field. Therefore we can neglect the change of the strength of the wiggler field when the quasi-photon number is changed by one. With that, the action of the creation and annihilation operators $\hat{a}_W^\dagger$ and $\hat{a}_W$ can be neglected. Hence, we replace them both by their approximate eigenvalue $\sqrt{n_W}$. With the new coupling constant

$$g \equiv g \sqrt{n_W} = \frac{e^2 A_W A_L}{\hbar mc^2} \sqrt{n_W}$$

we get from Eq. (1.8)

$$\hat{H} = \hbar g \int_{-\infty}^{\infty} dp' \ e^{i\phi(p')t} |p' + q\rangle \langle p' + q| + \text{h.c.}$$

as an expression for the Hamiltonian. We already chose the representation of $\exp[i2k\hat{z}]$ as ladder operator from Eq. (A.3) derived in appendix A.1. Shifting the variable of
integration $p' \to p' - q/2$, we can write with the definition of the phase

$$\phi_0(p) = \frac{2k}{m}$$  \hspace{1cm} (2.3)

Eq. (2.2) as

$$\hat{H} = \hbar g \int_{-\infty}^{\infty} dp' \left[ \hat{a}_L e^{i\phi_0(p')t} \left| p' + \frac{q}{2} \right\rangle \left\langle p' - \frac{q}{2} \right| + \hat{a}_L^* e^{-i\phi_0(p')t} \left| p' - \frac{q}{2} \right\rangle \left\langle p' + \frac{q}{2} \right| \right].$$  \hspace{1cm} (2.4)

We now divide up the integration over the momentum $p'$ in intervals of length $q$ to bring the Hamiltonian to the form

$$\hat{H} = \sum_{\nu = -\infty}^{\nu q + q/2} \int_{-q/2}^{+q/2} dp' \hbar g \left[ \hat{a}_L e^{i\phi_0(p' - \nu q)t} \left| p' - \nu q + \frac{q}{2} \right\rangle \left\langle p' - \nu q - \frac{q}{2} \right| + h.c. \right].$$  \hspace{1cm} (2.5)

In the last step we shifted the variables of integration $p' \to p' - \nu q$ in every integral of the sum.

The action of this Hamiltonian is shown in Figure 2.1. The momentum space is divided up into intervals of length $q$. An electron with momentum $p'$ will climb up or down on a set of equidistant momentum levels, due to the recoil $q$ picked up during the scattering process, as depicted in Figure 2.1(a).

But since there is an integral over $p'$ in Eq. (2.5), we do not have merely a multi-level system, but, in a manner of speaking, a superposition of infinitely many multi-level systems. Each point $p'$ in momentum space is connected to its set of levels, a momentum of $p' + \varepsilon$ has its different set. The variation of the set of levels due to the integral is shown in Figure 2.1(b).

Let us now recall the interaction of a two-level atom with a cavity. The Hamiltonian reads

$$\hat{H}_{JC} = \hbar g \left( \hat{a}_L e^{-i\Delta t} |e\rangle \langle g| + h.c. \right),$$

where $\Delta$ denotes the detuning, $|e\rangle$ the exited and $|g\rangle$ the ground state of the atom. This model is called Jaynes-Cummings (JC) model. If we compare Eq. (2.5) with this Hamiltonian, we see that the problem is similar in its structure. The atomic states are
2.1 Rotating Wave Approximation

(a) Infinite set of levels in momentum space separated by the recoil $q$, as deduced in Eq. (2.5). When a photon is absorbed, the electron climbs up the ladder in momentum space. When a photon is emitted, the electron falls down one level. Every transition has a different detuning.

(b) Here the meaning of the integral in Eq. (2.5) is shown. The ladder of momentum states is varied from $-q/2$ to $q/2$. So each momentum lies on a spoke and is linked to an infinite set of momenta, one in each interval.

Figure 2.1: Visualization of the action of the Hamiltonian when divided up into interaction intervals.

now momentum eigenstates, and there are not two, but infinitely many levels. In addition to that, we have a superposition of many JC Hamiltonians through the integration over $p'$, which corresponds to an integration over the ground state energy of the atom. Even the phase $\phi_0$ has its correspondence in the JC model: The detuning $\Delta$. This phase will now play an important role for justifying the model developed in this thesis.

In the JC model, transitions between the two energy levels can be suppressed if there is a large detuning between the energy difference of the atomic levels and the energy of one photon in the cavity. Only with sufficiently small detuning we observe effects like Rabi oscillations. In the problem of an FEL we integrate over the momentum and by that over the detuning. Thus, we have a superposition of interactions with detunings of all magnitudes. Hence, it is suggestive to neglect interactions with large phases.
When we look at the definition of the phase in Eq. (2.3), we see that
\( \phi_0(p' - \nu q) \propto p' - \nu q \) and \( \phi_0(0) = 0 \). In addition to that, \( p' \) is limited to the interval \( (-q/2, q/2) \). Hence, the absolute value of this phase is increasing for increasing \(|\nu|\) and the smallest detuning occurs for \( \nu = 0 \).

Neglecting terms with a rapidly rotating phase is called \textit{rotating wave approximation} (RWA). It is used, for example, to deduce the JC Hamiltonian \cite{18}. Since the Schrödinger equation is a differential equation of first order in time, we need to integrate in time to solve it. Performing this integrations, the phases are brought into the denominator. So the dominating summands are those with the smallest absolute value of the phase. Now performing the RWA and just keeping the summands with \( \nu = 0 \), we get

\[
\hat{H}_{\text{eff}} = \int_{-q/2}^{+q/2} dp' \ h g \left[ \hat{a}_L e^{i\phi_0(p') t} \left| p' + \frac{q}{2} \right| \langle p' - \frac{q}{2} \rangle + \text{h.c.} \right], \tag{2.6}
\]

which is, apart from the integral, reminiscent of the JC model, which can be solved exactly.

But how good is this approximation? In the JC model we assumed to have a two-level atom. With more than two levels, it would be much more complicated to solve. Of course this assumption is a crude approximation since such atoms do not exist. Nevertheless, the model plays a very important role in cavity quantum electrodynamics. All energy levels far off resonance are neglected, the same way we here neglect the momenta leading to high detuning. So if it is justified to assume just one atomic transition, we can also see that there is some validity to this RWA. But still there remains one important difference: The phase factors vary continuously. In a real atom, we have one transition close to resonance and the others are far off resonance. In our problem, we have some momenta on resonance, some close to resonance, some off resonance and some far off resonance. It is not clear, however, why to keep some of the levels and why to neglect others. We will focus on this question in the next section, where we derive this Hamiltonian in a different way.

To get a better feeling for the effective Hamiltonian, we again shift the integrals in Eq. (2.6) to bring it to the form

\[
\hat{H}_{\text{eff}} = h g \left[ \hat{a}_L \int_{-q}^{0} dp' \ e^{i\phi_+(p') t} \left| p' + q \right| \langle p' \rangle + \hat{a}_L^\dagger \int_{0}^{q} dp' \ e^{-i\phi_-(p') t} \left| p' - q \right| \langle p' \rangle \right], \tag{2.7}
\]

which is straightforward to interpret:
2.1 Rotating Wave Approximation

An electron with a initial momentum $p \in (0,q)$ creates a photon of the laser field. This corresponds to an electron with an energy larger than the resonance energy in classical FEL theory, as discussed in section 1.2.3. In the following, we will call this interval in momentum space gain interval. But in contrast to the classical theory there is no interaction if the momentum is larger than the recoil $q$. An electron with a momentum of $p \in (-q,0)$ annihilates a photon of the laser field. Obviously it is now below the classical resonance energy. We will call this interval loss interval. Again, no interaction occurs for momenta smaller than $-q$. Figure 2.2 visualizes these two intervals.

In addition to this classical resonance condition, we already see a quantum feature: For the momentum of $q/2$ there is no detuning in the gain interval, and for $-q/2$ in the loss interval, respectively. Hence, quantum resonances will occur at these momenta. This was already observed in [9,13].

In the RWA we have a two-level type Hamiltonian. It is not surprising, that this results in Rabi equations as we show in the following. We now write an arbitrary state as

\[
|\psi(t)\rangle = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} dp \ c_n(p) |n,p\rangle = \sum_{n=0}^{\infty} \left( \int_{-\infty}^{-q} dp + \int_{-q}^{0} dp + \int_{0}^{q} dp + \int_{q}^{\infty} dp \right) c_n(p) |n,p\rangle
\]

\[=
\sum_{n=0}^{\infty} \left( \int_{-\infty}^{-q} dp + \int_{-q}^{q} dp \right) c_n(p) |n,p\rangle + \int_{-q}^{0} dp \ c_0(p) |0,p\rangle
\]

\[+ \sum_{n=0}^{\infty} \int_{-q/2}^{q/2} dp \ \left\{ c_{n+1} \left( p - \frac{q}{2} \right) |n+1,p - \frac{q}{2}\rangle + c_n \left( p + \frac{q}{2} \right) |n,p + \frac{q}{2}\rangle \right\}, \quad (2.8)
\]

where we again shifted the index of integration to get a form, where the action of the
2 Effective Two-Level Hamiltonian

effective Hamiltonian \((2.6)\) can be seen. This action reads

\[
\hat{H}_{\text{eff}} |\psi\rangle = \sum_{n=0}^{\infty} \frac{q}{2} \int_{-\frac{q}{2}}^{\frac{q}{2}} dp \ h g \left\{ \sqrt{n+1} e^{i\phi_0(p)t} c_{n+1} \left( p - \frac{q}{2} \right) |n, p + \frac{q}{2}\rangle + \sqrt{n+1} e^{-i\phi_0(p)t} c_n \left( p + \frac{q}{2} \right) |n+1, p - \frac{q}{2}\rangle \right\}.
\]

The time derivative of the state can be taken by differentiating every coefficient in Eq. \((2.8)\) with respect to time. If we do this, and use the Schrödinger equation

\[
i\hbar \frac{\partial}{\partial t} |\psi\rangle = \hat{H}_{\text{eff}} |\psi\rangle
\]

to equate the coefficients, we arrive at differential equations

\[
\dot{c}_n(p) = 0 \quad \forall p \in (-\infty, -q) \cup (q, \infty)
\]

\[
\dot{c}_0(p) = 0 \quad \forall p \in (-q, 0)
\]

that show a constant behavior in time. This result is not surprising, since there is no interaction if the electron has a momentum outside the interaction intervals. With the results from the adiabatic approximation in section 2.2, we will better understand the validity of this approximation. If the electron enters a cavity with no photons in the loss interval, there is also no interaction. These coefficients decouple from the dynamics.

The interesting differential equations

\[
\dot{c}_{n+1} \left( p - \frac{q}{2} \right) = -i g \sqrt{n+1} e^{-i\phi_0(p)t} c_n \left( p + \frac{q}{2} \right) \quad \forall p \in \left( -\frac{q}{2}, \frac{q}{2} \right) \quad \text{and} \quad \forall n \in \mathbb{N}
\]

\[
\dot{c}_n \left( p + \frac{q}{2} \right) = -i g \sqrt{n+1} e^{i\phi_0(p)t} c_{n+1} \left( p - \frac{q}{2} \right) \quad \forall p \in \left( -\frac{q}{2}, \frac{q}{2} \right) \quad \text{and} \quad \forall n \in \mathbb{N}_0
\]

are the familiar Rabi equations. We have just a coupling of two momentum levels, one is always in the loss and one always in the gain interval. This is a system of coupled differential equations of first order with time-dependent prefactors. The well known solutions of the Rabi equations are

\[
c_n \left( p + \frac{q}{2} ; t \right) = e^{-i\phi_0(p)t} \left\{ -i g \sqrt{n+1} \frac{\sin \Omega_n^0 t}{\Omega_n^0} c_{n+1} \left( p - \frac{q}{2} ; 0 \right) + \left[ \cos \Omega_n^0 t + i \frac{\phi_0(p)}{2} \sin \Omega_n^0 t \right] c_n \left( p + \frac{q}{2} ; 0 \right) \right\}
\]

\[
c_{n+1} \left( p - \frac{q}{2} ; t \right) = e^{i\phi_0(p)t} \left\{ -i g \sqrt{n+1} \frac{\sin \Omega_n^0 t}{\Omega_n^0} c_n \left( p + \frac{q}{2} ; 0 \right) + \left[ \cos \Omega_n^0 t - i \frac{\phi_0(p)}{2} \sin \Omega_n^0 t \right] c_{n+1} \left( p - \frac{q}{2} ; 0 \right) \right\},
\]
2.2 Adiabatic Approximation

as discussed in [18]. These coefficients describe the dynamics of the coupled system of electron and laser field. The quantity

$$\Omega_{n}^{0}(p) \equiv \sqrt{\left(\frac{\phi_{0}(p)}{2}\right)^{2} + g(n + 1)}$$  \hspace{1cm} (2.12)

is called generalized Rabi frequency. We see, the phase $\phi_{0}(p)$, i.e. the deviation from the quantum resonance, is taking the role of detuning in the JC model. We will use this dynamics to investigate the time evolution of the laser field in section 3.1.

In this section, we have given a somewhat heuristic motivation of the effective Hamiltonian. The question is still open, under which circumstances the RWA is valid. It will become more clear when we look at the adiabatic approximation in section 2.2.

2.2 Adiabatic Approximation

Investigating the RWA in the FEL-Hamiltonian above gives an intuitive understanding of the physics behind the scattering processes. In this section we want to look at it from a different angle, that is less heuristic. We will follow the ideas of Khan and Zubairy [22, 23] as well as Marte and Stenholm [24]. They investigated atomic scattering at light waves in the Bragg regime, i.e. in the regime where the recoil of the scattered atoms is not negligible. For that, they used the so called adiabatic approximation to get an effective two-level behavior. We will use the same ansatz but do not enforce such a restrictive resonance condition as they did.

2.2.1 System of Differential Equations

In section 1.3 we derived in Eq. (1.8) the Bambini-Renieri Hamiltonian of the FEL in the interaction picture. We now use the Bambini-Renieri Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \hbar q \left( \hat{a}_{L} \int_{-\infty}^{\infty} dp' \mid p' + q \rangle \langle p' \mid + \text{h.c.} \right)$$  \hspace{1cm} (2.13)

in a picture where we do not transform the kinetic energy. We write an arbitrary state in this picture as

$$\left| \tilde{\psi}(t) \right> = e^{-\frac{i}{\hbar} \frac{\hat{p}^2}{2m} t} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} dp \hat{c}_{n}(p) \mid n, p \rangle$$
Effective Two-Level Hamiltonian

with the inclusion of a global phase \(\varphi^2/(2m\hbar)\), for the sake of convenience. The meaning of \(\varphi\) will be discussed later.

If we let the Hamiltonian (2.13) act on this state, we get

\[
\hat{H} \left\langle \tilde{\psi} \right\rangle = e^{-i \frac{\varphi^2}{2m} t} \sum_{n=-\infty}^{\infty} \int dp \tilde{c}_n(p) \left\{ \frac{p^2}{2m} |n,p\rangle + \sqrt{n} |n-1,p+q\rangle + \sqrt{n+1} |n+1,p-q\rangle \right\}
\]

\[
= e^{-i \frac{\varphi^2}{2m} t} \sum_{n=-\infty}^{\infty} \int dp \left\{ \frac{p^2}{2m} \tilde{c}_n(p) + \sqrt{n+1} \tilde{c}_{n+1}(p-q) + \sqrt{n} \tilde{c}_{n-1}(p+q) \right\} |n,p\rangle.
\]

In the last step we shifted the indices \(n \to n+1\) and \(p \to p-q\) in the second summand and \(n \to n-1\) and \(p \to p+q\) in the third summand. We keep in mind, that since \(\hat{a}_L |n=0\rangle = 0\) we define \(|n=-1\rangle \equiv 0\) and with this \(\tilde{c}_{-1}(p) \equiv 0\). Taking the time derivative

\[
\frac{i}{\hbar} \frac{\partial}{\partial t} \left\langle \tilde{\psi} \right\rangle = e^{-i \frac{\varphi^2}{2m} t} \sum_{n=0}^{\infty} \int dp \tilde{c}_n(p) \left\{ \frac{p^2}{2m} \right\} |n,p\rangle + \frac{\partial}{\partial t} \sum_{n=0}^{\infty} \int dp \tilde{c}_n(p) |n,p\rangle
\]

of this state we can use the Schrödinger equation (2.9) and equate the coefficients, to get the differential equations

\[
\frac{i}{\hbar} \frac{\partial}{\partial t} \tilde{c}_n(p) = \frac{p^2}{2m} \tilde{c}_n(p) + \hbar g \left\{ \sqrt{n+1} \tilde{c}_{n+1}(p-q) + \sqrt{n} \tilde{c}_{n-1}(p+q) \right\}
\]

(2.14)

for the coefficients. This is an infinite set of coupled differential equations with constant prefactors in time. We again see the ladder structure that also occurred when we discussed the RWA in section 2.1. A coefficient with momentum \(p\) is coupled to itself and those with \(p+q\) and \(p-q\).

2.2.2 Conditions for Neglecting the Dynamics of Coefficients

We now turn to the adiabatic approximation. If \(g\sqrt{n} \ll \left| (p^2 - \varphi^2)/(2m\hbar) \right|\), the first summand in Eq. (2.14) will dominate the time evolution. Since the coupling to the other levels is small compared to the coupling to the same level, there will just be a rapid oscillation of this level. So we can neglect the time derivative of these coefficients. On the other hand, we keep all time derivatives if

\[
\frac{\left| \frac{p^2 - \varphi^2}{2m\hbar} \right|}{g\sqrt{n}} < \frac{\epsilon_0 V c^3 k m}{2\epsilon_0 V c^3} \sqrt{mnW} \Rightarrow q^2 \left| \frac{(p/q)^2 - (\varphi/q)^2}{2} \right| < \frac{\epsilon_0 V c^3 \sqrt{mnW}}{\epsilon_0 V c^3} \equiv \eta^2.
\]

(2.15)
The condition is schematically plotted in Figure 2.3 for two different values of $\varphi$. This plot can be interpreted as follows: For a momentum $p$ at a given recoil $q$ the plot is colored, if the left side of the inequality is smaller than $\eta^3$. In that case, we keep the dynamics of the coefficients corresponding to this momentum. If the left side exceeds that value, the area is not colored and we neglect the dynamics of these momenta. Where the left side of the inequality is equal to zero, a vertical line is drawn. At these momenta resonances occur, as we will see later.

![Diagram showing the adiabatic approximation](image)

Figure 2.3: Schematic plot of Eq. (2.15) for $\varphi = q/2$ (orange) and $\varphi = 0.8q$ (gray). The inequality is fulfilled, i.e., the dynamics of the coefficients is kept, where the plots are colored. The resonances are at the vertical lines at $\pm \varphi$, where the left side of Eq. (2.15) is equal to 0. The separation of the resonances is always $2\varphi$.

There is one thing we can see immediately: The larger the recoil $q$ the more is the interaction limited. In the extreme quantum limit, we have just small intervals where we keep the dynamics. In addition to that, the width of these interval decreases for increasing $\varphi$. Further, we see that for small recoils, the interaction intervals of both values of $\varphi$ overlap. We will come back to that aspect later.

We always see two resonances in this plot for each $\varphi$. This can be explained with the quadratic nature of the inequality. For all recoils the left side of the inequality is exactly equal to zero if $p = \pm \varphi$. Since we just keep those differential equations where Eq. (2.15) is fulfilled, we have interaction areas around these resonances. But, as we see from Eq. (2.14), there is just a coupling between coefficients $\tilde{c}$ that are separated in
momentum space by \( q \). For high recoils, we more and more see the quantum character of the system, since the areas of interaction get limited. But in general for arbitrary \( \varphi \), these intervals are not separated by multiples of \( q \), but by \( 2 \varphi \) and thus we have no coupling of the differential equations that are kept. In Figure 2.3 we see that for \( \varphi = 0.8q \) we have no integer multiple of \( q \) as the difference between the interaction intervals. Just if we choose

\[
\varphi = \frac{\mu}{2}q \quad \mu \in \mathbb{N}
\]

(2.16)
a coupling between the kept coefficients occurs. We call this the resonance condition. For \( \mu = 1 \) we have a transition between \( q/2 \) and \(-q/2\), for \( \mu = 2 \) between \( q \) and \(-q\) and so on. These transitions correlate to a \( \mu \)-photon transition between two levels. Figure 2.4 shows the inequality for various \( \mu \). The resonances occur at momenta with multiples of \( q/2 \).

![Figure 2.4: Schematic plot of Eq. (2.15) for \( \varphi = \mu q/2 \). As marked in the plot, the interaction intervals are separated by multiples of \( q \) and we have scattering back and forth between these intervals. For \( \varphi = q/2 \) (orange), we get a one-photon, for \( \varphi = q \) (gray) a two-photon, and for \( \varphi = 3q/2 \) (yellow) a three-photon transition.](image)

However, so far it is not clear which \( \varphi \) to choose. Depending on the choice of \( \varphi \) a \( \mu \)-photon transition is selected. Of course in general multi-photon transitions can occur simultaneously. But we now define the quantum regime of the FEL as the regime, where just one transition is possible. As we see in Figure 2.4 the recoil \( q \) must be sufficiently large for that.
2.2 Adiabatic Approximation

Figure 2.5: Restriction to the electron’s initial momentum. If the initial momentum distribution overlaps with more than one colored area, there is a multi-level behavior. The solid blue curve shows a momentum distribution where we have a two-level behavior, the dashed blue momentum distributions overlap with more than one interaction interval.

But in addition to that, we get another restriction: The electron’s momentum needs to be close to one resonance and just one resonance. If not, more than one choice of $\varphi$ give possible dynamics and more than one transition occur. But this enforces a restriction on the width $\Delta p$ of the initial momentum distribution. If it is larger than $q/2$, it overlaps with more than one resonance, since they are separated by this distance in momentum space. But if

$$\Delta p < \frac{q}{2}$$

(2.17)

it is possible to have just an overlap with one. This is visualized in Figure 2.5. The blue curves symbolize initial momentum probability distributions. If this distribution is sharp enough and close to a resonance, as the solid curve shows, we get interaction between two momentum levels, since there is just an overlap with the orange area. If the width of the initial momentum distribution is too wide, there are more than just one resonance realized and we cannot make use of a two-level approximation. This situation is shown by the dashed blue curve in the upper left of Figure 2.5 where we have an overlap with the orange and gray areas. Even if the electron has a very definite momentum and is close to a resonance, the recoil still needs to be large enough. This is shown by the
dashed blue curve in the lower right of Figure 2.5. Here we have an overlap with the green, orange, and gray areas.

This shows the limits of our model and approximation. By defining the quantum regime of the FEL as the regime, where the two-level approximation is valid, we also get conditions for this regime, which we discuss in section 2.4 in more detail. However, the electron has to be strongly localized in momentum space and close to a resonance. Hence, the meaning of \( \varphi \) becomes clear. It is approximately the mean initial momentum of the electron.

We now know under which conditions we can apply the adiabatic approximation and which coefficients can be neglected.

### 2.2.3 Simplification of the Differential Equations

In section 2.2.1 we derived an infinite system of differential equations for the coefficients of an arbitrary state. In section 2.2.2 we investigated under which circumstances the dynamics of some of these coefficients can be neglected and we get an effective two-level behavior.

We now try to see this two-level behavior in the equations. We therefore shift the arguments of the coefficients \( p \to p + q/2 \) in Eq. (2.14) and consider the set of differential equations

\[
\begin{align*}
\hbar \frac{\dot{c}_{n-1}}{n} (p + \frac{3q}{2}) &= \frac{(p + 3q/2)^2 - \varphi^2}{2m} c_{n-1} (p + \frac{3q}{2}) \\
+ \hbar g \left\{ \sqrt{n} \, c_n (p + \frac{q}{2}) + \sqrt{n - 1} \, c_{n-2} (p + \frac{5q}{2}) \right\} \\
\hbar \frac{\dot{c}_n}{n} (p + \frac{q}{2}) &= \frac{(p + q/2)^2 - \varphi^2}{2m} c_n (p + \frac{q}{2}) \\
+ \hbar g \left\{ \sqrt{n + 1} \, c_{n+1} (p - \frac{q}{2}) + \sqrt{n} \, c_{n-1} (p + \frac{3q}{2}) \right\}
\end{align*}
\]
2.2 Adiabatic Approximation

\[ i \hbar \frac{\dot{c}_{n+1}}{2m} = \frac{(p - q/2)^2 - \psi^2}{2m} c_{n+1} \left( p - \frac{q}{2} \right) \]

\[ + \ h g \left\{ \sqrt{n + 2} \ c_{n+2} \left( p - \frac{3q}{2} \right) + \sqrt{n + 1} \ c_{n} \left( p + \frac{q}{2} \right) \right\} \]

\[ i \hbar \frac{\dot{c}_{n+2}}{2m} = \frac{(p - 3q/2)^2 - \psi^2}{2m} c_{n+2} \left( p - \frac{3q}{2} \right) \]

\[ + \ h g \left\{ \sqrt{n + 3} \ c_{n+3} \left( p - \frac{5q}{2} \right) + \sqrt{n + 2} \ c_{n+1} \left( p - \frac{q}{2} \right) \right\} . \]

This system is quite cumbersome, but if we take a closer look at these equations, the coupling between two neighboring levels in momentum space comes out clearly.

Let us now assume that the initial electron momentum is close to \( \pm q/2 \) and sufficiently sharp. Hence, we choose \( \psi = q/2 \). Further, we assume to have a sufficiently large recoil so that \( |(p \pm q/2)^2 - \psi^2|/(2m \hbar) \ll g \sqrt{n} \) but \( |(p \pm 3q/2)^2 - \psi^2|/(2m \hbar) \gg g \sqrt{n} \). This can just be true simultaneously for \( p \in (-q/2, q/2) \). This interval is the maximal possible interaction interval.

We now perform the adiabatic approximation as in \[22, 23, 24\]. We first neglect all terms double underlined since they will just yield correction terms from higher levels, so all \( c_n(p \pm 5q/2) \) are set to zero. Corrections occurring from higher levels are discussed in section 2.2.4. The underlined terms are zero, too, since the factors \( |(p \pm 3q/2)^2 - \psi^2|/(2m \hbar) \) are large and thus the time dependence can be neglected. This is the main part of the adiabatic approximation. We now get from the first and last equation in the system given above the relations

\[ \tilde{c}_{n-1} \left( p + \frac{3q}{2} \right) = - \ \frac{hg \sqrt{n}}{(p + 3q/2)^2 - \psi^2} \ c_{n} \left( p + \frac{q}{2} \right) \]

\[ \tilde{c}_{n+2} \left( p - \frac{3q}{2} \right) = - \ \frac{hg \sqrt{n + 2}}{(p - 3q/2)^2 - \psi^2} \ c_{n+1} \left( p - \frac{q}{2} \right) , \]

where we keep in mind, that the factors are much smaller than one.
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With these equations, we get the two coupled differential equations

\[
\begin{align*}
i\hbar \dot{\tilde{c}}_n (p + \frac{q}{2}) &= \left\{ \frac{(p + q/2)^2 - \psi^2}{2m} - \frac{2m\hbar^2 q^2 n}{(p + 3q/2)^2 - \psi^2} \right\} \tilde{c}_n (p + \frac{q}{2}) \\
&\quad + \hbar g \sqrt{n + 1} \tilde{c}_{n+1} \left( p - \frac{q}{2} \right)
\end{align*}
\]

\[
\begin{align*}
i\hbar \dot{\tilde{c}}_{n+1} \left( p - \frac{q}{2} \right) &= \left\{ \frac{(p - q/2)^2 - \psi^2}{2m} - \frac{2m\hbar^2 q^2 (n + 2)}{(p - 3q/2)^2 - \psi^2} \right\} \tilde{c}_{n+1} \left( p - \frac{q}{2} \right) \\
&\quad + \hbar g \sqrt{n + 1} \tilde{c}_n \left( p + \frac{q}{2} \right)
\end{align*}
\]

with a coupling to the same and to the other level. These two coupled differential equations can be solved and the solution is given in appendix B. To compare this solution to the one of the RWA we transform in appendix B the coefficients into the same interaction picture. For that, we use

\[
|\psi\rangle = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} dp \ c_n(p) |n,p\rangle \equiv e^{i\hat{H}_{\text{kin}} t} |\tilde{\psi}\rangle
\]

and get from Eq. (B.6)

\[
\begin{align*}
c_n \left( p + \frac{q}{2}; t \right) &= e^{-i\frac{\phi_0(p)+\Delta_1+\Delta_2}{2} t} \left\{ -i\hbar \sqrt{n + 1} \frac{\sin \tilde{\Omega} t}{\tilde{\Omega}} c_{n+1} \left( p - \frac{q}{2}; 0 \right) \\
&\quad + \left[ \cos \tilde{\Omega}_n t - i\frac{\phi_0(p) + \Delta_2 - \Delta_1}{2} \hbar \sin \tilde{\Omega}_n t \right] c_n \left( p + \frac{q}{2}; 0 \right) \right\}
\end{align*}
\]

\[
\begin{align*}
c_{n+1} \left( p - \frac{q}{2}; t \right) &= e^{i\frac{\phi_0(p)-\Delta_1-\Delta_2}{2} t} \left\{ -i\hbar \sqrt{n + 1} \frac{\sin \tilde{\Omega} t}{\tilde{\Omega}} c_n \left( p + \frac{q}{2}; 0 \right) \\
&\quad + \left[ \cos \tilde{\Omega}_n t - i\frac{\phi_0(p) + \Delta_2 - \Delta_1}{2} \hbar \sin \tilde{\Omega}_n t \right] c_{n+1} \left( p - \frac{q}{2}; 0 \right) \right\},
\end{align*}
\]

where we defined

\[
\tilde{\Omega}_n \equiv \sqrt{\left( \frac{\phi_0(p) + \Delta_2 - \Delta_1}{2} \right)^2 + g^2(n + 1)}
\]
2.2 Adiabatic Approximation

and

\[ \Delta_1 = 2 \frac{g^2 n}{(p + 3q/2)^2 - \varphi^2} \]
\[ \Delta_2 = 2 \frac{g^2 (n + 2)}{(p - 3q/2)^2 - \varphi^2} \]

as phase corrections. We now see again Rabi oscillations between the two levels. The similarity to the results of the RWA is obvious. We get the dynamics of Eq. (2.11), with some corrections \( \Delta_1 \) and \( \Delta_2 \) to the phase \( \phi_0(p) \), which are proportional to \( g^2 n \). In the denominator of these corrections are kinetic terms. They correspond to phases that are large within the adiabatic approximation. These corrections come from higher levels with their resonance \( \varphi = \pm q \).

When the JC model is solved for large detunings, very similar terms occur \[18\]. This is called the dynamical Stark effect. In our model the dynamical Stark effect of higher levels in momentum influences the dynamics of our approximated two-level system and gives corrections to the phase, i.e. to the detuning.

We already discussed that the approximation is only possible for \( p \in (-q/2, q/2) \). In this interval, the denominator of \( \Delta_j \) cannot become zero and hence there is no momentum, where the phase corrections diverge. On the other hand, the condition for the adiabatic approximation was \(|[(p \pm 3q/2)^2 - \varphi^2]/(2m\hbar)| \gg g\sqrt{n}\). Directly from that follows \(|\Delta_1| \ll g\sqrt{n} \) and \(|\Delta_2| \ll g\sqrt{n + 2} \), but they will not become zero. Since \(|\phi_0(0)| = 0 \) we see, that in exact resonance the phase corrections cannot be ignored but for non-resonant momenta it is a good approximation.

2.2.4 Higher-Level Corrections

In the derivation given above we have neglected corrections from \( \tilde{c}_n(p \pm 5q/2) \) and higher levels. We now take them into consideration. For that, we first look at the differential equation

\[
\frac{i\hbar}{2} \frac{d}{dp} \left( \frac{5q}{2} \right) \left( \frac{p + 5q/2}{2m} \tilde{c}_{n-2} \left( \frac{p + 5q}{2} \right) \right) = \frac{(p + 5q/2)^2 - \varphi^2}{2m} \tilde{c}_{n-2} \left( \frac{p + 5q}{2} \right) + h g \left\{ \sqrt{n - 1} \tilde{c}_{n-1} \left( \frac{p + 3q}{2} \right) + \sqrt{n - 2} \tilde{c}_{n-3} \left( \frac{p + 7q}{2} \right) \right\}
\]
2 Effective Two-Level Hamiltonian

that would be the next higher level. We again neglect the double underlined coupling to the level above and the underlined time derivative. So we get

\[ \tilde{c}_{n-2} \left( p + \frac{5q}{2} \right) = -\frac{g\sqrt{n} - 1}{(p+5q/2)^2 - \varphi^2} \tilde{c}_{n-1} \left( p + \frac{3q}{2} \right), \]

which we can plug into the differential equation for \( \tilde{c}_{n-1} \left( p + \frac{3q}{2} \right) \). Neglecting again the underlined time derivative of this level, we get the relation

\[ \tilde{c}_{n-1} \left( p + \frac{3q}{2} \right) = -\frac{g\sqrt{n}}{\frac{(p+3q/2)^2 - \varphi^2}{2m\hbar} - \frac{g^2(n-1)}{(p+5q/2)^2 - \varphi^2}} \tilde{c}_n \left( p + \frac{q}{2} \right). \]

The phase corrections

\[
\Delta_1 = 2 \frac{g^2n}{(p+3q/2)^2 - \varphi^2} - \frac{g^2(n-1)}{(p+5q/2)^2 - \varphi^2} - \frac{g^2(n-2)}{(p+7q/2)^2 - \varphi^2} - \ldots
\]

\[
\Delta_2 = 2 \frac{g^2(n+2)}{(p-3q/2)^2 - \varphi^2} - \frac{g^2(n+3)}{(p-5q/2)^2 - \varphi^2} - \frac{g^2(n+4)}{(p-7q/2)^2 - \varphi^2} - \ldots
\]

are the results if this procedure was done iteratively for all levels. The phase correction \( \Delta_2 \) was deduced analogously. Since we assume for all these terms the adiabatic approximation, we know that \( |(p \pm \nu q/2)^2 - \varphi^2|/(2m\hbar) \gg g\sqrt{n} \) for all odd \( \nu \geq 3 \). Hence, the correction is dominated by the first summand in the denominator, and neglecting the higher levels in the section above is a valid approximation.

2.3 Multi-Photon Transitions

So far, we have discussed two approaches to justify an effective two-level Hamiltonian. We focused on the one-photon transitions between loss and gain interval. In this section, we want to take a closer look at other resonances.

The adiabatic approximation is just valid if the electrons have a very definite initial momentum close to the resonance momenta \( \varphi = \mu q/2 \). As we have already seen, each
2.3 Multi-Photon Transitions

Figure 2.6: Two-photon transition occurring between the resonances at ±q in the adiabatic approximation. The intermediate levels can be adiabatically removed as shown in [24].

A resonance is associated with a µ-photon transition. We still have a two-level type behavior between the interval around µq/2 and the one around −µq/2, since the levels in between can be adiabatically eliminated, as shown in [24]. This is visualized in Figure 2.6. The phase corrections will read differently but the calculations can be done analogously. Hence, depending on the initial momentum, a resonance is chosen and a µ-photon transition selected, if the FEL is operated in the quantum regime.

We now also want to take into account multiple resonances in the RWA. For that, we divide up the integration over p’ of the Hamiltonian in a different way than in section 2.1 and chose intervals of (−q/4, q/4) symmetrically around the resonances:

\[
\hat{H} = \hbar g \int_{-\infty}^{\infty} dp' \left[ \hat{a}_L e^{i\phi(p')t} \left| p' + \frac{q}{2} \right\rangle \left\langle p' - \frac{q}{2} \right| + \text{h.c.} \right] \\
= \sum_{\nu = -\infty}^{\nu_q/2 + q/4} \int_{\nu_q/2 - q/4}^{\nu_q/2 + q/4} dp' \hbar g \left[ \hat{a}_L e^{i\phi(p')t} \left| p' + \frac{q}{2} \right\rangle \left\langle p' - \frac{q}{2} \right| + \text{h.c.} \right] \\
= \int_{-q/4}^{q/4} dp' \sum_{\nu = -\infty}^{\nu_q/2} \hbar g \left[ \hat{a}_L e^{i\phi(p' - \nu q/2)t} \left| p' + \frac{q}{2}(1 - \nu) \right\rangle \left\langle p' - \frac{q}{2}(1 + \nu) \right| + \text{h.c.} \right]
\]
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In the last step, we again shifted the index of integration $p' \rightarrow p' - \nu q/2$. When we now sum separately over odd and even $\nu$, we get

$$\hat{H} = \int \frac{q}{4} \sum_{\nu = -\infty}^{\infty} [\hat{H}^{\text{odd}}_{\nu}(p) + \hat{H}^{\text{even}}_{\nu}(p)]$$

with

$$\hat{H}^{\text{odd}}_{\nu}(p) = \hbar g \left[ \hat{a}_L e^{i\phi_0(p' - q\nu + 1/2)t} \left| p' - q\nu \right\rangle \left\langle p' - q(\nu + 1) \right| + \text{h.c.} \right]$$

$$\hat{H}^{\text{even}}_{\nu}(p) = \hbar g \left[ \hat{a}_L e^{i\phi_0(p' - q\nu)t} \left| p' - q \left( \nu - \frac{1}{2} \right) \right\rangle \left\langle p' - q \left( \nu + \frac{1}{2} \right) \right| + \text{h.c.} \right].$$

The action of these operators is depicted in Figure 2.7. As we can see, there are two separate interaction ladders, one that mediates the interaction around the resonances at $q\mu$ and one around the resonances at $q(\mu + 1/2)$. Hence, when we solve the Schrödinger equation (2.9), we get two sets of differential equations that are decoupled. This is no surprise, since the intervals have a width of $q/2$, but the coupling is always between momenta separated by $q$.

![Figure 2.7: Action of the Hamiltonian.](image)

Figure 2.7: Action of the Hamiltonian. $\hat{H}^{\text{odd}}_{\nu}(p)$ mediates interaction between the resonances that are multiple integers of $q$. In contrast to that, $\hat{H}^{\text{even}}_{\nu}(p)$ mediates the interaction between the resonances at $q(\mu + 1/2)$.

In lowest order of the phase, we just keep $\hat{H}^{\text{even}}_{0}$. So we just have a two-level behavior as above due to the interaction between two intervals in momentum space. If we keep the summands with the smallest phase of the $\hat{H}^{\text{odd}}_{\nu}$, we still have two terms, namely $\hat{H}^{\text{odd}}_{0}$ and $\hat{H}^{\text{odd}}_{-1}$. In this approximation, momenta close to the resonances at $\pm q$ and close to 0 interact. Due to the sum of the two operators, we have two one-photon transitions between three levels. However, we know from the adiabatic approximation, that the intermediate level will almost not be occupied, and the transition takes place between the outer two levels. Hence, we also get an effective two-photon transition.
These resonances at $\pm q$ have been already discovered in the numerical simulations of [8]. If we just keep the summands with the smallest phase and let them act on an arbitrary state, we can analogously to section 2.1 derive differential equations for the coefficients. From $\hat{H}_0^{\text{even}}$, we regain the usual Rabi equations for $p \in (-q/4, q/4)$

\begin{align}
  i\dot{c}_{n+1}(p - \frac{q}{2}) &= g\sqrt{n + 1}e^{i\phi_0 t}c_n\left(p + \frac{q}{2}\right), \quad (2.18a) \\
  i\dot{c}_n(p + \frac{q}{2}) &= g\sqrt{n + 1}e^{-i\phi_0 t}c_{n+1}\left(p - \frac{q}{2}\right), \quad (2.18b)
\end{align}

but for $\hat{H}_0^{\text{odd}}$ and $\hat{H}_{-1}^{\text{odd}}$ we get a system

\begin{align}
  i\dot{c}_{n+2}(p - q) &= g\sqrt{n + 2}e^{-i\phi_0 (p - \frac{q}{2}) t}c_{n+1}(p), \quad (2.19a) \\
  i\dot{c}_{n+1}(p) &= g\sqrt{n + 2}e^{i\phi_0 (p - \frac{q}{2}) t}c_{n+2}(p - q) + g\sqrt{n + 1}e^{-i\phi_0 (p + \frac{q}{2}) t}c_n(p + q), \quad (2.19b) \\
  i\dot{c}_n(p + q) &= g\sqrt{n + 1}e^{i\phi_0 (p + \frac{q}{2}) t}c_{n+1}(p). \quad (2.19c)
\end{align}

of three coupled differential equations. These systems of differential equations have been solved numerically for different $p$. The amplitudes $|c_n(p)|^2$ are plotted in Figure 2.8. In the common two-level system we get the expected Rabi oscillations between the photon states $|n\rangle$ and $|n + 1\rangle$. This goes along with a shift in momentum space by $q$. That is why the amplitudes of the photon state are shifted in Figure 2.8(a). As we see, the resonance occurs at $\pm q/2$.

For this set of parameters, which have been chosen to be far in the quantum regime, this resonance is quite narrow. In Figure 2.8(c), we zoomed into the resonance at $q/2$. In this figure it becomes clear, that a deviation from this resonance in momentum space acts as detuning and suppresses the Rabi oscillations.

In Figure 2.8(b) we plotted the solution of the three-level differential equations. As we can see, oscillations occur between momenta at $\pm q$. So we get a transition from the state $|n\rangle$ to $|n + 2\rangle$, which is a two photon transition. The intermediate level is not occupied at all. If we had used the formalism of adiabatic approximation, we could have eliminated this level adiabatically. So the approximation used in section 2.2 is numerically verified.

It is important to realize that the oscillations of the two-photon transition occur on a much larger time scale than the one of the two-level system. For sufficiently short interaction times, these higher resonances can be neglected as well.
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In addition to that, Figure 2.8(d) shows that the resonance occurring at \( \pm q \) is by some orders of magnitude narrower than the one at \( \pm q/2 \). This is no surprise, since as we saw in the discussion of the adiabatic approximation, the resonances become narrower for higher photon transitions. Overall, we see, that multi-photon processes are possible, but occur on a different time scale and the resonances become narrower.

In the following, we will concentrate just on a one-photon transition between the resonances \( \pm q/2 \), since we have seen, that this is the most dominating resonance for small interaction times. If the initial momentum is very close to one of the other resonances, we of course cannot make this limitation.
2.3 Multi-Photon Transitions

(a) Two-level system formed by the states $|n+1\rangle, |n+2\rangle$. We see Rabi oscillations between the resonances at $\pm q/2$ when we solve Eq. (2.19) numerically. Note that the time scale of these oscillations is much smaller than the one of Figure 2.8(b).

(b) Three-level system formed the states $|n\rangle, |n\rangle, |n+1\rangle$, when Eq. (2.19) is solved numerically. A resonance at $q$ can be seen, and the electron is scattered to $-q$. The intermediate level is not occupied.

(c) Close-up of the resonance at $q/2$ from Figure 2.8(a). Even though the resonance is fairly narrow, it is much broader than the resonance at $q$, shown in Figure 2.8(d).

(d) Close-up of the resonance at $q$ from Figure 2.8(b). The resonance is for this choice of parameters very narrow.

Figure 2.8: Numerical solutions of the differential equations resulting from the lowest order of $\hat{H}_\nu^{\text{even}}$ (blue) and $\hat{H}_\nu^{\text{odd}}$ (red). We plotted the amplitudes $|c_j(p)|^2$ with the parameters $m = 10^{-31}$ kg, $g = 5 \cdot 10^{13}$ s$^{-1}$, $\tau = 10^{-16}$ s, $k = 10^{10}$ m$^{-1}$, and $n = 10^3$. 

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2 Effective Two-Level Hamiltonian

2.4 Quantum Regime

Figure 2.9: Conditions for quantum behavior. For larger recoil and sufficiently focused electron momentum we see a two-level behavior in the adiabatic approximation. Resonances occur at the black horizontal lines.

So far, we have developed a simplified model for the FEL under certain approximations, in which we have an effective two-level behavior. This also gives our definition of the quantum regime: The dynamics of the FEL can be described by two interaction intervals in momentum space, that behave like a two-level system. Let us now recall the necessary assumptions for that.

In the adiabatic approximation we keep the dynamics of the terms that fulfill the condition Eq. (2.15)

\[ q^3 \left| \frac{(p/q)^2 - (\varphi/q)^2}{2} \right| \lesssim \frac{e^2 \hbar^2}{\varepsilon_0 V c^3 \sqrt{nmW}} \equiv \eta^3. \]

For a two-level behavior, i.e. for transitions back and forth between two states, the interaction intervals kept need to be separated by multiple integers of \( q \). To enforce this, we use the resonance condition Eq. (2.16)

\[ \varphi = \frac{\mu}{2} q, \quad \mu \in \mathbb{N}. \]

These two conditions are plotted in Figure 2.9. Here, the dynamics of those momenta
are kept, where the graphs are colored. The first thing we notice is that the dynamics get more and more limited for increasing recoil $q$. If the recoil approaches zero, there is no limitation at all, and we cannot derive the effective behavior. This corresponds to the classical case, where we have many multi-photon transitions simultaneously. But with a sufficiently large $q$ we get resonances at multiples of $q/2$, which are caused by quantum effects. Thus we call them quantum resonances. In contrast to that, there is no gain of intensity for electrons with initial momentum $p = 0$. We called this condition classical resonance. As we can see in the figure, this condition is also preserved in the quantum regime, since the interval around $p = 0$ does not split up into two parts. This is because for $\varphi = 0$ the quadratic nature is lost and we get a degenerated resonance. Hence, there is no dynamics.

As the intensity of the laser field increases, so does the right side of the inequality. The stronger the laser field, the more momenta are involved into the dynamics. So the FEL evolves from the quantum regime to the classical regime during the time of operation, even if the recoil cannot be neglected.

To display a two-level behavior, the initial electron momentum must be close to one and only one resonance. If the electron momentum distribution is not sharp enough, more than just one resonance are put into effect and we obtain a multi-level system. Hence, the electrons need momenta centered close to $\varphi$, i.e.

$$\langle p \rangle_{t=0} \approx \varphi = \frac{\mu}{2} q,$$

and the width of the initial electron distribution has to be so narrow that just one resonance is dominant, as already mentioned in Eq. (2.17), i.e.

$$\Delta p^2(t = 0) \equiv \langle (p - \langle p \rangle)^2 \rangle_{t=0} < \left( \frac{q}{2} \right)^2.$$

This can be easily seen from Figure 2.9.

With this thoughts we now also know, under which circumstances the RWA is a good approximation. The initial momentum needs to be close to the resonances at $\pm q/2$. Then, the dynamics of higher momenta are suppressed due to the large detuning. If the initial momentum is outside the interaction areas, we cannot neglect their interaction, and the system shows a multi-level dynamics. But the adiabatic approximation tells us, even in this case, that the intermediate levels can be eliminated and we have an effective two-level behavior, but with a multi-photon transition instead of a one-photon transfer.

In the following sections we will use the effective Hamiltonian of the RWA, i.e. we neglect the corrections to the phase $\phi_0(p)$, for the sake of simplicity. But by replacing
2 Effective Two-Level Hamiltonian

\[ \phi_0(p) \rightarrow \phi_0(p) + \Delta_2 - \Delta_1, \]
we can easily obtain the results from the adiabatic approximation. The only problem appears in the phase factors, which occur in the off-diagonal elements of the density matrix. But, as we will discuss later, they can be neglected. In addition to that we will see in section 4.1.2 that even without these phase corrections, the results in this approximation are pretty good and match the findings when solving the time evolution with perturbation theory.

In section 1.2.1 we already discussed whether the name FEL is justified. We argued, that it emits coherent radiation and the emission of light is stimulated by a self-consistent ponderomotive potential caused by the radiation of all electrons, even though it can be treated classically.

With that in mind, the quantum regime gives further insight into the properties of an FEL. The electrons make transitions between different levels in momentum space, that correspond to the transitions between energy levels of a bound electron in an conventional laser. In the quantum regime certain momentum levels are suppressed. In analogy to an atom, where we have discrete energy levels, the main interaction occurs between momenta where the electrons fulfill the resonance condition. So, in a matter of saying, the momenta of the electrons are quantized because the classical interaction is restricted. Hence in this regime, the term laser is justified, since it actually has the properties that define a conventional laser.

To treat a conventional laser, one uses some kind of time averaging [21, 25]. By that, some quantum features are kept, such as stimulated emission. On the other hand, some features are lost, such as trapping states [18]. Without the time average the device is often called a one-atom laser, which already has been realized experimentally [26].

In the quantum regime, an FEL behaves like a one-atom laser, and, as we will see in section 4.2.1, the characteristic classical gain curve is just a result of the special properties of such a quantum laser. The FEL in the quantum regime cannot be considered as a semi-classical device, since, in contrast to the atomic levels of a conventional laser, the lifetime of a momentum state is not finite. That is why the quantum features of this system are preserved, and there is no averaging process, as in conventional laser theory [21].
3 Properties of the Laser Field

In the previous chapter we have derived an effective two-level Hamiltonian for the quantum regime. This Hamiltonian leads to a model where the time evolution can be calculated exactly. We will derive in this chapter an equation of motion for the radiation field by tracing out the electrons’ momenta. Once we found a difference-differential equation of the photon statistics we look at its properties. Since this model is highly nonlinear in the photon number \( n \), we can find a steady state solution. The intensity and the variance are calculated for this steady state and by that it is shown that the photon distribution can display a sub-Poissonian behavior.

3.1 Equation of Motion of the Radiation Field

We now calculate the evolution of the laser field in the quantum regime, where we can use an effective Hamiltonian, as discussed in chapter 2. Applying the Schrödinger equation (2.9) leads to differential equations which can be solved analytically, as we have seen in section 2.1 and appendix B. We now use the solution of the Rabi equations (2.11) without the correction terms of the adiabatic approximation occurring from higher resonances, to get the equation of motion of the field.

We start with an arbitrary initial state

\[
|\psi(t)\rangle = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} dp \ c_n(p; t) |n, p\rangle,
\]

which has evolved after an interaction time \( \tau \) into

\[
|\psi(t + \tau)\rangle = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} dp \ c_n(p; t + \tau) |n, p\rangle.
\]

We know the coefficients \( c_n(p; t + \tau) \) of the time evolved state, since they follow directly from the solution of the differential equations from section 2.1. To calculate the reduced
3 Properties of the Laser Field

density matrix of the laser field $\hat{\rho}_L$ we trace over the electron momenta and arrive at

$$\hat{\rho}_L(t + \tau) = \text{Tr}_p \{ |\psi(t + \tau)\rangle \langle \psi(t + \tau)| \} = \int_{-\infty}^{\infty} dp \langle p|\psi(t + \tau)\rangle \langle \psi(t + \tau)|p \rangle$$

$$= \sum_{n,m=0}^{\infty} \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dp' \int_{-\infty}^{\infty} dp'' c_n(p';t + \tau)c_m^*(p'';t + \tau) \langle p|n,p'\rangle \langle m,p''|p \rangle$$

$$= \sum_{n,m=0}^{\infty} \int_{-\infty}^{\infty} dp c_n(p;t + \tau)c_m^*(p;t + \tau) \langle n|\langle m| \equiv \sum_{n,m=0}^{\infty} \varrho_{n,m}(t + \tau) |n\rangle \langle m|.$$ 

We now take a closer look at the time evolution of the matrix elements $\varrho_{n,m}(t + \tau)$. For that, we divide the integration over $p$ up into four intervals, and then the matrix elements read

$$\varrho_{n,m}(t + \tau) = \int_{-\infty}^{\infty} dp c_n(p;t + \tau)c_m^*(p;t + \tau)$$

$$= (\int_{-q}^{0} dp + \int_{0}^{q} dp + \int_{0}^{\infty} dp + \int_{-\infty}^{-q} dp) c_n(p;t + \tau)c_m^*(p;t + \tau).$$

As we know from section 2.1, there is a different solution of the time evolution of the coefficients in each interval. If we assume that the systems of electrons and photons are separable at the beginning of the interaction at time $t$, we can define

$$c_n(p;t)c_m^*(p';t) = \varrho_{n,m}(t) \varrho(p,p')$$

as the initial matrix elements. The time evolution is calculated in appendix D. There, we make the assumption that we are in the quantum regime, and the electron’s momentum is so sharp that we can neglect off-diagonal terms of the initial momentum density matrix with $\varrho(p \pm q, p)$. Analogously to Eq. (2.12),

$$\Omega^\pm_n \equiv \sqrt{\frac{\phi^\pm}{2} + g^2(n + 1)}$$
defines the generalized Rabi frequency. According to appendix D,
\[ \delta \varrho_{n,n} \equiv \varrho_{n,n}(t + \tau) - \varrho_{n,n}(t) \]
\[ = - \varrho_{n,n}(t) \left\{ g_n^2 \int_{-q}^{0} dp \, \varrho(p) \frac{\sin^2 \Omega_{n-1}^+ \tau}{\Omega_{n-1}^+} + g_n^2 (n + 1) \int_0^q dp \, \varrho(p) \frac{\sin^2 \Omega_{n}^- \tau}{\Omega_{n}^-} \right\} \]
\[ + \varrho_{n+1,n+1}(t) g_n^2 (n + 1) \int_{-q}^{0} dp \, \varrho(p) \frac{\sin^2 \Omega_{n-1}^+ \tau}{\Omega_{n-1}^+} \]
\[ + \varrho_{n-1,n-1}(t) g_n^2 n \int_0^q dp \, \varrho(p) \frac{\sin^2 \Omega_{n-1}^- \tau}{\Omega_{n-1}^-} \]  
(3.3)
is the change of the matrix elements expressed through the Rabi frequency Eq. (3.2). With that, we can develop a Scully-Lamb theory [21] of the FEL analogously to [6]. If electrons are injected with a rate \( r = 1/\Delta t \), and the field varies slowly during the interaction time \( \tau \), we can approximate the change of the density matrix elements during a time interval \( \Delta t \) by
\[ \Delta \varrho_{n,n} \approx r \Delta t \delta \varrho_{n,n} , \]
if \( \tau \lesssim \Delta t \). This approximation is possible if the change of the matrix elements caused by only one electron is not significant [6]. Taking now the coarse-grained derivative
\[ \frac{d \varrho_{n,n}}{dt} \approx \frac{\Delta \varrho_{n,n}}{\Delta t} = r \delta \varrho_{n,n} \]
as in [21] for the conventional laser theory, this leads to the equation of motion
\[ \dot{W}_n = - W_n \left\{ n \mathcal{R}_n^+ + (n + 1) \mathcal{R}_{n+1}^- \right\} + W_{n+1} (n + 1) \mathcal{R}_{n+1}^+ + W_{n-1} n \mathcal{R}_n^- \]  
(3.4)
for the radiation field. In this equation, we renamed \( W_n \equiv \varrho_{n,n} \) to emphasize on the interpretation of the diagonal elements of the radiation field, since \( W_n \) corresponds to the probability of finding \( n \) photons in the cavity.

In Eq. (3.4) the rate coefficients
\[ \mathcal{R}_n^- \equiv r g_n^2 \int_0^q dp \, \varrho(p) \left( \frac{\sin \Omega_{n-1}^- \tau}{\Omega_{n-1}^-} \right)^2 \]  
(3.5a)
\[ \mathcal{R}_n^+ \equiv r g_n^2 \int_{-q}^0 dp \, \varrho(p) \left( \frac{\sin \Omega_{n-1}^+ \tau}{\Omega_{n-1}^+} \right)^2 \]  
(3.5b)
3 Properties of the Laser Field

were defined. The equation of motion is visualized in Figure 3.1, where the interpretation of this equation becomes clearer: The population of the state $|n\rangle$ changes by the probability flow from and to the two neighboring levels $|n+1\rangle$ and $|n-1\rangle$. The flow from and to all other levels is suppressed, since in our approximation we only consider one-photon transitions. We can interpret the summands of Eq. (3.4) as follows:

$$\begin{pmatrix}
\text{Probability} \\
\text{flow from state} \\
|n\rangle \text{ to } |n+1\rangle
\end{pmatrix}
= \frac{W_n}{ \text{Occupation of the initial level}} \times \frac{\langle n+1 \rangle}{\text{Photons in the cavity}} \times \frac{R_{n+1}^-}{\text{Rate coefficient of emission to level } |n+1\rangle}$$

Hence, we see that the probability flow to the upper level by emission depends on the initial occupation of the level times the number of photons present plus one, times the rate coefficient. Stimulated emission is always proportional to the number of photons present, the plus one can be explained with spontaneous emission. We see from the definition (3.5a) of the rate coefficient $R^-$, that it is just nonzero in the gain interval. This is evident, since the occupation of $|n\rangle$ decreases due to an increase of $|n+1\rangle$. By that, the mean photon number increases and we have a gain of laser intensity.
3.1 Equation of Motion of the Radiation Field

The process of absorption can be seen as:

\[
\left( \text{Probability flow from state } |n\rangle \text{ to } |n-1\rangle \right) = W_n \times \frac{n}{n} \times R^+ \times \frac{R^+}{R^-}.
\]

The only difference is that we have no process corresponding to spontaneous emission and so we just multiply by the photon number. The rate coefficient describing the loss of photons is now \( R^+ \), and by that just nonzero in the loss interval below classical resonance, as we see from Eq. (3.5b).

We now take a closer look at these rate coefficients. For that, we assume the electron to be initially in a momentum eigenstate, i.e. \( \varrho(p) = \delta(p - p_0) \). Hence, the integrations in Eq. (3.5) can be carried out easily.

![Figure 3.2: Plot of the rate coefficients for \( n = 11 \) photons and electron momentum eigenstates. The loss rate coefficient is blue, the gain rate coefficient is red. The parameters are: \( m = 10^{-31} \text{ kg}, g = 5 \cdot 10^{13} \text{ s}^{-1}, \tau = 10^{-16} \text{ s}, \) and \( k = 10^{10} \text{ m}^{-1}. \)](image)

Throughout this thesis we will use always the same set of parameters, so we can compare all figures. We choose \( m = 10^{-31} \text{ kg}, g = 5 \cdot 10^{13} \text{ s}^{-1}, \tau = 10^{-16} \text{ s}, \) and \( k = 10^{10} \text{ m}^{-1}. \) These parameters correspond to the quantum regime, because we have a fairly small coupling constant and a large wave vector, which implies a large recoil.

With this choice of parameters, we get a separation of the gain and the loss interval,
3 Properties of the Laser Field

as we see in Figure 3.2 and 3.3. For a small coupling constant, the Rabi frequency \( \Omega_n^{\pm}(p) \) does not vary much for small \( n \). We plotted the rate coefficients \( R_n^{\pm} \) for \( n = 11 \) in Figure 3.2. The coefficients behave in momentum like \( \sin(p)/p \) centered around the resonances at \( \pm q/2 \). The probability flow is the maximal for momenta at those resonances. For our choice of parameters in the quantum limit we see that these functions decline rapidly in \( p \) and there is almost no probability flow any more, when we approach the border of our interaction intervals. Even if we extended the integrals in Eq. (3.5) to \( \pm \infty \), there would be no significant different behavior and interaction outside of \( (-q,q) \) in comparison to the maxima.

![Figure 3.3: Loss rate coefficient (blue) and gain rate coefficient (red) for different momenta and photon numbers. The main maxima decrease and momenta far from the resonance become more important. The parameters are: \( m = 10^{-31} \text{kg}, g = 5 \cdot 10^{13} \text{s}^{-1}, \tau = 10^{-16} \text{s}, \) and \( k = 10^{10} \text{m}^{-1} \).](image)

With increasing intensity the Rabi frequencys \( \Omega_n^{\pm} \) change and by that the rate coefficients behave differently. Their dependence on \( n \) and \( p \) is plotted in Figure 3.3. For small photon numbers, we see the behavior as discussed above. But as the photon number increases, the main maxima at the resonances become less important. At very high \( n \) the interaction close to the borders of the interaction intervals cannot be neglected any longer in comparison to the resonance. As already mentioned in section 2.4, when we discussed the limits of our model, the two-level approximation breaks down for very large intensities. On the other hand, this dependence of the coefficients on \( n \) is important, since it is nonlinear and only because of that we can construct a steady state. We will discuss this steady state in section 3.4 in more detail.
3.2 Time Evolution of the Photon Statistics

So far, we derived the equation of motion for the laser field with the effective Hamiltonian motivated in chapter 2. In this consideration we neglected all effects due to cavity losses. To get a full and more accurate description, we of course need to include these losses.

So we add the usual term with a cavity damping of $\omega/Q$ to the equation of motion. To get a better understanding of the appearance of these cavity losses, we recapitulated them in appendix C. So when we add the right side of Eq. (C.2) to the right side of Eq. (3.4), we arrive at

$$
\dot{W}_n = - W_n \left[ \frac{\omega}{Q} \left( \frac{1}{2} + \frac{1}{2} \right) n \right] + W_{n+1}(n+1) \left[ \frac{\omega}{Q} (n_{th} + 1) \right] + W_{n-1}(n) \left[ \frac{\omega}{Q} n_{th} \right],
$$

where $n_{th}$ denotes the thermal photon number and $Q$ the quality of the cavity.

We now have a description of the development of the FEL’s radiation field in the quantum regime with inclusion of cavity losses. Prepared with that, we can turn to the further investigation of the photon statistics in the next section.

3.2 Time Evolution of the Photon Statistics

The difference-differential equation cannot be solved analytically in an exact way for arbitrary initial conditions. In this section, we show the time evolution based on an iterative method. Remembering that Eq. (3.6) was obtained by using a coarse-grained derivative, we can write

$$
W_n(t + \Delta t) = W_n(t) - W_n(t) \Delta t \left[ n \mathcal{R}_n^+ + (n + 1) \mathcal{R}_{n+1}^- + \frac{\omega}{Q} \left( \frac{1}{2} + \frac{1}{2} \right) n \right] + W_{n+1}(t) \Delta t (n + 1) \left[ \mathcal{R}_{n+1}^+ + \frac{\omega}{Q} (n_{th} + 1) + W_{n-1}(t) \Delta t n \left[ \mathcal{R}_n^- + \frac{\omega}{Q} n_{th} \right]
$$

for the time evolution of the laser field. Starting from the vacuum, i.e. $W_n(0) = \delta_{n,0}$, we iteratively solve this equation and calculate $W_n(N \cdot \Delta t)$ for an initial electron momentum eigenstate with $p = 0.45 \, q$. Figure 3.4 shows the result. As we can see, the vacuum state is quickly broadened and evolves into a peaked photon distribution. At some point in time, this distribution does not change any more. This gives rise to the conjecture, that a steady state solution exists. We will find an analytical expression for that photon distribution in section 3.4.
3 Properties of the Laser Field

Figure 3.4: Build-up of the laser field from the vacuum state at zero temperature. The difference-differential equation (3.6) is iteratively solved for $\Delta t = 15/r = 15\tau$, $\omega/(Qr) = 1.745 \cdot 10^{-5}$, and $p = 0.45 q$. We see, that the photon statistics evolves into a steady state. The parameters are: $m = 10^{-31}$ kg, $g = 5 \cdot 10^{13}$ s$^{-1}$, $\tau = 10^{-16}$ s, and $k = 10^{10}$ m$^{-1}$.

Note that this iteration was done at zero temperature, i.e. $n_{th} = 0$. In this case, trapping states occur, as for one-atom lasers [18]. These states are characterized by a very sharp photon distribution. Since at zero temperature the probability flows can become zero at certain photon numbers, a state is trapped between those points and cannot evolve into other regions of $n$.

If we had taken a nonzero temperature, the iteration scheme would have broken down for this set of parameters. Since the terms due to cavity damping were obtained with perturbation theory and thus have just a linear behavior in $n$, they would become too dominant in contrast to the terms from the interaction of the electron with the laser field.

3.3 Photon Number and Variance

One of the main properties of a radiation field is its intensity, i.e. the mean photon number. So far, we did not compute a steady state. But the time evolution of the laser field before it reaches a steady state is of importance, too, since it describes the small signal gain.
3.3 Photon Number and Variance

So the change of the mean photon number in time is with the help of Eq. (3.6)

\[
\frac{d}{dt} \langle n(t) \rangle = \sum_{n=0} W_n n
\]

\[
= \sum_{n=0} -W_n n \left[ n \mathcal{R}_n^+ + (n+1) \mathcal{R}_{n+1}^- + \frac{\omega}{Q} \left( 2n_{th} \left( n + \frac{1}{2} \right) + n \right) \right]
\]

\[
+ \sum_{n=0} W_{n+1} n(n+1) \left[ \mathcal{R}_{n+1}^+ + \frac{\omega}{Q} (n_{th} + 1) \right] + \sum_{n=0} W_{n-1} n^2 \left[ \mathcal{R}_n^- + \frac{\omega}{Q} n_{th} \right].
\]

Performing a shift \( n \rightarrow n - 1 \) of the index of summation in the second sum and \( n \rightarrow n + 1 \) in the third sum we get the equation

\[
\frac{d}{dt} \langle n \rangle = \sum_{n=0} W_n \left\{ \left[ -n^2 + n(n-1) \right] \mathcal{R}_n^+ + \left[ -n(n+1) + (n+1)^2 \right] \mathcal{R}_{n+1}^- \right\}
\]

\[
+ \sum_{n=0} W_n \frac{\omega}{Q} \left\{ -n_{th} n(2n+1) + n^2 + n_{th} (n+1)^2 + (n_{th} + 1) n(n-1) \right\}
\]

\[
= \sum_{n=0} W_n \left\{ -n \mathcal{R}_n^+ + (n+1) \mathcal{R}_{n+1}^- + \frac{\omega}{Q} (-n + n_{th}) \right\}
\]

\[
= -\langle n \mathcal{R}_n^+ \rangle + \langle (n+1) \mathcal{R}_{n+1}^- \rangle - \frac{\omega}{Q} \langle n \rangle + \frac{\omega}{Q} n_{th}
\]

(3.7)

of the mean photon number. It has to be mentioned that this is not a differential equation in the usual way, since it depends on averages of functions of \( n \) rather than on functions of averages of \( n \). Thus, we cannot calculate a simple solution of this equation. Nevertheless, this form will turn out to be useful when we look at the steady state statistics. Furthermore, this equation is very catchy, since it can be interpreted easily:

\[
\frac{d}{dt} \langle n \rangle = -\langle n \mathcal{R}_n^+ \rangle + \langle (n+1) \mathcal{R}_{n+1}^- \rangle - \frac{\omega}{Q} \langle n \rangle + \frac{\omega}{Q} n_{th}
\]

(3.8)

Change of mean photon number

Mean losses due to absorption

Mean gain due to stimulated + spontaneous emission

Mean loss due to cavity losses

Gain due to thermal photons

Note that the gain due to thermal photons is independent of the photons in the cavity present, but just depends on the temperature that defines the thermal photon number \( n_{th} \).

This change of the photon number in time corresponds to the change of intensity of the laser field. Thus, we can call this function the \textit{gain function}

\[
\mathcal{G} \equiv \frac{d}{dt} \langle n \rangle
\]
in accordance to its classical counterpart $G_c$ from section 1.2.3. Note, that in contrast to that we now directly calculated the gain, and not by using the loss of kinetic energy of the electrons.

Figure 3.5: Evolution of the mean photon number $\langle n \rangle$ in time. To calculate it numerically, we used the time evolution of the photon statistics of Figure 3.4 with an initial momentum of $p = 0.45 q$ and $\omega/(Qr) = 1.745 \cdot 10^{-5}$ at zero temperature. In section 3.4 we will show that a steady state solution exists. This is in accordance to the saturation of the photon number in this figure at $\langle n \rangle^\text{st}$. The parameters are: $m = 10^{-31}$ kg, $g = 5 \cdot 10^{13}$ s$^{-1}$, $\tau = 10^{-16}$ s, and $k = 10^{10}$ m$^{-1}$.

In Figure 3.5, we plotted the time evolution of the mean photon number. For that, we used the same iteration as performed in section 3.2. Not surprisingly, the mean photon number and by that the intensity first increases rapidly, until it saturates. This is one more hint that we expect a steady state solution.

The change in time of the variance $\langle \Delta n^2 \rangle$ with $\Delta n \equiv n - \langle n \rangle$ can be calculated as in [28]. With Eq. (3.6) and the sum shifts as above when we derived the time evolution of
the photon number, we get

$$\frac{d}{dt} \langle \Delta n^2 \rangle = \sum_{n=0} W_n \Delta n^2$$

$$= - \sum_{n=0} W_n \Delta n^2 \left[ n R_n^+ + (n+1) R_{n+1}^- + \frac{\omega}{Q} \left( 2n_{th} \left( n + \frac{1}{2} \right) + n \right) \right]$$

$$+ \sum_{n=0} W_{n+1} \Delta n^2 (n+1) \left[ R_{n+1}^+ + \frac{\omega}{Q} (n_{th} + 1) \right]$$

$$+ \sum_{n=0} W_{n-1} \Delta n^2 n \left[ R_n^- + \frac{\omega}{Q} n_{th} \right]$$

$$= \sum_{n=0} W_n \left[ n (-\Delta n^2 + \Delta(n-1)^2) R_n^+ + (n+1) (-\Delta n^2 + \Delta(n+1)^2) R_{n+1}^- \right]$$

$$+ \sum_{n=0} W_n \frac{\omega}{Q} \left[ -\Delta n^2 2n_{th} \left( n + \frac{1}{2} \right) - \Delta n^2 n \right.$$  

$$+ \Delta(n-1)^2 n(n_{th} + 1) + \Delta(n+1)^2 (n_{th} + 1) \left. \right]$$

for the change of the variance in time. Since we know that $\langle n \rangle$ is independent of the summation index, we obtain the relation

$$\Delta(n \pm 1)^2 = (n \pm 1 - \langle n \rangle)^2 = 1 + (n - \langle n \rangle)^2 \pm 2(n - \langle n \rangle^2) = \Delta n^2 + 1 \pm 2 \Delta n$$

and can rewrite the time derivative of the variance as

$$\frac{d}{dt} \langle \Delta n^2 \rangle = \langle n R_n^+ (-2 \Delta n + 1) \rangle + \langle (n+1) R_{n+1}^- (2 \Delta n + 1) \rangle$$

$$+ \frac{\omega}{Q} \langle -2n \Delta n + 2n_{th} \Delta n + 2(n_{th} + 1)n + n_{th} \rangle$$

$$= -2 \langle n R_n^+ \Delta n \rangle + 2 \langle (n+1) R_{n+1}^- \Delta n \rangle + \langle n R_n^+ \Delta n \rangle + \langle (n+1) R_{n+1}^- \Delta n \rangle$$

$$-2 \frac{\omega}{Q} \langle n^2 - n \langle n \rangle \rangle + 2n_{th} \frac{\omega}{Q} \langle n - \langle n \rangle \rangle + (2n_{th} + 1) \frac{\omega}{Q} \langle n \rangle + \frac{\omega}{Q} n_{th}$$

$$= -2 \langle n R_n^+ \Delta n \rangle + 2 \langle (n+1) R_{n+1}^- \Delta n \rangle + \langle n R_n^+ \Delta n \rangle + \langle (n+1) R_{n+1}^- \Delta n \rangle$$

$$-2 \frac{\omega}{Q} \langle \Delta n^2 \rangle + (2n_{th} + 1) \frac{\omega}{Q} \langle n \rangle + \frac{\omega}{Q} n_{th}. \quad (3.9)$$

Though Eq. (3.9) is not a differential equation in the common sense, these results are used to investigate the steady state photon statistics in the next section.
3 Properties of the Laser Field

3.4 Steady State of the Photon Statistics

Since the rate coefficients $R_n^\pm$ are highly nonlinear in $n$, we can find a steady state solution, as shown in [21, 25]. For that, we first rearrange the terms of Eq. (3.6) to

$$
\dot{W}_n = (n + 1) \left\{ -W_n \left[ R_{n+1}^- + \frac{\omega}{Q} n_{\text{th}} \right] + W_{n+1} \left[ R_{n+1}^+ + \frac{\omega}{Q} (n_{\text{th}} + 1) \right] \right\} \\
- n \left\{ -W_{n-1} \left[ R_n^- + \frac{\omega}{Q} n_{\text{th}} \right] + W_n \left[ R_n^+ + \frac{\omega}{Q} (n_{\text{th}} + 1) \right] \right\} \\
\equiv S_{n+1} - S_n 
$$

where $S_n$ is defined as

$$
S_n \equiv \left\{ -W_{n-1} \left[ R_n^- + \frac{\omega}{Q} n_{\text{th}} \right] + W_n \left[ R_n^+ + \frac{\omega}{Q} (n_{\text{th}} + 1) \right] \right\}.
$$

One condition to get a steady state solution, i.e. $\dot{W}_n = 0$, is to have $S_{n+1} = S_n$ for all $n$. Since obviously $S_0 = 0$, this means $S_n = 0$ for all $n$. This condition is called detailed balance, since the flow between the different levels is balanced out and there is no net change of the probability distribution. It describes a state where the probability flow between two neighboring photon states is exactly equal to zero. The condition $S_n = 0$ can be written as

$$
W_n = \frac{R_n^- + \frac{\omega}{Q} n_{\text{th}}}{R_n^+ + \frac{\omega}{Q} (n_{\text{th}} + 1)} W_{n-1} \equiv \Lambda_n W_{n-1} \quad \forall n \in \mathbb{N}.
$$

Simple induction of this recursion formula yields the steady state photon statistics

$$
W\text{_{stat}}^n = W\text{_{stat}}^0 \prod_{n'=1}^{n} \Lambda_{n'} \quad (3.10)
$$

and the normalization condition

$$
W\text{_{stat}}^0 = \frac{1}{1 + \sum_{n=1}^{\infty} \prod_{n'=1}^{n} \Lambda_{n'}} \quad (3.11)
$$

gives the vacuum probability. So we have found a steady state solution, as predicted from the iteration in section 3.2 where the time evolution stopped at a certain photon statistics. Figure 3.6 shows the photon statistics after the iteration compared to the analytical expression Eq. (3.10). As we can see, the iteratively found photon distribution
3.4 Steady State of the Photon Statistics

corresponds in fact to our analytical expression, even though we did just do a finite set of iteration steps. For more iteration steps, both curves would match even better.

\[
W_{\text{st}}^n = \frac{\gamma_n^{\text{th}}}{\mathcal{R}_n^+ + \gamma_n^{\text{th}}(\gamma_{\text{th}} + 1)} < 1.
\]

At zero temperature, \( \Lambda_n = 0 \) and hence, just the vacuum state will be occupied. Even for finite temperatures we get a thermal-type photon distribution with its maximum at \( n = 0 \).
3 Properties of the Laser Field

![Figure 3.7: Steady state photon statistics $W_{n}^{\text{st}}$ for electrons in a momentum eigenstate $p$ in the gain interval for an detuning of $\omega/(Qr) = 10^{-6}$ at a finite temperature with $n_{\text{th}} = 5 \cdot 10^4$. Sharp maxima occur for momenta close to the resonance at $q/2$. The parameters are: $m = 10^{-31}$ kg, $g = 5 \cdot 10^{13}$ s$^{-1}$, $\tau = 10^{-16}$ s, and $k = 10^{10}$ m$^{-1}$.]

Of course, the main purpose of a laser is to gain light intensity and to have a mean photon number much larger than zero. So the case of electrons initially in the gain interval is more interesting, since usually the laser will operate in this regime. We see, that for this set of parameters, the photon distributions in Figure 3.7 have maxima and are dominated by one peak. In the following sections we try to find these features analytically.

At zero temperature, i.e. $n_{\text{th}} = 0$, we see that $\Lambda_{n}$ can turn zero at certain values of $n$ for momentum eigenstates, since the numerator is just $R_{n}^{-} \propto \sin^{2} \Omega_{n} \tau$. In this case, the previously mentioned trapping states occur, as described in [18] for the one-atom laser.

3.4.1 Maxima

In this section, the extrema of the steady state are determined as in [28, 29, 30, 31]. We can write the photon statistics from Eq. (3.10) as

$$
\frac{W_{n}^{\text{st}}}{W_{0}^{\text{st}}} = \exp \left[ \ln \prod_{n' = 1}^{n} \Lambda_{n'} \right] = \exp \left[ \sum_{n' = 1}^{n} \ln \Lambda_{n'} \right] \approx \exp \left[ \int_{1}^{n} \ln \Lambda(n') \, dn' \right], \quad (3.12)
$$
where in the last step we used a natural continuous approximation and replaced the sum by an integral. To get the extrema $n_j$ of this photon statistics, we set the derivative with respect to the photon number $n$ equal to zero

$$
\frac{d}{dn} \left. W_{n_j}^{\text{st}} \right|_{n_j} = \exp \left[ \int_1^{n_j} \ln \Lambda(n') \, dn' \right] \ln \Lambda(n_j) = 0.
$$

This is just true for $\Lambda(n_j) = 1$, i.e.

$$
\mathcal{R}_{n_j}^- - \mathcal{R}_{n_j}^+ = \frac{\omega}{Q}.
$$

(3.13)

For a given initial electron momentum distribution, this equation can be solved numerically for $n_j$.

Even though the quantum condition restricts ourselves to momentum distributions sharply peaked around $\pm q/2$, we can suppose that extrema just occur if $\mathcal{R}_{n_j}^- > \mathcal{R}_{n_j}^+$. So in general, we can say that there need to be more electrons above than below classical resonance or, in a matter of speaking, more electrons in the gain than in the loss region. This corresponds to the condition of inversion in a conventional laser.

Figure 3.8: Graphical method to determine the extrema of the steady state distribution. For each horizontal plane $\omega/(Qr)$ we find intersections with $\mathcal{R}_{n_j}^+/r$. At those points, extrema occur. The better the quality $Q$ of the cavity, the lower is the horizontal plane and more extrema may occur due to the oscillatory behavior in $n$. The parameters are: $m = 10^{-31}$ kg, $g = 5 \cdot 10^{13}$ s$^{-1}$, $\tau = 10^{-16}$ s, and $k = 10^{10}$ m$^{-1}$. 

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Let us now assume electrons having momenta in the gain interval, i.e. \( R^+ = 0 \). Condition (3.13) then reads \( R_j^{-} = \omega/Q \). Unfortunately, this equation also cannot be solved analytically.

We want to discuss a special case where it is possible to find a solution of this equation graphically. This is the case for initial electron momentum eigenstates. So we set \( \varrho(p) = \delta(p - p_0) \) and get \( R_j^{-} = r g^2 [\sin(\Omega_j \tau)/\Omega_j]^2 \), as already discussed in section 3.1. This function is plotted in Figure 3.8 in dependence of both variables \( p \) and \( n \). As we see, it is declining and oscillating for increasing photon number \( n \). If this function intersects a horizontal plane in the height \( \omega/Q \), we get the extrema of the steady state distribution. This intersection is shown for \( p = 0.45 q \) in Figure 3.9.

Due to the periodicity of the sine-function in \( n \), it is possible to have more than just one extremum, as the figure shows. To determine whether the extrema are maxima or...
3.4 Steady State of the Photon Statistics

minima, we now take the second derivative with respect to \( n \) and arrive at

\[
\frac{d^2}{dn^2} \left. W_{\text{st}}^n \right|_{n_j} = \exp \left[ \int_1^{n_j} \ln \Lambda(n') \right] \left[ \ln^2 \Lambda(n_j) + \frac{1}{\Lambda(n_j)} \frac{d}{dn} \Lambda(n) \right]_{n_j},
\]

where in the last step we used the identity \( \Lambda(n_j) = 1 \). For electrons with momentum in the gain interval we get

\[
\left. \frac{d}{dn} \Lambda(n) \right|_{n_j} = \frac{1}{\sqrt{2}(n_{\text{th}} + 1)} \left. \frac{d}{dn} R_n \right|_{n_j}.
\]

This means, that if

\[
\left. \frac{d}{dn} R_n \right|_{n_j} > 0
\]

the extrema are minima and if

\[
\left. \frac{d}{dn} R_n \right|_{n_j} < 0
\]

maxima occur at \( n_j \). Graphically speaking, we get minima at intersections where \( R_n^- \) is ascending, and maxima at intersections where \( R_n^- \) is descending. This is also shown in Figure 3.9 for our case of \( p = 0.45 q \).

We can also see in this figure, that for a momentum close to resonance more maxima occur, the better the quality \( Q \) of the cavity is. If the quality is too bad, there is no intersection at all. Hence, in general it is possible to have no maximum, if the cavity damping is too large as we see from Figure 3.3. We can also see that just one maximum may occur, e.g. for a cavity quality \( Q_1 \) in Figure 3.9.

To verify this result, we plotted the photon statistics \( W_{\text{st}}^n \) from Eq. (3.10) in Figure 3.10. It shows, that both maxima occur as predicted by our scheme. But, as we see, the first maximum is much larger than the second one and by that it dominates the properties of the laser field. We will investigate this further in the next section, where we try to see directly from Eq. (3.10) whether one of the peaks is dominating.
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Figure 3.10: The orange curve shows the photon statistics $W_n$ from Eq. (3.10) for an initial momentum of $p = 0.45 q$ and a cavity damping of $\omega/(Qr) = 10^{-6}$ at a finite temperature with $n_{th} = 5 \cdot 10^4$. We see, that the maxima occur at the predicted photon numbers from Figure 3.9. To visualize this, the green and red curve are qualitatively drawn again. The parameters are: $m = 10^{-31}$ kg, $g = 5 \cdot 10^{13}$ s$^{-1}$, $\tau = 10^{-16}$ s, and $k = 10^{10}$ m$^{-1}$.

3.4.2 Dominating Maxima

We now take a closer look at a cavity damping where we have more than one maximum. Let two neighboring peaks be at $n_1$ and $n_2$, without loss of generality $n_1 < n_2$. We try to see whether one of them can be dominating. For that, we take the ratio

$$\frac{W_{n_2}}{W_{n_1}} = \frac{W_n^{st}}{W_n^{st}} \prod_{n'=1}^{n_2} \Lambda_{n'} \approx \exp \left[ \int_{1}^{n_2} dn' \ln \Lambda(n') - \int_{1}^{n_1} dn' \ln \Lambda(n') \right]$$

$$= \exp \left[ \int_{n_1}^{n_2} dn' \ln \Lambda(n') \right],$$

where we again used the continuous approximation. If the exponent is much larger than zero, maximum $n_2$ is dominating, if it is much smaller, maximum $n_1$ is dominating.
3.4 Steady State of the Photon Statistics

Figure 3.11: The black curve shows the integrand \( \ln \Lambda(n) \) for the momentum eigenstate with \( p = 0.45q \) and a cavity damping of \( \omega/(Qr) = 10^{-6} \) at a finite temperature with \( n_{th} = 5 \cdot 10^4 \). The integration goes from one maximum to the other (marked through the intersection of the green and red curves from Figure 3.9, which are drawn qualitatively). We see that the extrema of the photon statistics correspond to \( \ln \Lambda(n_j) = 0 \). If \( \ln \Lambda(n_j) \) is descending, we have maxima, if it is ascending, we get a minimum. For this choice of parameters, the exponent will be negative, and hence the first maximum dominating. The parameters are: \( m = 10^{-31} \text{ kg}, \ g = 5 \cdot 10^{13} \text{ s}^{-1}, \ \tau = 10^{-16} \text{ s}, \) and \( k = 10^{10} \text{ m}^{-1} \).

The integrand \( \ln \Lambda(n) \) is plotted in Figure 3.11 for our example of \( p = 0.45q \). In this figure, we see that the negative area under the filled curve is larger than the positive area, and thus we know that the exponent is much smaller than zero and hence the first maximum is dominating. This is in accordance with Figure 3.10 where the first maximum is much larger than the second one.

Looking at the figure, we see the connection between the function \( R_n^- \) and the integrand \( \ln \Lambda_n \). If \( R_n^- \) is crossed by \( \omega/Q \), the integrand is zero, and hence we expect a maximum.

In general, we conclude that it is possible to have more than one maximum, but one maximum might be dominating. This will be of importance in the next sections where we develop the interesting variables around a dominating peak.
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3.4.3 Steady State Photon Number

In this section, we assume the electrons to be above the classical resonance energy, i.e. \( R_n^+ = 0 \). In steady state, the mean photon number does not change. So we can see at once

\[
\langle (n + 1)R_{n+1}^- \rangle^{\text{st}} = \frac{\omega}{Q}\langle n \rangle^{\text{st}} - \frac{\omega}{Q}n_{\text{th}}
\]  

(3.14)

from the time evolution of the mean photon number Eq. (3.7). As discussed in 3.4.1, more than just one maximum might occur because of the periodicity of \( \sin^2 \Omega - n \tau \). We now look at the case where there is only one dominating maximum, as discussed in the previous section. It is intuitive to assume the steady state photon number to be close to this maximum and expand

\[
\langle (n + 1)R_{n+1}^- \rangle^{\text{st}} = \left( \langle n \rangle^{\text{st}} + 1 \right)R_{n+1}^- + \frac{d}{dn} \left( \langle n \rangle^{\text{st}} + 1 \right)R_{n+1}^-
\]

(3.15)

around the steady state photon number \( \langle n \rangle^{\text{st}} \). This just holds true under the condition of a dominating peak \[28\]. If there is a distribution of more than one peak of comparable height, the expansion is not a good approximation because there are contributions of peaks not close to the steady state photon number.

Within this approximation, we get

\[
\langle n \rangle^{\text{st}} \approx \frac{Q}{\omega} \left( \langle n \rangle^{\text{st}} + 1 \right)R_{n+1}^- + \frac{d}{dn} \left( \langle n \rangle^{\text{st}} + 1 \right)R_{n+1}^- \left|_{\langle n \rangle^{\text{st}}} \right. \langle n - \langle n \rangle^{\text{st}} \rangle
\]

(3.16)

for the steady state photon number. In the last step, we used \( \langle \Delta n \rangle = \langle n - \langle n \rangle \rangle = 0 \). This equation is similar to the condition for the extrema of the steady state photon statistic Eq. (3.13), which reads \( n_j = Q/\omega n_j R_{n_j}^- \). But we see, that we get a slight shift due to thermal photons. Again, this equation cannot be solved analytically, but for initial momentum eigenstates we get a similar graphical solution: The steady state photon number is at the intersection of \( (\langle n \rangle^{\text{st}} + 1)R_{n+1}^- \) with the line of \( (\langle n \rangle^{\text{st}} - n_{\text{th}})\omega/(Qr) \). As we can see in Figure 3.12 there are more than one intersection for this set of parameters. But as discussed in section 3.4.2 with Figure 3.11 the first one describes the dominating peak, and thus the first intersection determines the steady state photon number. This intersection is magnified in Figure 3.12. The orange line is not a line through the origin because of the thermal photons \( n_{\text{th}} \) that occur due to a finite temperature. The red line
3.4 Steady State of the Photon Statistics

Figure 3.12: Graphical method to find the steady state photon number for $p = 0.45$ and $\omega/(Qr) = 10^{-6}$ at a finite temperature with $n_{th} = 5 \cdot 10^4$: The approximate mean photon number $\langle n \rangle_{st}$ is at the intersection of the black and orange curve, which denotes the solution of Eq. (3.16). The red line marks the exact value of the mean photon number, without the approximation done above. The parameters are: $m = 10^{-31}$ kg, $g = 5 \cdot 10^{13}$ s$^{-1}$, $\tau = 10^{-16}$ s, and $k = 10^{10}$ m$^{-1}$.

marks the calculated mean photon number. As we see, our approximate photon number marked by the intersection does not agree with this value, but is close to it. The reason for this is obvious. The scheme presented in this section just concentrates on the dominating maximum and other maxima are neglected. Since a second smaller maximum of the photon statistics occurs at a higher photon number for this set of parameters, the true mean photon number is slightly shifted to higher numbers in comparison to our prediction.
3 Properties of the Laser Field

3.4.4 Steady State Variance

With the result from section 3.3 we can easily derive the variance of a steady state. We again assume $R_n^+ = 0$ and a photon distribution dominated by one peak. With the same expansion as in Eq. (3.15) we get

$$\langle (n+1)R_{n+1}^{-}\Delta n \rangle_{st} \approx \langle n \rangle_{st} + 1 - \langle n \rangle_{st} + 1 \langle \Delta n \rangle_{st} + \frac{d}{dn} (n+1)\left|_{\langle n \rangle_{st}} \right. \langle \Delta n^2 \rangle_{st}$$

$$\frac{d}{dn} (n+1)R_{n+1}^- \left|_{\langle n \rangle_{st}} \right. \langle \Delta n^2 \rangle_{st}.$$

If we use Eq. (3.14), which holds true for a steady state, plug this into Eq. (3.9), and use $d\langle \Delta n^2 \rangle_{st}/dt = 0$, we get

$$\langle \Delta n^2 \rangle_{st} = \frac{n_{th} + 1}{1 - \frac{Q}{\omega} \frac{d}{dn} (n+1)R_{n+1}^- \left|_{\langle n \rangle_{st}} \right.} \langle n \rangle_{st} \equiv \sigma^2 \langle n \rangle_{st},$$

(3.17)

where we call $\sigma^2$ the normalized variance, which is in complete agreement with [28].

This expression proves that we can achieve a sub-Poissonian distribution, i.e. $\sigma^2 < 1$. A necessary, though not sufficient, condition for that is

$$\frac{d}{dn} (n+1)R_{n+1}^- \left|_{\langle n \rangle_{st}} \right. < 0.$$  

For a given electron distribution, this can be easily checked. In Figure 3.12 the black curve shows the function $(n+1)R_{n+1}^-$ for a momentum eigenstate. The periodicity of the sine-function gives intervals where this function is ascending, and intervals where it is descending. To achieve a sub-Poissonian distribution, the function needs to be descending at the steady state photon number. As we see in the figure, this is always true for the set of parameters used. But this condition is, as already mentioned, just necessary.

The sufficient condition for a sub-Poissonian behavior reads

$$-\frac{Q}{\omega} \frac{d}{dn} (n+1)R_{n+1}^- \left|_{\langle n \rangle_{st}} \right. > n_{th}.$$  

(3.18)

This condition is more restrictive, and we see that the derivative of $(n+1)R_{n+1}^-$ does not only have to be negative, but its absolute value needs to be bigger than $Q/\omega n_{th}$. But, with the right choice of parameters, this can be achieved, especially for low temperatures and small thermal photon numbers. Even for our set of parameters, we have such a behavior: The width of the first peak is smaller than the mean photon number.
3.4.5 Gaussian Approximation

In the preceding sections, we have discussed the properties of the steady state photon statistics with one dominating peak. We now try to find an approximate photon distribution that has similar properties, as shown in [31]. For that, we expand the exponent of the steady state photon distribution Eq. (3.12)

\[ \int_1^n \! dn' \ln \Lambda(n') \approx \int_1^n \! dn' \ln \Lambda(n') - \frac{(n - n_j)^2}{2 \delta n^2} \]

into a Taylor series around a maximum, where

\[ \delta n^2 = -\frac{1}{\frac{\partial}{\partial n} \Lambda(n)|_{n_j}} = \frac{\partial}{\partial n} R_n|_{n_j} \]

is the variance. With that, the photon statistics can be written as

\[ W_{n_j}^{st} \approx W_0^{st} \exp \left[ \int_1^n \! dn' \ln \Lambda(n') \right] \exp \left[ -\frac{(n - n_j)^2}{2 \delta n^2} \right] \]

\[ = W_{n_j}^{st} \exp \left[ -\frac{(n - n_j)^2}{2 \delta n^2} \right]. \quad (3.19) \]

This is a Gaussian distribution centered around \( n_j \) with its mean photon number \( \langle n \rangle^{st} = n_j \) and the variance \( \delta n^2 \). As we already discussed in [3.4.3], the steady state photon number is approximately Eq. (3.16) without the thermal photon number. Of course, this approximation just holds true close to the maximum under consideration if there is a multi-peak distribution. But, if there is one dominating maximum, this approximation is fairly good.

Since

\[ \frac{\partial}{\partial n} n R_n|_{n_j} = n_j \frac{\partial}{\partial n} R_n|_{n_j} + R_{n_j} \]

we can rewrite the variance as

\[ \delta n^2 = \frac{\frac{\partial}{\partial n} R_n|_{n_j}}{-\frac{\partial}{\partial n} n R_n|_{n_j} + R_{n_j}} n_j = \frac{n_{th}}{\frac{\partial}{\partial n} n R_n|_{n_j} + R_{n_j}} n_j \]

\[ = \frac{n_{th} + 1}{1 - \frac{\partial}{\partial n} n R_n|_{n_j}} n_j \quad (3.20) \]

with the help of Eq. (3.13). This is the same expression as Eq. (3.17) derived in section
3.4.4 So there is the same sub- or super-Poissonian characteristics as discussed in that section.

In Figure 3.13 this Gaussian approximation is plotted for both peaks and compared the photon statistics Eq. (3.10). Each peak is fitted by a Gaussian distribution fairly well, even though some deviations occur, mainly in between both peaks. But especially around the maxima, this is a good approximation. In addition to that, we can see that

![Figure 3.13: The exact photon distribution $W_n^{\text{st}}$ (orange) of Eq. (3.10) is compared to the Gaussian approximations (black) from Eq. (3.19) for $p = 0.45 q$ with a cavity damping $\omega/(\Omega r) = 10^{-6}$ at a finite temperature with $n_{\text{th}} = 5 \cdot 10^4$. The black solid curves shows the Gaussian approximation of the dominating peak, the black dashed curve the Gaussian approximation of the second one. The parameters are: $m = 10^{-31} \text{kg}$, $g = 5 \cdot 10^{13} \text{s}^{-1}$, $\tau = 10^{-16} \text{s}$, and $k = 10^{10} \text{m}^{-1}$.]

the first peak has a sub-Poissonian behavior, since the width of the curve is much smaller than the mean value. We discussed the possibility of such a behavior in the previous section. Even though the Gaussian does not fit the first peak exactly, the width is of comparable size.
3.5 Intrinsic Linewidth

To calculate the intrinsic linewidth of FELs, we have to take a closer look at the electric field. In the derivation of the Hamiltonian in section 1.2.2, we used the ansatz

\[ \hat{A}_L = \hat{e} A_L \hat{a}_L e^{-i(\omega_L t - k_L z)} + \text{h.c.} \]

for the vector potential of the laser field in the interaction picture. The electric field of the laser light then reads

\[ \hat{E}_L = -\frac{\partial \hat{A}_L}{\partial t} = i \hat{e} A_L \omega \hat{a}_L e^{-i(\omega t - k z)} + \text{h.c.} \]

in the Bambini-Renieri frame. Differentiating the expectation value of the electric field \( \langle \hat{E}_L \rangle = \vec{E}_L e^{-i(\omega t - k z)} \sum_{n=0}^{\infty} \langle n | \hat{a}_L \hat{b}_L | n \rangle + \text{c.c.} = \vec{E}_L e^{-i(\omega t - k z)} \sum_{n=0}^{\infty} \sqrt{n} \varrho_{n-1,n} + \text{c.c.} \]

with respect to time yields

\[ \frac{d}{dt} \langle \hat{E}_L \rangle = \vec{E}_L e^{-i(\omega t - k z)} \left( -i\omega \sum_{n=0}^{\infty} \sqrt{n + 1} \varrho_{n,n+1} + \sum_{n=0}^{\infty} \sqrt{n + 1} \varrho_{n+1,n} \right) + \text{c.c.} \quad (3.21) \]

and we get a dependence on the time derivative of the matrix elements \( \varrho_{n+1,n} \). To calculate the linewidth, we use the ideas of [32, 33]. We already have derived the time evolution of the density matrix elements in appendix D and we can write them with the same coarse-grained derivative as in chapter 3.1. Together with Eq. (C.2) occurring from the cavity losses from appendix C, we arrive at the difference-differential equation

\[ \dot{\varrho}_{n+1,n} = r (\varrho_{n+1,n}(t + \tau) - \varrho_{n+1,n}) \]

\[ = - \frac{\mu_n}{2} \left( \varrho_{n+1,n} + \xi_{n-1} \varrho_{n,n-1} + \xi_{n+1} \varrho_{n+2,n+1} \right) \]

\[ = - \frac{\mu_n}{2} \left( \varrho_{n+1,n} + \left( \xi_{n-1} \varrho_{n,n-1} - \xi_n \varrho_{n+1,n} \right) + \left( \xi_{n+1} \varrho_{n+2,n+1} - \xi_n \varrho_{n+1,n} \right) \right) \]

(3.22)
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with

\[
\xi_n = \sqrt{n + 1} \sqrt{n + 2} \left[ r g^2 \int_0^q dp \, \varrho(p) \frac{\sin \Omega_{n+1}^+ \tau}{\Omega_{n+1}^-} \frac{\sin \Omega_{n}^- \tau}{\Omega_n^-} + \frac{\omega}{Q} \mu_{th} \right]
\]

\[
\zeta_n = \sqrt{n + 1} \sqrt{n} \left[ r g^2 \int_{-q}^0 dp \, \varrho(p) \frac{\sin \Omega_n^+ \tau}{\Omega_n^+} \frac{\sin \Omega_{n-1}^- \tau}{\Omega_{n-1}^-} + \frac{\omega}{Q} (n_{th} + 1) \right]
\]

and

\[
\mu_n = \mu_n - 2\xi_n - 2\zeta_n
\]

\[
= 2r \int_0^q dp \, \varrho(p) \left[ 1 - \cos \Omega_{n+1}^- \tau \cos \Omega_n^- \tau - \sin \Omega_{n+1}^- \tau \sin \Omega_n^- \tau \left( \frac{\phi_+}{\tau} \right)^2 + g^2 \sqrt{n + 2\sqrt{n + 1}} \right] \frac{\Omega_{n+1}^- \Omega_n^-}{\Omega_{n+1}^+ \Omega_n^+}
\]

\[
+ 2r \int_{-q}^0 dp \, \varrho(p) \left[ 1 - \cos \Omega_n^+ \tau \cos \Omega_{n-1}^- \tau - \sin \Omega_n^+ \tau \sin \Omega_{n-1}^- \tau \left( \frac{\phi_+}{\tau} \right)^2 + g^2 \sqrt{n + 2\sqrt{n + 1}} \right] \frac{\Omega_n^+ \Omega_{n-1}^-}{\Omega_n^+ \Omega_{n-1}^-}
\]

\[
+ 2 \frac{\omega}{Q} \left[ (n_{th} + 1) \left( n + \frac{1}{2} - \sqrt{n + 1} \sqrt{n} \right) + n_{th} \left( n + \frac{3}{2} - \sqrt{n + 1} \sqrt{n} \right) \right]
\]

\[
+ i r \left[ \int_{-q}^0 dp \, \varrho_+ \left( \frac{\sin \Omega_n^+ \tau}{\Omega_n^+} \cos \Omega_{n-1}^- \tau - \frac{\sin \Omega_{n-1}^+ \tau}{\Omega_{n-1}^+} \cos \Omega_n^- \tau \right) \right] + i r \left[ \int_0^q dp \, \varrho_- \left( \frac{\sin \Omega_n^- \tau}{\Omega_n^-} \cos \Omega_{n+1}^- \tau - \frac{\sin \Omega_{n+1}^- \tau}{\Omega_{n+1}^-} \cos \Omega_n^- \tau \right) \right]. \tag{3.23}
\]

This last expression, although lengthy, will turn out to be the approximate linewidth, as we see in the following.

The coefficients \(\xi_n\) and \(\zeta_n\) can be interpreted as probability flow between the off-diagonal elements of the density matrix. This way of thinking is illustrated in Figure 3.14. We can now, guided by the thoughts that have lead to the steady state photon statistics, impose a detailed balance condition between those off-diagonal elements, i.e.

\[
\xi_{n-1} \varrho_{n,n-1} - \zeta_n \varrho_{n+1,n} = 0.
\]
3.5 Intrinsic Linewidth

\[ \begin{array}{ccc}
\rho_{n,n-1} & \rho_{n,n} & \rho_{n,n+1} \\
\xi_{n-1} & \zeta_n & \\
\rho_{n+1,n-1} & \rho_{n+1,n} & \rho_{n+1,n+1} \\
\xi_n & \zeta_{n+1} & \\
\rho_{n+2,n-1} & \rho_{n+2,n} & \rho_{n+2,n+1} \\
\end{array} \]

Figure 3.14: Interpretation of the coefficients \( \xi_n \) and \( \zeta_n \) as flow between the off-diagonal elements of the density matrix.

We use this condition to recursively define the off diagonal elements

\[ \varrho_{n+1,n}(0) = \varrho_{1,0}(0) \prod_{n' = 1}^{n} \frac{\xi_{n'-1}}{\zeta_{n'}} \]

for \( t = 0 \). With this expression we follow [32] and use the ansatz

\[ \varrho_{n+1,n}(t) = e^{-\frac{D_n(t) + i L_n(t)}{2}} \varrho_{n+1,n}(0) \]

with the real valued functions \( D_n(t) \) and \( L_n(t) \). If we plug them into Eq. (3.22) we get the differential equation

\[ \dot{\varrho}_{n+1,n} = -\frac{\dot{D}_n}{2} - i \frac{\dot{L}_n}{2} e^{-\frac{D_n(t) + i L_n(t)}{2}} \varrho_{n+1,n}(0) \]

\[ = -\frac{\mu_n}{2} e^{-\frac{D_n + i L_n}{2}} \varrho_{n+1,n}(0) \]

\[ + \left( e^{-\frac{D_{n-1} + i L_{n-1}}{2}} \xi_{n-1} \varrho_{n,n-1}(0) - e^{-\frac{D_n + i L_n}{2}} \zeta_n \varrho_{n+1,n}(0) \right) \]

\[ + \left( e^{-\frac{D_{n+1} + i L_{n+1}}{2}} \zeta_{n+1} \varrho_{n+2,n+1}(0) - e^{-\frac{D_n + i L_n}{2}} \xi_n \varrho_{n+1,n}(0) \right) \]

\[ = \left[ -\frac{\mu_n}{2} e^{-\frac{D_n + i L_n}{2}} + \zeta_n \left( e^{-\frac{D_{n-1} + i L_{n-1}}{2}} - e^{-\frac{D_n + i L_n}{2}} \right) \right. \]

\[ + \xi_n \left( e^{-\frac{D_{n+1} + i L_{n+1}}{2}} - e^{-\frac{D_n + i L_n}{2}} \right) \varrho_{n+1,n}(0), \quad (3.24) \]
3 Properties of the Laser Field

where in the last step we used the relations

\[ \xi_{n-1} \varrho_{n,n-1}(0) = \xi_{n-1} \varrho_{1,0}(0) \prod_{n'=1}^{n-1} \frac{\xi_{n'-1}}{\xi_{n'}} = \zeta_n \varrho_{1,0}(0) \prod_{n'=1}^{n} \frac{\xi_{n'-1}}{\xi_{n'}} = \zeta_n \varrho_{n+1,n}(0) \]

\[ \zeta_{n+1} \varrho_{n+2,n+1}(0) = \zeta_{n+1} \varrho_{1,0}(0) \prod_{n'=1}^{n+1} \frac{\xi_{n'-1}}{\xi_{n'}} = \xi_n \varrho_{1,0}(0) \prod_{n'=1}^{n} \frac{\xi_{n'-1}}{\xi_{n'}} = \xi_n \varrho_{n+1,n}(0). \]

Taking now the real part of Eq. (3.24) we arrive at

\[ \dot{D}_n = \Re\{\mu_n\} + 2\xi_n \left( 1 - e^{-\frac{D_n - D_{n-1}}{2}} \right) + 2\xi_n \left( 1 - e^{-\frac{D_{n+1} - D_n}{2}} \right), \tag{3.25} \]

the difference-differential equation for the functions \( D_n \). In lowest order of the exponents,

\[ D_n(t) = \Re\{\mu_n\}t \]

is the solution of this differential equation. This is just true, if \( D_n \) does not change much in \( n \) and

\[ |D_n - D_{n-1}| \approx |\Re\{\mu_n - \mu_{n-1}\}|t \approx \left| \frac{\partial \Re\{\mu_n\}}{\partial n} \right| t \ll 1 \]

is fulfilled. So either the dependence of \( \Re\{\mu_n\} \) on \( n \) is small, or the times \( t \) are small, as explained in [32]. This can be analogously done for \( L_n \). If we use this approximate solution of the differential equation to calculate the decay of the electric field, we get from Eq. (3.21)

\[ \frac{d}{dt} \langle \hat{E}_L \rangle = \hat{E}_L e^{-i(\omega t - k z)} \left( -\frac{\Re\{\mu(n)\}}{2} - i\omega + \frac{\dot{L}(n)}{2} \right) \sum_{n=0}^{\infty} \sqrt{n+1} \varrho_{n,n+1} + \text{c.c.} \tag{3.26} \]

where we assumed a photon distribution peaked around its mean photon number as in [18, 32]. The possibility of one dominating maximum was discussed in section 3.4.2, where we have seen that for the right parameters, this is a good approximation. The real part of the term in the parentheses corresponds to the linewidth \( D \) [32] and we get
3.5 Intrinsic Linewidth

the result

\[ D \equiv D_{(n)} = \Re \{ \mu_{(n)} \} \]

\[ = 2r \int_0^q dp \, \varrho(p) \left[ 1 - \cos \Omega_{(n)}^- \cos \Omega_{(n)}^- \tau \right. \]

\[ \left. - \sin \Omega_{(n)}^- \sin \Omega_{(n)}^- \frac{\left( \frac{\phi_-}{2} \right)^2 + g^2 \sqrt{\langle n \rangle} + 2 \sqrt{\langle n \rangle} + 1}{\Omega_{(n)}^- + 1} \right] \]

\[ + 2r \int_{-q}^0 dp \, \varrho(p) \left[ 1 - \cos \Omega_{(n)}^+ \cos \Omega_{(n)}^+ \tau \right. \]

\[ \left. - \sin \Omega_{(n)}^+ \sin \Omega_{(n)}^+ \frac{\left( \frac{\phi_+}{2} \right)^2 + g^2 \sqrt{\langle n \rangle} + 1 \sqrt{\langle n \rangle}}{\Omega_{(n)}^+ \Omega_{(n)}^+ - 1} \right] \]

\[ + 2 \frac{\omega}{Q} \left( n_{th} + 1 \right) \left( \langle n \rangle + \frac{1}{2} - \sqrt{\langle n \rangle} + \sqrt{\langle n \rangle} \right) \]

\[ + n_{th} \left( \langle n \rangle + \frac{3}{2} - \sqrt{\langle n \rangle} + \sqrt{\langle n \rangle} + 2 \right) \]

(3.27)

This expression is quite cumbersome and since it is an approximation, we expand the square roots according to appendix E for sufficiently large \( \langle n \rangle \) and get

\[ D \approx 4r \left[ \int_0^q dp \, \varrho(p) \sin^2 \left( \frac{\Omega_{(n)}^- + 1 - \Omega_{(n)}^-}{2} \right) + \int_{-q}^0 dp \, \varrho(p) \sin^2 \left( \frac{\Omega_{(n)}^+ - \Omega_{(n)}^+ - 1}{2} \right) \right] \]

\[ + \frac{rg^2}{4} \left[ \frac{1}{\langle n \rangle + 1} \int_0^q dp \, \varrho(p) \left( \frac{\phi_-}{\Omega_{(n)}^- + 1} \right)^2 \sin \Omega_{(n)}^- \sin \Omega_{(n)}^- \tau \right. \]

\[ \left. + \frac{1}{\langle n \rangle} \int_{-q}^0 dp \, \varrho(p) \left( \frac{\phi_+}{\Omega_{(n)}^+ \Omega_{(n)}^+ - 1} \right)^2 \sin \Omega_{(n)}^+ \sin \Omega_{(n)}^- \tau \right] \]

\[ + \frac{\omega}{Q} \left( n_{th} + 1 \right) \left( \langle n \rangle + \frac{1}{4} \right) + \frac{n_{th}}{4(\langle n \rangle + 1)} \]

(3.28)

for the linewidth. When we look at these terms, we see that the term in the first brackets corresponds to the linewidth of the one-atom laser as in [32] and the term in the third brackets occurs due to cavity losses. The terms in the second brackets turns up because of the detuning.
3 Properties of the Laser Field

Figure 3.15: Linewidth $D/r$ without the cavity damping terms. Two peaks occur around the resonances, declining rapidly with increasing photon number, which is in a logarithmic scale. The parameters are: $m = 10^{-31}$ kg, $g = 5 \cdot 10^{13}$ s$^{-1}$, $\tau = 10^{-16}$ s, and $k = 10^{10}$ m$^{-1}$.

The linewidth is plotted in Figure 3.15 without cavity damping terms. We see that it depends on the mean photon number $\langle n \rangle$ and decays with increasing laser field intensity. In momentum space, we have maxima at $\pm q/2$. This multi-peaked linewidth is highly non-classical.

In Figure 3.16, we investigate which terms of Eq. (3.28) are dominating. The red lines denote the terms of the first brackets. For increasing photon number, they become dominant. This can be explained in an easy analytic way. The terms of the first line of Eq. (3.28) behave like

$$\sin^2(\sqrt{n + 1} - \sqrt{n}) \approx \sin^2 \left( \frac{\partial}{\partial n} \sqrt{n + 1} \right) \approx \sin^2 \frac{1}{\sqrt{n}} \approx \frac{1}{n}$$

in contrast to the one in the second with

$$\frac{1}{n} \left( \frac{\phi_{\pm}/2}{\sqrt{n + 1}} \right)^2 \sin \sqrt{n} \sin \sqrt{n + 1} \approx \frac{(\phi_{\pm}/2)^2}{n^3} \sin^2 \sqrt{n}.$$ 

We see, that for large photon numbers, the terms in the first line are dominating, for small photon numbers the terms in the second line. The behavior of Figure 3.16 can be explained by that. But, even at higher intensities there are still two peaks around the resonance momenta.
Figure 3.16: In these plots, $D/r$ (black) is plotted for various photon numbers. For higher $\langle n \rangle$, the axis of the ordinate is displayed on a smaller scale, since the linewidth is decreasing. The red line shows the term $4 \sin^2 \left[ \left( \Omega_{(n)+1}^\pm - \Omega_{(n)}^\pm \right) \tau/2 \right]$. The parameters are: $m = 10^{-31}$ kg, $g = 5 \cdot 10^{13}$ s$^{-1}$, $\tau = 10^{-16}$ s, and $k = 10^{10}$ m$^{-1}$. 
4 Embedding the Results into the Context of other Theories

4.1 Link to Perturbation Theory

So far, we have derived an equation of motion for the radiation field, found a steady state in the detailed balance, and investigated some properties of this solution. However, this equation of motion is just valid in our quantum regime and under the conditions summarized in section 2.4. The calculations have been performed in the adiabatic approximation without the corrections of higher levels to the phase and interaction has been assumed in the whole gain and loss interval. To check, whether these assumptions and approximations are appropriate, we try to find a connection of this model to the perturbation theory of [13] and to the classical theory.

4.1.1 Results of Perturbation Theory

In order to discuss the connection between our model and [13], we have to briefly look into the results obtained in that paper. There, the problem of solving the Schrödinger equation with the time-dependent Hamiltonian in the interaction picture was faced with perturbation theory, but, as in contrast to other approaches [6, 12], the time evolution operator was expanded up to forth order in $g\tau \sqrt{n}$, rather than in orders of the recoil $q$. Forth order is necessary to see nonlinearities and thus to get a steady state. The expansion into second order was already done in [9]. The calculations in both works have been performed for initial electron momentum eigenstates and a cavity at zero temperature.

First of all, it is noted that there are no contributions to the diagonal elements of the reduced density matrix of the laser field from odd orders of $g\tau \sqrt{n}$.

The difference-differential equation of the photon statistics takes the form

$$\dot{W}_n = W_n r \{ P(n \to n) - 1 \} + W_{n+1} r P(n + 1 \to n) + W_{n-1} r P(n - 1 \to n)$$
$$+ W_{n+2} r P(n + 2 \to n) + W_{n-2} r P(n - 2 \to n),$$

where the same coarse-grained derivative as in this thesis was introduced. One of the main results is that we can neglect the multi-photon transitions Eq. (F.5) in the quantum regime, if $g^2 \tau^2 n$ is sufficiently small. For the same reason, we can neglect some terms in
4 Embedding the Results into the Context of other Theories

the other probability flows, as we see later. As discussed in [13], the equation of motion reads in good approximation

$$\dot{W}_n = W_n \, r[P(n \rightarrow n) - 1] + W_{n+1} \, rP(n + 1 \rightarrow n) + W_{n-1} \, rP(n - 1 \rightarrow n). \quad (4.1)$$

This approximation has nothing to do with a step back to second order perturbation theory, since the probability flows $rP$ between the levels are now nonlinear in $n$ as we see in Eq. (4.3) and in contrast to second order perturbation theory, terms proportional to $n^2$ are included. These nonlinearities make it possible to find a steady state solution.

But is there any connection to our model? In order to determine that, we have to take a closer look at the specific form of the probability flows $rP$.

We define the function

$$S_\pm \equiv \frac{\sin \frac{\phi_\pm \tau}{2}}{\phi_\pm / 2} \quad (4.2)$$

and get after some algebra from the results of [13] the probability flows

$$r[P(n \rightarrow n) - 1] = -n \left[ r g^2 S_+^2 + \frac{\omega}{Q} - \{ \cdots \} \right] - n^2 \left[ r g^4 \frac{\partial S_+^2}{\partial (\phi_+ / 2)} + \frac{\omega}{Q} - \{ \cdots \} \right]$$

$$- (n + 1) \left[ r g^2 S_-^2 - \{ \cdots \} \right] - (n + 1)^2 \left[ r g^4 \frac{\partial S_-^2}{\partial (\phi_- / 2)} - \{ \cdots \} \right] \quad (4.3a)$$

$$rP(n + 1 \rightarrow n) = n \left[ - \{ \cdots \} \right] + n^2 \left[ - \{ \cdots \} \right]$$

$$+ (n + 1) \left[ r g^2 S_+^2 + \frac{\omega}{Q} - \{ \cdots \} \right] + (n + 1)^2 \left[ r g^4 \frac{\partial S_+^2}{\partial (\phi_+ / 2)} - \{ \cdots \} \right] \quad (4.3b)$$

$$rP(n - 1 \rightarrow n) = n \left[ r g^2 S_-^2 - \{ \cdots \} \right] + n^2 \left[ r g^4 \frac{\partial S_-^2}{\partial (\phi_- / 2)} - \{ \cdots \} \right]. \quad (4.3c)$$

For the sake of simplicity, we expressed these probability flows in terms proportional to $n$, $n + 1$, $n^2$ and $(n + 1)^2$. The full expressions including the ones abbreviated by the dots in the curly braces can be found in appendix [F]. In addition to that, we made use of the relation

$$\frac{\partial S_\pm^2}{\partial (\phi_\pm / 2)} = \frac{1}{2} \frac{S_\pm}{(\phi_\pm / 2)^2} \left( \tau \cos \frac{\phi_\pm}{2} - S_\pm \right) \quad (4.4)$$

to find this form.
4.1 Link to Perturbation Theory

When we look at the terms in the curly braces in appendix F, two things attract attention: First of all, they are proportional to $g^4 \tau^2$. Hence, we can neglect those terms if they occur linear to $n$ or $n + 1$, since already $g^2 \tau^2 n$ was assumed to be small. Without these terms, the summands proportional to $n$ and $n + 1$ correspond exactly to the ones occurring in second order perturbation theory \[9\].

Moreover, they consist of a mixture of phases $\phi_-, \phi_+, \phi_{-3}$, and $\phi_{+3}$. The last two phases are defined as $\phi_{\pm 3} \equiv \phi_{\pm} (p \pm q)$ and correspond to the resonances at $\pm 3q/2$.

In the quantum limit, where we have a very definite electron momentum close to one resonance, the phase corresponding to this resonance is small, and the other phases are much larger. They occur in the expressions \[F.1\] in the curly braces in such a way, that always at least one of the factors is small. So it is intuitive that the braces do not contribute much to the terms proportional to $n^2$ and $(n + 1)^2$.

In figures 4.1 and 4.2 we plotted the probability flows. We see the terms with and without the correction of the curly braces of each summand. As discussed, the terms proportional to $n$ or $n + 1$ are very good approximated by setting the curly braces equal to zero. This is no surprise, since the correction terms are proportional to $g^4 \tau^4$ and therefore much smaller than the other summand in the order of $g^2 \tau^2$. Even the terms proportional to $n^2$ or $(n + 1)^2$ are good approximated if we set the curly braces to zero. There are some resonances at multiples of $q/2$, which are neglected if we drop the braces, but these resonances are small compared to the main resonance at $\pm q/2$.

One of the major results is that the interaction mainly takes place around this main resonance. From these maxima, the probability flows decay fast and become almost equal to zero outside the interaction intervals. Hence, in the quantum regime, multiphoton transitions are suppressed. Here, the first connection to our effective two-level model appears: This thought has lead to our effective Hamiltonian in the first place and it is not that surprising, that we get similar results.

So let us now neglect these terms in curly braces. We find

\[
\begin{align*}
\text{rP}(n \rightarrow n - 1) &= - n \left[ rg^2 S^2_+ + \frac{\omega}{Q} \right] - n^2 \left[ rg^4 2 \frac{\partial S^2_+}{\partial (\phi_+/2)} \right] \\
&\quad - (n + 1) \left[ rg^2 S^2_+ \right] - (n + 1)^2 \left[ rg^4 2 \frac{\partial S^2_+}{\partial (\phi_+/2)} \right] \quad (4.5a) \\
\text{rP}(n + 1 \rightarrow n) &= (n + 1) \left[ rg^2 S^2_+ + \frac{\omega}{Q} \right] + (n + 1)^2 \left[ rg^4 2 \frac{\partial S^2_+}{\partial (\phi_+/2)} \right] \quad (4.5b) \\
\text{rP}(n - 1 \rightarrow n) &= n \left[ rg^2 S^2_- \right] + n^2 \left[ rg^4 2 \frac{\partial S^2_-}{\partial (\phi_-/2)} \right] \quad (4.5c)
\end{align*}
\]

as an approximation for the probability flows.
4 Embedding the Results into the Context of other Theories

Figure 4.1: Influence of the correction terms \{\ldots\} on $P(n - 1 \to n)$ (a),(b) and $P(n + 1 \to n)$ (c),(d). For the sake of simplicity we assumed in (c) and (d) both summands proportional to $n^2$ instead of one to $n^2$ and one to $(n + 1)^2$ and added the functions. In a good approximation, the braces can be neglected. The parameters are: $m = 10^{-31}$ kg, $g = 5 \cdot 10^{13}$ s$^{-1}$, $\tau = 10^{-16}$ s, and $k = 10^{10}$ m$^{-1}$. 

(a) Terms of $P(n - 1 \to n) \propto n$.

(b) Terms of $P(n - 1 \to n) \propto n^2$.

(c) Terms of $P(n + 1 \to n) \propto n + 1, n$.

(d) Terms of $P(n + 1 \to n) \propto (n + 1)^2, n^2$. 

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4.1 Link to Perturbation Theory

Figure 4.2: Influence of the correction terms \{\ldots\} on $P(n \rightarrow n)$ \((a)-(d)\). The parameters are: $m = 10^{-31}$ kg, $g = 5 \cdot 10^{13}$ s$^{-1}$, $\tau = 10^{-16}$ s, and $k = 10^{10}$ m$^{-1}$.
4 Embedding the Results into the Context of other Theories

4.1.2 Taylor Expansion of the Rate Coefficients $nR^\pm_n$ in $g^2\tau^2n$

To see the connection between these results of perturbation theory and our model, we now make a Taylor expansion of the rate coefficients $nR^\pm_n$. In our model, we have an exact solution of the Schrödinger equation with an approximate Hamiltonian, in contrast to the approximate solution of the exact Schrödinger equation in perturbation theory. This approximate solution was found by expanding the time evolution operator into orders of $g\tau\sqrt{n}$. So in order to regain these results, we also make a Taylor expansion of the transition rates in $g^2\tau^2n$ and in $g^2\tau(n+1)$, respectively, around zero and get

$$
g^2n \frac{\sin^2 \Omega^\pm_n}{\Omega^\pm_{n-1}} \approx n g^2S^2_n + n^2 \frac{g^4}{(\phi_\pm/2)^2} \cos \frac{\phi_\pm}{2} (\tau - S_\pm) \tag{4.6a}
$$

$$
g^2(n+1) \frac{\sin^2 \Omega^\pm_n}{\Omega^\pm_n} \approx (n+1) g^2S^2_n + (n+1)^2 \frac{g^4}{(\phi_\pm/2)^2} \cos \frac{\phi_\pm}{2} (\tau - S_\pm) \tag{4.6b}
$$

where we used the definition (4.2) and the relation (4.4). If we use this expansion to rewrite the rate coefficients $nR^\pm_n$ in the equation of motion (3.6), the probability flows take the form

$$
r [P(n \rightarrow n) - 1] = -(n+1)R^-_{n+1} - n \left[ R^+_n + \frac{\omega}{Q} \right] 
$$

$$
\approx -n \left[ rg^2 \int_{-q}^0 dp \ g(p)S^2_n + \frac{\omega}{Q} \right] - n^2 \left[ rg^42 \int_{-q}^0 dp \ g(p) \frac{\partial S^2_n}{\partial (\phi_+/2)} \right] 
$$

$$
- (n+1) \left[ rg^2 \int_{-q}^q dp \ g(p)S^2_n \right] - (n+1)^2 \left[ rg^42 \int_{-q}^0 dp \ g(p) \frac{\partial S^2_n}{\partial (\phi_-/2)} \right] \tag{4.7a}
$$

$$
rP(n+1 \rightarrow n) = (n+1) \left[ R^+_n + \frac{\omega}{Q} \right] 
$$

$$
\approx (n+1) \left[ rg^2 \int_{-q}^0 dp \ g(p)S^2_n + \frac{\omega}{Q} \right] + (n+1)^2 \left[ rg^42 \int_{-q}^0 dp \ g(p) \frac{\partial S^2_n}{\partial (\phi_+/2)} \right] \tag{4.7b}
$$

$$
rP(n-1 \rightarrow n) = nR^-_n 
$$

$$
\approx n \left[ rg^2 \int_{-q}^q dp \ g(p)S^2_n \right] + n^2 \left[ rg^42 \int_{-q}^0 dp \ g(p) \frac{\partial S^2_n}{\partial (\phi_-/2)} \right] \tag{4.7c}
$$
4.1 Link to Perturbation Theory

at zero temperature, i.e. with $n_{th} = 0$. Since the perturbation theory was done for initial momentum eigenstates, we replace $g(p)$ by a Dirac-function and regain the results from Eq. (4.5), if the momentum is inside the interaction intervals $(-q, 0)$ or $(0, q)$. So this model’s probability flows correspond to the red lines in figures 4.1 and 4.2 within the interaction intervals, and zero outside. We still have in mind, that the results of our model were obtained through the adiabatic approximation for momenta close to $\pm q/2$. If we wanted to investigate the interaction of momenta close to $\pm q, \pm 3q/2$, we would have to make a different approximation to get an effective two-level system. But even if we drop this restriction to those momenta close to the two resonances, we we are in good consistency with perturbation theory.

We see, that the theory with the effective Hamiltonian gives the same results as perturbation theory, if we expand into small $g^2 \tau^2 n$. The only features we did loose in this approximation are multiple resonances and interaction outside the intervals $(-q, 0)$ and $(0, q)$. But this was exactly our ansatz, so it is not surprising that we do not find these effects in our theory. Of course, if we performed an adiabatic approximation for intervals around every resonance, we would get a better description in every interval.

We have to keep in mind, that perturbation theory is just valid for small $g^2 \tau^2 n$, i.e. for small intensities and the right set of parameters. And exactly in this limit we can perform the Taylor expansion of the rate coefficients. But, with increasing intensity our model is still valid and higher orders of the expansion become important. Nevertheless, at very large photon numbers, our model breaks down, too, as discussed in section 2.4.

So for small laser fields, perturbation theory is more accurate since more resonances are kept. But in contrast to perturbation theory, we have not such a restrictive limitation of the validity of our solution for increasing $n$.

In section 2.4 we discussed that there are correction terms $\Delta_j$ to the phase, when we perform the adiabatic approximation. They scale with $g^2 n$ and are thus in the Taylor expansion small anyway. But with increasing intensity, they become more important and cannot be neglected any more.

However, we see that both theories are linked and lead to very similar results. Even perturbation theory shows that the main interaction occurs close to $\pm q/2$. This can be taken as further justification for the adiabatic approximation. We also see, that if we expand the interaction intervals to $\pm \infty$, we still get good results in our model.
4 Embedding the Results into the Context of other Theories

4.2 Connection to Classical FEL Theory

Up until now, we have discussed the FEL in the quantum regime. The model used is just valid for electrons with a sharp momentum and a sufficiently large recoil. So of course, the results just hold true for these conditions. In this section, we want to investigate what happens if the electrons’ recoils become less important. Since \( q = 2\hbar k \), this goes along with neglecting higher orders in \( \hbar \). Even though this classical limit is beyond our model, a link to the results from classical theory can be seen.

4.2.1 Small Signal Gain

In this section, we look at the connection between the gain function derived in the quantum regime and the classical one. For that, we use the results from section 3.3 where the gain was written in Eq. (3.7) as

\[
\mathcal{G} = -\langle nR_n^+ \rangle + \langle (n+1)R_{n+1}^- \rangle = -\langle nR_n^+ - (n+1)R_{n+1}^- \rangle,
\]

where we have already neglected the terms originating in the cavity losses. To describe the small signal gain, we again use the Taylor series Eq. (4.6) in \( g^2 \tau^2 n \). But now, in contrast to the previous chapter, we are not interested in the saturation terms and thus we just use the expansion in first order. For \( \langle n+1 \rangle \approx \langle n \rangle \) we get the result

\[
\mathcal{G} \approx -g^2 r \langle n \rangle (S_+^2 - S_-^2) \quad (4.8)
\]

with the use of the definition of \( S_{\pm} \) from Eq. (4.2), an expansion of the limits of the integrals to \( \pm \infty \), and a momentum eigenstate of the electron as initial condition. The extension of limits of integration is a fairly good approximation in the small signal case, as we saw in section 4.1.2. So far, we have just discussed the weak field limit of the gain, but now we turn to the classical regime, by expanding

\[
S_{\pm}^2 \approx S_{\pm}\Big|_{\hbar=0} + \frac{\partial}{\partial \hbar} S_{\pm}\Big|_{\hbar=0} \hbar \quad (4.9)
\]

for small \( \hbar \). With \( S_{\pm}\Big|_{\hbar=0} \equiv S^{2} \) we perform the derivative and find

\[
\frac{\partial}{\partial \hbar} S_{\pm}\Big|_{\hbar=0} \approx \left( \pm \frac{2k^2}{m} \right) \left( \frac{4r \cos \frac{W \tau}{2} \sin \frac{W \tau}{2} - 8 \sin^2 \frac{W \tau}{2}}{W^2} \right) = \left( \pm \frac{2k^2}{m} \right) \frac{\partial}{\partial W} \left( \frac{\sin \frac{W \tau}{2}}{W/2} \right)^2.
\]
4.2 Connection to Classical FEL Theory

Here, we used the definition of the scaled momentum $W = 2kp/m$ from section 1.2.3. Hence, in this approximation, we get for the gain the expression

$$G \approx -g^2 \langle n \rangle \left( S^2 + \frac{2\hbar k^2}{m} \frac{\partial}{\partial W} S^2 - S^2 + \frac{2\hbar k^2}{m} \frac{\partial}{\partial W} S^2 \right) = -g^2 \langle n \rangle r \frac{qk}{m} \frac{\partial}{\partial W} \left( \frac{\sin W\tau}{W} \right)^2.$$

With the definition of $\varpi^2$ from section 1.2.3 and the coupling constant $g$ from section 1.3 the relation

$$\frac{g\sqrt{\langle n \rangle}}{\varpi^2} = \frac{e^2 A_W \sqrt{\omega W A_L \langle n \rangle}}{hmc^2} \cdot \left( \frac{8e^2 A_W A_L k^2}{(mc)^2} \right)^{-1} = \frac{m}{8\hbar k^2} = \frac{1}{4 \, qk}$$

can be seen. Here, we defined $A_j = A_j \sqrt{n_j}$. With that, the gain function takes the form

$$G = -\varpi^4 \frac{mr}{2 \, qk} \frac{\partial}{\partial W} \left( \frac{\sin W\tau}{W} \right)^2 = -\alpha \varpi^4 \frac{\partial}{\partial W} \left( \frac{\sin W\tau}{W} \right)^2$$

and the connection to the classical gain function becomes clear. In the last step, we have introduced Becker’s quantum parameter $\alpha$ defined in Eq. (1.10) with $r = 1/\tau$. We do immediately see that this gain is proportional to the classical gain function $G_{cl}$ from section 1.2.3.

The proportionality factor is this quantum parameter. As discussed in 1.3.1, the parameter is small in the quantum limit, i.e. the gain in the quantum regime is smaller than the classical one. Even though the classical regime is defined in [12] for $\alpha \gg 1$, we cannot generalize our results beyond $\alpha = 1$, since our model is not valid for small recoils. But we see, that for $\alpha = 1$, our gain merges into the classical one.

To show the connection between both regimes, Eq. (4.8) is plotted in Figure 4.3. To see the transfer between the quantum and the classical regime, we replaced $h$ by $\chi \cdot h$ and varied $\chi \in (0, 1)$. In this plot, we see the functional behavior of the classical gain curve for $\chi$ approaching zero, which is emphasized by the black curve in the figure. This result is quite interesting: Even though our model is not valid in the classical regime, the gain curve evolves into its classical counterpart.

For increasing $\chi$, we see that the gain drops rapidly, as predicted by Eq. (4.10). This is not a surprising fact, since in the quantum regime, we have just one-photon processes in a limited interval of momenta. In this case, we have just a two-level behavior, whereas
4 Embedding the Results into the Context of other Theories

Figure 4.3: Gain function $\mathcal{G}$ against momentum $p$ and $\hbar$ for a photon number of $n = 10^5$. In the gain function, all $\hbar$ were replaced by $\chi \cdot \hbar$. As $\chi$ approaches zero, the functional behavior corresponds to the one of Figure 1.5 in the classical theory, as shown by the black line. For increasing $\chi$, the gain decreases rapidly. The maxima and minima are plotted red and blue. As $\chi$ approaches 1, these resonances drift apart, as we can see in the density plot on the bottom of the figure. So loss and gain interval become more separated and there is no interaction in between. The parameters are: $m = 10^{-31}$ kg, $g = 5 \cdot 10^{13}$ s$^{-1}$, $\tau = 10^{-16}$ s, and $k = 10^{10}$ m$^{-1}$.

in classical theory we do not have such a restriction and simultaneous multi-photon processes are possible (if there were any photons in classical theory).

To investigate the behavior of the gain when approaching the quantum regime, a density plot was plotted on the bottom of Figure 4.3. Even though the maxima and minima decrease, we can see their behavior in the quantum regime. The two peaks, which form classically the derivative of the sinc-function, drift apart and become more and more separated. So here we see again how the interaction areas arise. The recoil is increasing, and the gain is concentrated around $q/2$ and the loss around $-q/2$. Between those two resonances, there is no interaction at all and by that there is no gain or loss in intensity.
4.2 Connection to Classical FEL Theory

4.2.2 Saturation

In section 3.2, we did performed an iteration of the time evolution of the photon distribution, and we have seen it evolving from the vacuum into a steady state. This steady state depends only on the rate coefficients $R_{\pm n}$, as we saw in Eq. (3.10). Thus, it is of no importance from which initial distribution the iteration starts, it will always evolve into this steady state.

Hence, in the quantum regime we have the same behavior as in the classical regime: The intensity saturates at some point independently from the initial seeding intensity. Of course, the small oscillation seen in Figure 1.6 of the intensity in the saturated regime cannot occur in the quantum case. There, a steady state is responsible for the saturation, and the photon distribution does not change at all.

4.2.3 Linewidth

To see the connection of the linewidth calculated in Eq. (3.28) to the classical linewidth, we again go to the small signal regime. For that, we develop $D$ into orders of $g^2 \tau^2 \langle n \rangle$ up until the first non vanishing terms occur. Since $g^2 \tau^2 n$ is small, and $n$ is a positive integer, usually larger than one, we assume $g^2 \tau^2 \langle n \rangle + 1 \approx g^2 \tau^2 \langle n \rangle \ll 1$. By that, the terms with $\sin^2 \left( (\Omega_{n+1} - \Omega_{n}) \tau \right)$ occurring in Eq. (3.28) are equal to zero. So we get for the linewidth

$$D \approx \frac{rg^2}{4} \left[ \frac{1}{\langle n \rangle} \left( \frac{\phi_-/2}{\Omega_{(n)}^-} \right)^2 \sin^2 \Omega_{(n)}^- \tau + \frac{1}{\langle n \rangle} \left( \frac{\phi_+/2}{\Omega_{(n)}^+} \right)^2 \sin^2 \Omega_{(n)-1}^+ \tau \right],$$

when we neglect cavity damping and drop the integration over $p$ as in section 4.2.1. If we expand this expression into a Taylor series around $g^2 \tau^2 \langle n \rangle + 1$, this yields with

$$\left( \frac{\sin \Omega_{(n)} \tau}{\Omega_{(n)}^\pm \langle n \rangle} \right)^2 \approx \left( \frac{\sin \Omega_{(n)} \tau}{\Omega_{(n)}^\pm \langle n \rangle} \right)^2 \left|_{g^2 \tau^2 \langle n \rangle + 1} = \frac{\sin^2 \phi_+/2 \tau}{\left( \phi_+/2 \right)^2} = \frac{S_+^2}{\left( \phi_+/2 \right)^2} \right.$$  

an expression corresponding to the linewidth in second order perturbation theory [9]

$$D \approx \frac{rg^2}{4 \langle n \rangle} \left( S_+^2 + S_-^2 \right).$$
To compare this to the classical linewidth, we use the Taylor expansion (4.9) for small $\hbar$ and write the linewidth as

$$D = \frac{rg^2}{4\langle n \rangle} \left( S^2 + \frac{qk}{m} \frac{\partial}{\partial W} S^2 + S^2 \frac{qk}{m} \frac{\partial}{\partial W} S^2 \right) = \frac{rg^2}{2\langle n \rangle} S^2.$$ 

This result is the same as obtained in $[10, 12]$. In this short analytic discussion we saw that for small $\hbar$, both peaks of the amplitudes $S^2_\pm$ at $\pm q/2$ approach each other and merge to a peak of $2S^2$, which is twice as large as the two separated ones. To show this evolution, we once again replace $\hbar$ by $\chi \cdot \hbar$ in Eq. (3.28) and vary $\chi$ from zero to one, which is depicted in Figure 4.4. Here, we see the analytic results in a very clear way. The linewidth divides up into two peaks of the half height for increasing $\chi$. The linewidth does not decay more than that, because $g^2$ is independent of $\hbar$ (since $g \propto A^2/\hbar$ and $A \propto \sqrt{\hbar}$).

In $[9]$, the linewidth was used to define the quantum regime, as the regime, where the recoil is large enough if one can distinguish between both peaks of the linewidth. This
4.2 Connection to Classical FEL Theory

leads again to the quantum parameter $\alpha$.

We now follow the discussion of [12] to compare the linewidth to one of a conventional laser. For that, we first write the linewidth

$$D = \frac{1}{4 \langle n \rangle} \left[r g^2 \left(S^2 + S^2_+\right) + \frac{\omega}{Q}\right] = \frac{1}{2 \langle n \rangle} \left[r g^2 \left(S^2_+ - S^2_\mp\right) \frac{S^2_+ + S^2_\mp}{2 (S^2_+ - S^2_\mp)} + \frac{\omega}{2Q}\right]$$

with the inclusion of cavity losses at zero temperature. When we now look at Eq. (3.7), we see that in steady state

$$\frac{\omega}{Q} \langle n \rangle = \langle (n + 1)R_{n+1} - nR_n \rangle$$

and we can use this in our limit to find $rg^2 \left(S^2 - S^2_\pm\right) \approx \omega/Q$. If we use this and develop the $S^2_\pm$ up into first order of $\hbar$ according to Eq. (4.9), we find

$$D \approx \frac{\omega}{Q} \frac{1}{2 \langle n \rangle} \left[2S^2 + \frac{1}{2}\right] \approx \frac{\omega}{Q} \frac{1}{2} \left[\frac{2S^2}{\left(-2\frac{qk}{m}\frac{\partial}{\partial W}S^2\right)} + \frac{1}{2}\right].$$

Using the identity

$$\frac{\partial}{\partial W} \left\{ \frac{\sin(W\tau/2)}{W\tau/2} \right\}^2 = \frac{2\cos(W\tau\sin(W\tau/2)}{(W\tau/2)^2} - \frac{2\sin^2(W\tau/2)}{(W\tau/2)^3} = \frac{2\tau^2}{\tau^2} \frac{\partial}{\partial W} S^2$$

this can be written as

$$D \approx \frac{\omega}{Q} \frac{1}{2 \langle n \rangle} \left[\frac{m}{2qk} \frac{S^2}{\left(-\frac{\partial}{\partial W}S^2\right)} + \frac{1}{2}\right] = \frac{\omega}{Q} \frac{1}{2 \langle n \rangle} \left[\frac{m}{2qk} \frac{\tau^2 |\sin(x)/x|^2}{\left(-\frac{\partial}{\partial x} |\sin(x)/x|^2\right)} + \frac{1}{2}\right],$$

where we wrote $x = W\tau/2$ for the sake of simplicity. We now recall the definition of the quantum parameter $\alpha$, and use $-\partial [\sin(x)/x]^2 / \partial x \approx [\sin(x)/x]^2$ as in [12]. This last relation holds true at maximal small signal gain. With that, we get

$$D = \frac{\omega}{Q} \frac{2\alpha + \frac{1}{2}}{2 \langle n \rangle}.$$

If we compare this to the result for an conventional atomic laser with linewidth $D_{\text{atomic}}$
4 Embedding the Results into the Context of other Theories

from [21], we arrive at

\[ D = \left[ 2\alpha + \frac{1}{2} \right] D_{\text{atomic}}. \quad (4.11) \]

Since the quantum parameter \( \alpha \) is much smaller than unity in the quantum regime, we see that the quantum FEL has a narrower linewidth in comparison with a conventional laser. Its lower bound is half the linewidth of an conventional laser. On the other hand, for \( \alpha > 1/4 \), the linewidth is broader. With increasing recoil, i.e. with decreasing \( \alpha \), we get a much broader linewidth.
In this thesis, we have introduced a simple quantum model of FELs. We have found heuristic arguments to justify it, such as the rotating wave approximation. A more rigorous derivation was the adiabatic approximation, which is known from atomic Bragg scattering at light waves, but in contrast to that, we did allow momenta with a deviation from the resonances.

With these tools, we did motivate an effective Hamiltonian for which we could solve the Schrödinger equation analytically and that had a lot in common with the Jaynes-Cummings model. We therefore could easily derive the time evolution of the radiation field, which was nonlinear in the photon number $n$. This made it possible for us to find a steady state which we investigated closer. For a given initial electron momentum distribution, maxima and variance can be calculated numerically. We have also found that it is possible to have a sub-Poissonian steady state photon distribution.

In the last chapter, we have seen the connection to other theories: We can approach the low signal regime by expanding our solution in orders of small photon numbers. This limit brings us back to the results of perturbation theory. As $\hbar$ approaches zero, some classical features of the FEL appear, even though they are beyond our model. We therefore can see our results in the context of these two other theories.

The simplicity of this two-level model is intriguing and convincing. However, some crude approximations have been made. First of all, interaction outside of the gain or loss interval in momentum space has been neglected. This could be easily fixed by performing the adiabatic approximation around every resonance. By that, we would allow multi-photon transitions, but still have a two-level type behavior for every momentum. That way, we would get a more accurate description for momenta at other resonances, but at least in the small signal regime, it was shown by perturbation theory that the most important interaction takes place between the resonances we considered.

Of course, there are multiple possibilities to extend this model. We have to keep in mind, that we have just discussed a one-dimensional theory, which has to be extended to higher dimensions. Further, we have included cavity losses, but the interpretation of this cavity is not clear, since we have just developed a one-particle theory. So one step to improve this model would be to build up a many-particle theory. A promising approach could be to use Dicke’s superradiance. This has been investigated for two-level
5 Conclusions

atoms and it might be possible to transfer it to an effective two-level Hamiltonian in FEL theory. By that, one might be able to see directly that the intensity of the coherent radiation is proportional to the square of the number of electrons.
A

REWITING THE HAMILTONIAN

A.I Action of the Ladder Operator

In this section, we derive a different representation of the operator \( \exp[\pm i2k\hat{z}] \). This representation is quite important for the interpretation as ladder operator and by that for the derivation of the effective two-level Hamiltonian in this thesis.

When we look at the operator, as in \([9]\), the action

\[
\langle z| \exp[\pm i2k\hat{z}] |p\rangle = \exp[\pm i2kz] \langle z| p \rangle = \exp[\pm i2kz] \exp \left[ \frac{i}{\hbar}zp \right] \\
= \exp \left[ \frac{i}{\hbar}z(p \pm 2\hbar k) \right] = \langle z| p \pm 2\hbar k \rangle
\]

(A.1)

becomes clear. In this consideration we used the position representation of the momentum eigenstates. When we define the recoil

\[
q \equiv 2\hbar k
\]

(A.2)

we see, that the operator \( \exp[\pm i2k\hat{z}] \) acts as raising or lowering operator in momentum space, since

\[
\exp[\pm i2k\hat{z}] |p\rangle = |p \pm q\rangle.
\]

Hence, we find

\[
\exp[\pm i2k\hat{z}] = \int_{-\infty}^{\infty} dp \ |p \pm q\rangle \langle p|
\]

(A.3)

as a convenient representation of this operator.
A Rewriting the Hamiltonian

A.II Transformation into Interaction Picture

In this section, we transform the Hamiltonian from Eq. (1.6) into the interaction picture. When we look at the unitary transformation \( \exp \left[ i \left( \hat{H}_{\text{rest}} + \hat{H}_{\text{kin}} + \hat{H}_{\text{field}} \right) t / \hbar \right] \), we realize that the transformations with \( \hat{H}_{\text{rest}} \), \( \hat{H}_{\text{kin}} \), and \( \hat{H}_{\text{field}} \) can be done independently, since they all do commute pairwise.

A.II .1 Transformation with the Rest Energy

Since \( \hat{H}_{\text{rest}} = mc^2 \) is just a c-number, the commutation relations

\[
\left[ \hat{H}_{\text{rest}}, \hat{a}_j \right] = \left[ \hat{H}_{\text{rest}}, \hat{a}^\dagger_j \right] = 0 \\
\left[ \hat{H}_{\text{rest}}, \exp \left[ \pm i 2k \hat{z} \right] \right] = 0
\]

are obvious. So we immediately find

\[
\exp \left[ \frac{i}{\hbar} \hat{H}_{\text{rest}} t \right] \hat{H}_{\text{int}} \exp \left[ -\frac{i}{\hbar} \hat{H}_{\text{rest}} t \right] = \hat{H}_{\text{int}}
\]

and hence the transformation with the rest energy does not give a contribution.

A.II .2 Transformation with the Kinetic Energy

The kinetic part of the Hamiltonian is \( \hat{H}_{\text{kin}} = \hat{p}^2 / (2m) \). Of course one sees at once

\[
\left[ \hat{H}_{\text{kin}}, \hat{a}_j \right] = \left[ \hat{H}_{\text{kin}}, \hat{a}^\dagger_j \right] = 0
\]

and hence the position dependent phase is the only part that does contribute when we perform the unitary transformation. We can see the transformation

\[
e^{i \hat{H}_{\text{kin}} t} \exp \left[ \pm i 2k \hat{z} \right] e^{-i \hat{H}_{\text{kin}} t} = e^{i \frac{\hat{p}^2 \, t}{2m}} \exp \left[ \pm i 2k \hat{z} \right] e^{-i \frac{\hat{p}^2 \, t}{2m}}
\]

\[
= e^{i \frac{\hat{p}^2 \, t}{2m}} \int_{-\infty}^{\infty} dp \ |p \pm q \rangle \langle p| e^{-i \frac{\hat{p}^2 \, t}{2m}}
\]

\[
= \int_{-\infty}^{\infty} dp \ e^{i \frac{(p \pm q)^2 \, t}{2m}} |p \pm q \rangle \langle p| e^{-i \frac{\hat{p}^2 \, t}{2m}}
\]

\[
= \int_{-\infty}^{\infty} dp \ |p \pm q \rangle \langle p| \exp \left[ i \frac{\pm 2pq + q^2 \, t}{\hbar / 2m} \right]
\]
A.II Transformation into Interaction Picture

of the phase, where we used the ladder representation Eq. (A.3) of exp [±i2k\hat{z}]. If we now define the phases

$$\phi_{\pm}(p) = \frac{\pm 2pq + q^2}{2mh} = \frac{2k}{m} \left( p \pm \frac{q}{2} \right)$$

(A.5)

we get the relation

$$e^{\frac{i}{\hbar}\hat{H}_{\text{kin}}t} \exp [\pm i2k\hat{z}] e^{-\frac{i}{\hbar}\hat{H}_{\text{kin}}t} = \exp [\pm i2k\hat{z}] e^{i\phi_{\pm}(p)t}.$$  

(A.6)

A.II .3 Transformation with the Free Field Energy

The free field part of the Hamiltonian is \( \hat{H}_{\text{field}} = \hbar \omega_L \hat{a}_L^\dagger \hat{a}_L + \hbar \omega_W \hat{a}_W^\dagger \hat{a}_W \). Hence, we see the commutation relation

$$\left[ \hat{H}_{\text{field}}, \hat{z} \right] = 0$$

at once. From [18] and others, it is known that the annihilation in the interaction picture read

$$e^{i\omega_j \hat{a}_j^\dagger t} \hat{a}_j e^{-i\omega_j \hat{a}_j^\dagger t} = \hat{a}_j e^{-i\omega_j t}$$

$$e^{i\omega_j \hat{a}_j^\dagger t} \hat{a}_j^\dagger e^{-i\omega_j \hat{a}_j^\dagger t} = \hat{a}_j^\dagger e^{i\omega_j t}. $$

These are the last relations needed for the transformation in the interaction picture.

A.II .4 Putting the Parts Together

We now use the results from the equations derived in the sections above, to transform into the interaction picture. We get

$$\hat{H} \equiv \exp \left[ \frac{i}{\hbar} \left( \hat{H}_{\text{rest}} + \hat{H}_{\text{kin}} + \hat{H}_{\text{field}} \right) t \right] \hat{H}_{\text{int}} \exp \left[ -\frac{i}{\hbar} \left( \hat{H}_{\text{rest}} + \hat{H}_{\text{kin}} + \hat{H}_{\text{field}} \right) t \right]$$

$$= \hbar \tilde{g} \left( \hat{a}_L e^{-i\omega_L t} \hat{a}_L^\dagger e^{i\omega_W t} e^{i[(\omega_L - \omega_W) t + 2k\hat{z}]} e^{i\phi_{\pm}(p)t} + \text{h.c.} \right)$$

$$= \hbar \tilde{g} \left( \hat{a}_L \hat{a}_L^\dagger e^{i2k\hat{z}} e^{i\phi_{\pm}(p)t} + \text{h.c.} \right)$$

(A.7)

for the Hamiltonian in the interaction picture.
Solving Coupled Differential Equations

When performing the adiabatic approximation in section 2.2.3, differential equations of the coefficients occur in the form of

\[ \dot{c}_1 = a_1 c_1 + b c_2 \]
\[ \dot{c}_2 = a_2 c_2 + b c_1, \]

where the \( a_j, b \) are constant in time. The general solution of this system reads

\[ c_1(t) = e^{-i\frac{a_1+a_2}{2}t} \left\{ -ib \frac{\sin \tilde{\Omega} t}{\tilde{\Omega}} c_2(0) + \left[ \cos \tilde{\Omega} t + i \frac{a_2 - a_1 \sin \tilde{\Omega} t}{2} \right] c_1(0) \right\}, \]
\[ c_2(t) = e^{-i\frac{a_1+a_2}{2}t} \left\{ -ib \frac{\sin \tilde{\Omega} t}{\tilde{\Omega}} c_1(0) + \left[ \cos \tilde{\Omega} t - i \frac{a_2 - a_1 \sin \tilde{\Omega} t}{2} \right] c_2(0) \right\}, \]

where we defined

\[ \tilde{\Omega} = \sqrt{\left( \frac{a_2 - a_1}{2} \right)^2 + b^2}. \]

We now look at

\[ \frac{(p + \nu q/2)^2 - \varphi^2}{2m\hbar} = \frac{(p + \nu q/2)^2 - (q/2)^2}{2m\hbar} = \frac{p^2}{2m\hbar} + \frac{\phi_0 \left( \nu p + \frac{(\nu^2 - 1)q}{2} \right)}{2}, \tag{B.1} \]

where we recalled the definition of \( \phi_0(p) \), Eq. (A.5). With the use of Eq. (B.1) we identify

\[ a_1 = \left\{ \frac{(p + q/2)^2 - \varphi^2}{2m\hbar} - \frac{2m\hbar q^2 n}{(p + 3q/2)^2 - \varphi^2} \right\} = \frac{p^2}{2m\hbar} + \frac{\phi_0(p)}{2} + \frac{g^2 n}{2m\hbar + \phi_0(3p/2 + 2q)} \]
\[ \equiv \frac{p^2}{2m\hbar} + h \frac{\phi_0(p)}{2} + \frac{\Delta_1}{2}, \tag{B.2} \]
\[ a_2 = \left\{ \frac{(p - q/2)^2 - \varphi^2}{2m\hbar} - \frac{2m\hbar q^2 (n+2)}{(p - 3q/2)^2 - \varphi^2} \right\} = \frac{p^2}{2m\hbar} - \frac{\phi_0(p)}{2} + \frac{g^2 (n+2)}{2m\hbar - \phi_0(3p/2 - 2q)} \]
\[ \equiv \frac{p^2}{2m\hbar} - \frac{\phi_0(p)}{2} + \frac{\Delta_2}{2}. \tag{B.3} \]
and

\[ b = g\sqrt{n + 1}. \]

With that, the Rabi frequency reads

\[ \tilde{\Omega}_n = \sqrt{\left( \frac{\phi_0(p) + \Delta_2 - \Delta_1}{2} \right)^2 + g^2(n + 1)} \]  

(B.4)

and the solutions of the differential equations are

\[
\tilde{c}_n \left( p + \frac{q}{2}, t \right) = e^{-i\left( \frac{p^2}{2m} + \frac{\Delta_1 + \Delta_2}{2} \right)t} \left\{ -ig\sqrt{n + 1} \frac{\sin \tilde{\Omega}t}{\tilde{\Omega}} \tilde{c}_{n+1} \left( p - \frac{q}{2}, 0 \right) \right. \\
+ \left. \left[ \cos \tilde{\Omega}_n t + i \frac{\phi_0(p) + \Delta_2 - \Delta_1}{2} \frac{\sin \Omega_n t}{\Omega_n} \right] \tilde{c}_n \left( p + \frac{q}{2}, 0 \right) \right\}
\]

and

\[
\tilde{c}_{n+1} \left( p - \frac{q}{2}, t \right) = e^{-i\left( \frac{p^2}{2m} + \frac{\Delta_1 + \Delta_2}{2} \right)t} \left\{ -ig\sqrt{n + 1} \frac{\sin \tilde{\Omega}t}{\tilde{\Omega}} \tilde{c}_n \left( p + \frac{q}{2}, 0 \right) \right. \\
+ \left. \left[ \cos \tilde{\Omega}_n t - i \frac{\phi_0(p) + \Delta_2 - \Delta_1}{2} \frac{\sin \Omega_n t}{\Omega_n} \right] \tilde{c}_{n+1} \left( p - \frac{q}{2}, 0 \right) \right\}
\]

in this special picture.

We now transform this result into the same interaction picture where we solved the Schrödinger equation with the Hamiltonian in the RWA. We use the relation

\[ |\psi\rangle = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} dp \ c_n(p) |n, p\rangle \]

\[ \equiv e^{\frac{i}{\hbar} H_{\text{kin}} t} \tilde{|\psi\rangle} = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} dp \ e^{\frac{i}{\hbar} \frac{p^2 - \nu^2}{2m} t} \tilde{c}_n(p) |n, p\rangle \]

to equate the coefficients, and get the connection between both pictures via

\[ c_n(p) = e^{\frac{i}{\hbar} \frac{p^2 - \nu^2}{2m} t} \tilde{c}_n(p). \]  

(B.5)
In our special case with $\varphi = q/2$ and Eq. (B.1) the solutions of the dynamics read

\begin{align*}
  c_n \left( p + \frac{q}{2}; t \right) &= e^{-i\frac{\Omega_0(p) + \Delta_1 + \Delta_2}{2} t} \left\{ -ig\sqrt{n+1} \frac{\sin \bar{\Omega} t}{\bar{\Omega}} c_{n+1} \left( p - \frac{q}{2}; 0 \right) \\
  &\quad + \left[ \cos \bar{\Omega} n t + i\frac{\phi_0(p)}{2} - \frac{\Delta_2 - \Delta_1}{2} \sin \bar{\Omega} n t \right] c_n \left( p + \frac{q}{2}; 0 \right) \right\} \quad \text{(B.6a)}
\end{align*}

\begin{align*}
  c_{n+1} \left( p - \frac{q}{2}; t \right) &= e^{i\frac{\phi_0(p) - \Delta_1 - \Delta_2}{2} t} \left\{ -ig\sqrt{n+1} \frac{\sin \bar{\Omega} t}{\bar{\Omega}} c_n \left( p + \frac{q}{2}; 0 \right) \\
  &\quad + \left[ \cos \bar{\Omega} n t - i\frac{\phi_0(p)}{2} + \frac{\Delta_2 - \Delta_1}{2} \sin \bar{\Omega} n t \right] c_{n+1} \left( p - \frac{q}{2}; 0 \right) \right\} \quad \text{(B.6b)}
\end{align*}

in this picture.
Cavity Losses due to a Heat Bath

To get a complete description of the FEL, we do not only need to discuss the interaction of electrons with the radiation field of the cavity, but we also have to investigate the damping of the cavity. For that, we assume a one-mode cavity damped by the coupling to a thermal heat bath. We follow [25, 27] to discuss this situation. Let a cavity with creation and annihilation operators $\hat{a}_L^\dagger$ and $\hat{a}_L$ be coupled to a bath of harmonic oscillators with the creation operators $\hat{b}_j^\dagger$ and the annihilation operators $\hat{b}_j$. The Hamiltonian reads

$$\hat{H}_{\text{tot}} = \hat{H}_{\text{cav}} + \hat{H}_{\text{bath}} + \hat{H}_{\text{coup}} = \hbar \omega \hat{a}_L^\dagger \hat{a}_L + \sum_j \hbar \nu_j \hat{b}_j^\dagger \hat{b}_j + \hbar \sum_j (\kappa_j \hat{a}_L^\dagger \hat{b}_j + \text{h.c.})$$

as in [27]. This coupling term is easy to interpret: If the cavity loses one photon, the heat bath gains photons depending of the frequency $\nu_j$ and the strength of the coupling constant $\kappa_j$. Transforming this into the interaction picture, we get

$$\hat{H}_{\text{damp}} = \exp \left[ -\frac{i}{\hbar} \left( \hat{H}_{\text{cav}} + \hat{H}_{\text{bath}} \right) (t - t_0) \right] \hat{H}_{\text{coup}} \exp \left[ \frac{i}{\hbar} \left( \hat{H}_{\text{cav}} + \hat{H}_{\text{bath}} \right) (t - t_0) \right] = \hbar \sum_j (\kappa_j \hat{a}_L^\dagger \hat{b}_j e^{i(\omega - \nu_j)(t - t_0)} + \text{h.c.}) \equiv \hbar \left( \hat{a}_L^\dagger \hat{F}(\tau) + \text{h.c.} \right),$$

where we used the results from section [A.II.3]. In the last step, we defined $\tau = t - t_0$ and

$$\hat{F}(\tau) = \sum_j \kappa_j \hat{b}_j^\dagger e^{i(\omega - \nu_j)\tau}$$
as an operator just acting on the heat bath reservoir. We will now discuss the time evolution of the density matrix. The time evolution of the coupled system is

\[
\dot{\varrho}_{\text{cav,bath}}(t) \approx \varrho_{\text{cav,bath}}(t_0) - \frac{i}{\hbar} \int_0^\tau d\tau' \left[ \hat{H}_{\text{damp}}(\tau'), \varrho_{\text{cav,bath}}(t_0) \right] + \left( -\frac{i}{\hbar} \right)^2 \int_0^\tau d\tau' \int_0^{\tau'} d\tau'' \left[ \hat{H}_{\text{damp}}(\tau'), \left[ \hat{H}_{\text{damp}}(\tau''), \varrho_{\text{cav,bath}}(t_0) \right] \right]
\]

up to second order perturbation theory and in the interaction picture. We now take a trace over the bath to get the equation of motion of the field in the cavity

\[
\dot{\varrho}_{\text{cav}}(t) \approx -\frac{1}{\hbar \tau} \int_0^\tau d\tau' \text{Tr}_{\text{bath}} \left\{ \left[ \hat{H}_{\text{damp}}(\tau'), \varrho_{\text{cav,bath}}(t_0) \right] \right\} - \left( \frac{1}{\hbar \tau} \right)^2 \int_0^\tau d\tau' \int_0^{\tau'} d\tau'' \text{Tr}_{\text{bath}} \left\{ \left[ \hat{H}_{\text{damp}}(\tau'), \left[ \hat{H}_{\text{damp}}(\tau''), \varrho_{\text{cav,bath}}(t_0) \right] \right] \right\}.
\]

We assume \( \varrho_{\text{cav,bath}}(t_0) = \varrho_{\text{cav}}(t_0) \varrho_{\text{bath}}(t_0) \) and that \( \varrho_{\text{bath}}(t_0) \) is a thermal state, which is diagonal. Hence, traces of the form \( \text{Tr}_{\text{bath}} \{ \hat{F} \varrho_{\text{bath}} \} \) vanish and the first integral is zero. We now use \( [\hat{A}, [\hat{B}, \hat{C}]] = \hat{A} \hat{B} \hat{C} - \hat{A} \hat{C} \hat{B} + \text{ h.c.} \), remember that the operators under the trace can be permuted cyclically and that \( \text{Tr}_{\text{bath}} \{ \cdot \varrho_{\text{bath}} \} = \langle \cdot \rangle_{\text{bath}} \) and get

\[
\dot{\varrho}_{\text{cav}} = -\left( \frac{1}{\hbar \tau} \right)^2 \int_0^\tau d\tau' \int_0^{\tau'} d\tau'' \times \left\{ \hat{a}_L^\dagger \hat{a}_L \varrho_{\text{cav}} \langle \hat{F}(\tau') \hat{F}^\dagger(\tau'') \rangle_{\text{bath}} - \hat{a}_L \hat{a}_L^\dagger \varrho_{\text{cav}} \langle \hat{F}(\tau'') \hat{F}^\dagger(\tau') \rangle_{\text{bath}} + \hat{a}_L \hat{a}_L^\dagger \varrho_{\text{cav}} \langle \hat{F}^\dagger(\tau') \hat{F}(\tau'') \rangle_{\text{bath}} - \hat{a}_L^\dagger \hat{a}_L \varrho_{\text{cav}} \langle \hat{F}^\dagger(\tau'') \hat{F}(\tau') \rangle_{\text{bath}} + \hat{a}_L^\dagger \hat{a}_L \varrho_{\text{cav}} \langle \hat{F}(\tau') \hat{F}^\dagger(\tau'') \rangle_{\text{bath}} - \hat{a}_L^\dagger \hat{a}_L \varrho_{\text{cav}} \langle \hat{F}^\dagger(\tau'') \hat{F}(\tau') \rangle_{\text{bath}} \right\} + \text{h.c.}
\]
as in [25]. Since the initial density matrix of the heat bath is diagonal, the terms
\( \langle \hat{F}(\tau')\hat{F}^\dagger(\tau'') \rangle_{\text{bath}} \) and \( \langle \hat{F}^\dagger(\tau')\hat{F}(\tau'') \rangle_{\text{bath}} \) vanish. We can also write
\[
\langle \hat{F}(\tau')\hat{F}^\dagger(\tau'') \rangle_{\text{bath}} = \sum_j |\kappa_j|^2 \langle \hat{b}_j\hat{b}_j^\dagger \rangle_{\text{bath}} e^{i(\omega-\nu_j)(\tau'-\tau)},
\]
where we replaced the sum over modes \( \nu_j \) by an integral over frequencies \( \nu \) and included the density of modes \( D(\nu) \) [25]. To perform the Markoff approximation, we use the relation
\[
\lim_{\tau' \to \infty} \int_0^{\tau''} d\tau' e^{i(\omega-\nu)(\tau'-\tau)} \approx \pi \delta(\omega - \nu),
\]
where the Cauchy principle part is neglected, since it causes just a small frequency shift. The so called Markoff approximation is now [25]
\[
\int_0^\tau d\tau' \int_0^{\tau''} d\tau'' \langle \hat{F}(\tau')\hat{F}^\dagger(\tau'') \rangle_{\text{bath}} \approx \int_0^\tau d\tau' \int_0^\infty d\nu \pi \delta(\omega - \nu) D(\nu)|\kappa(\nu)|^2 \langle \hat{b}(\nu)\hat{b}(\nu)^\dagger \rangle_{\text{bath}},
\]
We know, that for a thermal state at temperature \( T \), the mean photon number is [21]
\[
\langle \hat{b}^\dagger(\omega)\hat{b}(\omega) \rangle_{\text{bath}} = \frac{1}{e^{\hbar\omega/(k_B T)} - 1} \equiv n_{\text{th}}, \quad (C.1)
\]
the thermal photon number. Hence, the equation of motion for the reduced density matrix
\[
\dot{\hat{\rho}}_{\text{cav}} = -\frac{\omega}{Q} (n_{\text{th}} + 1) \left[ \hat{a}_L^\dagger \hat{a}_L \hat{\rho}_{\text{cav}} - \hat{\rho}_{\text{cav}} \hat{a}_L \hat{a}_L^\dagger \right] - \frac{\omega}{Q} n_{\text{th}} \left[ \hat{\rho}_{\text{cav}} \hat{a}_L \hat{a}_L^\dagger - \hat{a}_L^\dagger \hat{a}_L \hat{\rho}_{\text{cav}} \right] + \text{h.c.}
\]
describes a laser field in a cavity coupled to a heat bath. We defined the quality \( Q \) of a cavity with frequency \( \omega \) as \( Q = \omega/(2\pi D(\omega)|\kappa(\omega)|^2) \). If we define \( \hat{\rho}_{\text{cav}} = \sum_{n,m} \rho_{n,m} |n \rangle \langle m| \) we get as an equation of motion for the matrix elements [18]
\[
\dot{\rho}_{n,m} = -\rho_{n,m} \frac{\omega}{Q} \left[ n_{\text{th}} (n + m + 1) + \frac{n + m}{2} \right] + \rho_{n-1,m-1} \frac{\omega}{Q} n_{\text{th}} \sqrt{n} \sqrt{m} + \rho_{n+1,m+1} \frac{\omega}{Q} (n_{\text{th}} + 1) \sqrt{n + 1} \sqrt{m + 1}. \quad (C.2)
\]
Time Evolution of the Matrix Elements

In this section, we calculate the time evolution of the matrix elements $\bar{\rho}_{n,m}$ from section 3.1 due to a Schrödinger equation with the effective Hamiltonian. There,

$$
\bar{\rho}_{n,m}(t + \tau) = \left( \int_{-\infty}^{-q} dp + \int_{-q}^{0} dp + \int_{0}^{q} dp + \int_{q}^{\infty} dp \right) c_n(p; t + \tau) c^*_m(p; t + \tau)
$$

was the form we used. We first look at the intervals $(-\infty, -q)$ and $(q, \infty)$. We know from section 2.1 that the coefficients fulfill the differential equation

$$
\dot{c}_n(p; t) = 0
$$

in these intervals. There is no interaction, and

$$
c_n(p; t + \tau) = c_n(p; t)
$$

is the trivial solution of this equation. Therefore, we get

$$
\left( \int_{-\infty}^{-q} dp + \int_{q}^{\infty} dp \right) c_n(p; t + \tau) c^*_m(p; t + \tau) = \bar{\rho}_{n,m}(t) \left( \int_{-\infty}^{-q} dp + \int_{q}^{\infty} dp \right) \bar{\rho}(p)
$$

(D.1)

with the diagonal elements $\bar{\rho}(p, p) = \bar{\rho}(p)$ and an initially decoupled electron and laser field.

We now turn to the solution of the coupled Rabi equations Eq. (2.10). In the gain interval $(0, q)$ the time evolution is, according to Eq. (2.11a),

$$
c_n(p; t + \tau) = e^{-i \phi \tau^+} \left\{ -ig \sqrt{n + 1} \sin \frac{\Omega^+_{n-1}}{\Omega_n} c_{n+1}(p - q; t) \\
+ \left[ \cos \frac{\Omega^+_{n-1}}{2} \Omega_n \right] c_n(p; t) \right\},
$$

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where the generalized Rabi frequency

\[ \Omega_n^\pm \equiv \sqrt{\frac{\phi \pm 2}{2}} + g^2(n + 1) \]  \hspace{1cm} (D.2)\]

was defined. Hence, we get

\[
\int_0^q dp \ c_n(p; t + \tau)c_m^*(p; t + \tau) = \\
\int_0^q dp \ \left\{ q_{n+1,m+1}(t)g(p - q) \ g^2\sqrt{n + 1}\sqrt{m + 1} \ \frac{\sin \Omega_n^-\tau \sin \Omega_m^-\tau}{\Omega_n^- \Omega_m^-} \right. \\
+ \ q_{n,m}(t) \ g(p) \ \left[ \cos \Omega_n^-\tau \cos \Omega_m^-\tau + \left( \frac{\phi - 2}{2} \right)^2 \frac{\sin \Omega_n^-\tau \sin \Omega_m^-\tau}{\Omega_n^- \Omega_m^-} \right. \\
+ i \frac{\phi - 2}{2} \left( \frac{\sin \Omega_n^-\tau}{\Omega_n^-} \cos \Omega_m^-\tau - \cos \Omega_n^-\tau \frac{\sin \Omega_m^-\tau}{\Omega_m^-} \right) \right] \\
+ \ q_{n+1,m}(t) \ g(p - q, p) g\sqrt{n + 1} \ \frac{\sin \Omega_n^-\tau}{\Omega_n^-} \left[ \cos \Omega_m^-\tau - i \frac{\phi - 2}{2} \frac{\sin \Omega_m^-\tau}{\Omega_m^-} \right] \\
+ \ q_{n,m+1}(t) \ g(p, p - q) \left[ \cos \Omega_n^-\tau + i \frac{\phi - 2}{2} \frac{\sin \Omega_m^-\tau}{\Omega_n^-} \right] \ g\sqrt{m + 1} \ \frac{\sin \Omega_m^-\tau}{\Omega_m^-} \} .
\]

This expression is quite complicated. We will simplify it further by using the condition of our quantum regime. As discussed in section 2.4, there is a condition that the initial electron momentum state must be sufficiently sharp, i.e. \( \Delta p \leq q/2 \), and centered at \( \pm q/2 \). So we can set \( g(p, p \pm q; t) \approx 0 \approx g(p \pm, p; t) \), since the center of the distributions are separated by \( q \) and each distribution has a width less than \( q/2 \). So we can neglect the overlap. That means, we only have a coupling between elements of the photon density matrix just along the diagonal.
When we in addition to that shift the index of integration of the first summand \( p \to p + q \),

\[
\int_{-q}^{q} dp \; c_n(p; t + \tau)c_m^*(p; t + \tau) =
\]

\[
\varrho_{n+1,m+1}(t) \int_{-q}^{q} dp \; g(p)g^2 \sqrt{n+1} \sqrt{m+1} \frac{\sin \Omega_{n+1}^+ \sin \Omega_{m+1}^+}{\Omega_n^+ \Omega_m^+} \]

\[
+ \varrho_{n,m}(t) \int_{0}^{q} dp \; g(p) \left[ \cos \Omega_{n-1}^- \cos \Omega_{m-1}^- + \left( \frac{\phi_-}{2} \right)^2 \frac{\sin \Omega_{n-1}^- \sin \Omega_{m-1}^-}{\Omega_n^- \Omega_m^-} \right. \\
+ \left. \frac{\phi_-}{2} \left( \frac{\sin \Omega_{n-1}^-}{\Omega_n^-} \cos \Omega_{m-1}^- - \cos \Omega_{n-1}^- \sin \Omega_{m-1}^- \Omega_m^- \right) \right] \]  \hspace{1cm} (D.3)

is the time evolution of the coefficients in the gain interval.

The solution of the Rabi equations in the loss interval \((-q, 0)\) are

\[
c_n(p; t + \tau) = e^{i \frac{\phi_+}{2} \tau} \left\{ -i g \sqrt{n} \frac{\sin \Omega_{n-1}^+}{\Omega_{n-1}^+} c_{n-1}(p + q; t) \\
+ \cos \Omega_{n-1}^+ + i \frac{\phi_+}{2} \frac{\sin \Omega_{n-1}^+}{\Omega_{n-1}^+} \right\} c_n(p; t) \}
\]

according to Eq. (2.11b). We will treat the case of \( n = 0 \) separately. We now get under the assumption of the quantum regime

\[
\int_{-q}^{0} dp \; c_n(p; t + \tau)c_m^*(p; t + \tau) =
\]

\[
\varrho_{n-1,m-1}(t) \int_{0}^{q} dp \; g(p)g^2 \sqrt{n} \sqrt{m} \frac{\sin \Omega_{n-1}^- \sin \Omega_{m-1}^-}{\Omega_{n-1}^- \Omega_{m-1}^-} \]

\[
+ \varrho_{n,m}(t) \int_{-q}^{0} dp \; g(p) \left[ \cos \Omega_{n-1}^+ \cos \Omega_{m-1}^+ + \left( \frac{\phi_+}{2} \right)^2 \frac{\sin \Omega_{n-1}^+ \sin \Omega_{m-1}^+}{\Omega_{n-1}^+ \Omega_{m-1}^+} \right. \\
+ \left. \frac{\phi_+}{2} \left( \frac{\sin \Omega_{n-1}^+}{\Omega_{n-1}^+} \cos \Omega_{m-1}^- - \cos \Omega_{n-1}^+ \sin \Omega_{m-1}^- \Omega_{m-1}^- \right) \right] \]  \hspace{1cm} (D.4)

analogously to the gain interval. Since if there is no interaction, if the electrons are shot...
D Time Evolution of the Matrix Elements

into the wiggler with a momentum in the loss regime, but there are no photons in the
cavity, no dynamics takes place. The differential equation reads

\[ \dot{c}_0(p; t) = 0 \]

and we get

\[ \int_{-q}^{0} dp \ c_0(p; t + \tau)c_0^*(p; t + \tau) = \varrho_{0,0}(t) \int_{-q}^{0} dp \ \varrho(p). \]  

(D.5)

As we will see from an easy trigonometric consideration, this corresponds to the change
of the diagonal elements from Eq. (D.4). The off-diagonal elements are with
\( n \neq 0 \)

\[ \int_{-q}^{0} dp \ c_n(p; t + \tau)c_0^*(p; t + \tau) = \varrho_{n,0}(t) \int_{-q}^{0} dp \ \varrho(p) e^{i \frac{\phi_{\pm}}{2} + 2 \Omega_{n-1}^+ \tau} \left[ \cos \Omega_{n-1}^+ + i \frac{\phi_{\pm} \sin \Omega_{n-1}^+}{\Omega_{n-1}^+} \right], \]  

(D.6)

where we again used the quantum condition to eliminate the off-diagonal elements \( \varrho(p + q, p) \). When we put these results together we get

\[ \varrho_{n,n}(t + \tau) = \varrho_{n,n}(t) \left( \int_{-\infty}^{-q} dp + \int_{0}^{q} dp \right) \varrho(p) + \varrho_{n,n}(t) \left( \int_{-q}^{0} dp + \int_{0}^{q} dp \right) \varrho(p) \]

\[ - \varrho_{n,n}(t) \ g^2 \left\{ n \int_{-q}^{0} dp \ \varrho(p) \left[ \cos^2 \Omega_{n-1}^+ - 1 + \left( \frac{\phi_{\pm}}{2} \right)^2 \frac{\sin^2 \Omega_{n-1}^+}{\Omega_{n-1}^+} \right] \right\} \]

\[ + (n + 1) \int_{0}^{q} dp \ \varrho(p) \left[ \cos^2 \Omega_{n}^- - 1 + \left( \frac{\phi_{\pm}}{2} \right)^2 \frac{\sin^2 \Omega_{n}^-}{\Omega_{n}^-} \right] \}

\[ + \varrho_{n+1,n+1}(t) \ g^2(n + 1) \int_{-q}^{0} dp \ \varrho(p) \frac{\sin^2 \Omega_{n}^+ \tau}{\Omega_{n}^+} \]

\[ + \varrho_{n-1,n-1}(t) \ g^2 n \int_{0}^{q} dp \ \varrho(p) \frac{\sin^2 \Omega_{n-1}^- \tau}{\Omega_{n-1}^-}, \]

where we added the last two integrals in the first line and subtracted corresponding terms
in the second and third line. We now use the trigonometric relation \( \sin^2 x + \cos^2 x = 1 \)
\[ \cos^2 \Omega_n^\pm \tau - 1 + \left( \frac{\phi_\pm}{2} \right)^2 \frac{\sin^2 \Omega_n^\pm \tau}{\Omega_n^\pm 2} = \frac{\sin^2 \Omega_n^\pm \tau}{\Omega_n^\pm 2} \left( -\Omega_n^\pm 2 + \left( \frac{\phi_\pm}{2} \right)^2 \right) \\
= -g^2 (n + 1) \frac{\sin^2 \Omega_n^\pm \tau}{\Omega_n^\pm 2}. \]

With that, and the normalization of \( \varrho(p) \), the time evolution reads

\[ \varrho_{n,n}(t + \tau) = \varrho_{n,n}(t) \\
- \varrho_{n,n}(t) \frac{g^2}{n} \left\{ \int_{-q}^{0} dp \, \varrho(p) \frac{\sin^2 \Omega_{n-1}^+ \tau}{\Omega_{n-1}^\pm 2} + (n + 1) \int_{0}^{q} dp \, \varrho(p) \frac{\sin^2 \Omega_n^+ \tau}{\Omega_n^\pm 2} \right\} \\
+ \varrho_{n+1,n+1}(t) \frac{g^2}{n} (n + 1) \int_{-q}^{0} dp \, \varrho(p) \frac{\sin^2 \Omega_n^+ \tau}{\Omega_n^\pm 2} \\
+ \varrho_{n-1,n-1}(t) \frac{g^2}{n} \int_{0}^{q} dp \, \varrho(p) \frac{\sin^2 \Omega_{n-1}^- \tau}{\Omega_{n-1}^- 2}. \]  

(D.7)
Expansion of Square Roots

If \( \langle n \rangle \) is sufficiently large, one can use the expansion of the square root \[^{[34]}\]

\[
\sqrt{\langle n \rangle} + 1 = \langle n \rangle + \left( \frac{1}{2 \langle n \rangle} - \frac{1}{8 \langle n \rangle^2} \right) = \langle n \rangle + \frac{1}{2} - \frac{1}{8 \langle n \rangle} \tag{E.1a}
\]

\[
\sqrt{\langle n \rangle} + 2\sqrt{\langle n \rangle} + 1 = (\langle n \rangle + 1) + \frac{1}{2} - \frac{1}{8 (\langle n \rangle + 1)} \tag{E.1b}
\]

to write the terms of Eq. (3.27) with the coefficient \( \omega/Q \) as

\[
\frac{\omega}{Q} \left[ \frac{n_{th} + 1}{4 \langle n \rangle} + \frac{n_{th}}{4(\langle n \rangle + 1)} \right].
\]

Let us now expand the expression

\[
\left( \frac{\phi_+}{2} \right)^2 + g^2 \sqrt{\langle n \rangle} + 1 \sqrt{\langle n \rangle} = \sqrt{\left( \frac{\phi_+}{2} \right)^4 + g^4 \langle n \rangle (\langle n \rangle + 1)} + \left( g \frac{\phi_+}{2} \right)^2 2 \sqrt{\langle n \rangle} \sqrt{\langle n \rangle + 1}
\]

\[
\approx \sqrt{\Omega_{\langle n \rangle}^+ \frac{g^4}{4 \langle n \rangle} (\langle n \rangle + 1)} + \left( g \frac{\phi_+}{2} \right)^2 \left( 2 \langle n \rangle + 1 \right) - \left( g \frac{\phi_+}{2} \right)^2 \frac{1}{4 \langle n \rangle}
\]

\[
= \Omega_{\langle n \rangle}^+ \frac{g^4}{4 \langle n \rangle} (\langle n \rangle + 1) - \left( g \frac{\phi_+}{2} \right)^2 \frac{1}{4 \langle n \rangle}
\]

\[
= \Omega_{\langle n \rangle}^+ \Omega_{\langle n \rangle}^+ (\langle n \rangle + 1) - \left( g \frac{\phi_+}{2} \right)^2 \frac{1}{4 \langle n \rangle} \Omega_{\langle n \rangle}^+ \Omega_{\langle n \rangle}^+ (\langle n \rangle + 1)
\]

\[
\approx \Omega_{\langle n \rangle}^+ \Omega_{\langle n \rangle}^+ (\langle n \rangle + 1) - \left( g \frac{\phi_+}{2} \right)^2 \frac{1}{8 \langle n \rangle} \Omega_{\langle n \rangle}^+ \Omega_{\langle n \rangle}^+ (\langle n \rangle + 1) \tag{E.2}
\]
as well. Since $\Omega^4 \propto \langle n \rangle^2$ and $\langle n \rangle$ is sufficiently large, the last approximation $\sqrt{1 - x} \approx 1 - x/2$ is valid. We get

$$
\left( \frac{\phi}{2} \right)^2 + g^2 \sqrt{\langle n \rangle} + 2 \sqrt{\langle n \rangle} + 1 \approx \Omega_{(n)+1}^2 \Omega_{(n)}^2 \left( 1 - \left( g \frac{\phi}{2} \right)^2 \frac{1}{8 (\langle n \rangle + 1) \Omega_{(n)+1}^2 \Omega_{(n)}^2} \right)
$$

(E.3)

analogously. Since these terms are multiplied by two sine functions, the identity

$$
1 - \cos \alpha \cos \beta - \sin \alpha \sin \beta = 1 - \frac{\cos(\alpha - \beta) + \cos(\alpha + \beta) + \cos(\alpha - \beta) - \cos(\alpha + \beta)}{2} = 1 - \cos(\alpha - \beta) = 2 \sin^2(\alpha/2 - \beta/2)
$$

(E.4)

turns out to be useful to simplify this expression even further.
Perturbative Results

The results of perturbation theory in fourth order for small $g\tau\sqrt{n}$ lead to the equation of motion \[ [13] \]

$\dot{W}_n = W_n \cdot r \[ P(n \rightarrow n) - 1 \] + W_{n+1} \cdot r \[ P(n + 1 \rightarrow n) + W_{n-1} \cdot r \[ P(n - 1 \rightarrow n) + W_{n+2} \cdot r \[ P(n + 2 \rightarrow n) + W_{n-2} \cdot r \[ P(n - 2 \rightarrow n) \]

with

$\dot{W}_n = - (n + 1) \left[ r g^2 S^2_+ - \left\{ r (g\tau)^4 2 N_1 \right\} \right]$

$- n \left[ r g^2 S^2_+ + \frac{\omega}{Q} - \left\{ r (g\tau)^4 2 \left( \frac{\tilde{S}^2_+ \tilde{S}^2_-}{4} \right) \right\} \right]$

$- (n + 1)^2 \left[ r g^4 2 \frac{\partial S^2_+}{\partial (\phi_+ / 2)} - \left\{ r (g\tau)^4 2 N_1 \right\} \right]$

$- n^2 \left[ r g^4 2 \frac{\partial S^2_+}{\partial (\phi_+ / 2)} + \frac{\omega}{Q} - \left\{ r (g\tau)^4 2 \left( \frac{\tilde{S}^2_+ \tilde{S}^2_-}{4} \right) \right\} \right]$

$- (n + 1)^2 \left[ r g^4 2 \frac{\partial S^2_+}{\partial (\phi_+/2)} - \left\{ r (g\tau)^4 2 \left( \frac{\tilde{S}^2_+ \tilde{S}^2_-}{4} \right) \right\} \right]$

$\frac{\partial S^2_+}{\partial (\phi_+/2)} - \left\{ r (g\tau)^4 2 \left( \frac{\tilde{S}^2_+ \tilde{S}^2_-}{4} \right) \right\} + N_3 + N_4 + N_6 \right) \right]$

(F.1a)

Similarly, for $(n+1 \rightarrow n)$

$\dot{W}_n = (n+1) \left[ r g^2 S^2_+ + \frac{\omega}{Q} - \left\{ r (g\tau)^4 2 \tilde{S}^-_+ M_1 \right\} \right]$

$+ n \left[ - \left\{ r (g\tau)^4 2 \tilde{S}^-_+ M_4 \right\} \right]$

$+ (n+1)^2 \left[ r g^4 2 \frac{\partial S^2_+}{\partial (\phi_+ / 2)} - \left\{ r (g\tau)^4 2 \tilde{S}^-_+ M_1 \right\} \right]$

$+ n^2 \left[ - \left\{ r (g\tau)^4 2 \tilde{S}^-_+ M_4 \right\} \right]$

(F.1b)

And for $(n-1 \rightarrow n)$

$\dot{W}_n = n \left[ r g^2 S^2_- - \left\{ r (g\tau)^4 2 \tilde{S}^-_+ (M_3 - M_6) \right\} \right]$

$+ n^2 \left[ r g^4 2 \frac{\partial S^2_-}{\partial (\phi_+/2)} - \left\{ r (g\tau)^4 2 \tilde{S}^-_+ (M_3 + M_6) \right\} \right]$

(F.1c)
This form was found from [13] after some algebra. We used the relation

\[
\frac{\partial S_+^2}{\partial (\phi_+ \pm 2)} = \frac{1}{2} \frac{S_-}{(\phi_\pm/2)^2} \left( \tau \cos \frac{\phi_- \tau}{2} - S_- \right)
\]

to bring it to this form. We used the abbreviations as in [13], namely

\[
S_\pm = \frac{\sin \frac{\phi_\pm \tau}{2}}{\phi_\pm/2} \quad \text{(F.2a)}
\]

\[
\tilde{S}_\pm = \frac{S_\pm}{\tau} \quad \text{(F.2b)}
\]

\[
\phi_{\pm j} = \frac{2k}{m} \left( p \pm j \frac{q}{2} \right) \quad \text{(F.2c)}
\]

and

\[
N_1 = \frac{1}{(\phi_{-3})^2 (\phi_{-3} + \phi_-)\tau} \left[ \cos((\phi_{-3} + \phi_-)\tau) - 1 + \frac{1 - \cos \phi_- \tau}{\phi_- \tau} \right]
\]

\[
+ \frac{1}{(\phi_- \tau)^2} \left\{ \cos \phi_- \tau - 1 \left( \phi_{-3} + \phi_- \right) \phi_- \tau + \frac{1}{\phi_- \tau} \left[ \cos \phi_- \tau - 1 + \sin \phi_- \tau \right] \right\} \quad \text{(F.3a)}
\]

\[
N_6 = \frac{1}{(\phi_{+3})^2 (\phi_{+3} + \phi_+)\tau} \left[ \cos((\phi_{+3} + \phi_+)\tau) - 1 + \frac{1 - \cos \phi_+ \tau}{\phi_+ \tau} \right]
\]

\[
+ \frac{1}{(\phi_+ \tau)^2} \left\{ \cos \phi_+ \tau - 1 \left( \phi_{+3} + \phi_+ \right) \phi_+ \tau + \frac{1}{\phi_+ \tau} \left[ \cos \phi_+ \tau - 1 + \sin \phi_+ \tau \right] \right\} \quad \text{(F.3b)}
\]

\[
N_3 = \frac{1}{\phi_+ \tau} \left\{ \frac{1}{(\phi_- \tau)^2} + \frac{\cos \phi_- \tau - 1}{\phi_- \tau} \right\}
\]

\[
- \frac{1}{\phi_- \tau} \left( \phi_- + \phi_+ \right) \frac{1 - \cos \phi_- \tau}{\phi_- \tau} + \frac{\cos \phi_- \tau - 1}{(\phi_- \tau)^2} \right\} \quad \text{(F.3c)}
\]

\[
N_4 = \frac{1}{\phi_- \tau} \left\{ \frac{1}{\phi_- \tau} + \frac{\cos \phi_- \tau - 1}{(\phi_- \tau)^2} \right\}
\]

\[
- \frac{1}{\phi_- \tau} \left( \phi_- + \phi_+ \right) \frac{1 - \cos \phi_- \tau}{\phi_- \tau} + \frac{\cos \phi_- \tau - 1}{(\phi_- \tau)^2} \right\}, \quad \text{(F.3d)}
\]

as well as

\[
M_1 = \frac{1}{\phi_- \tau} \left\{ \frac{1}{\phi_+ \tau} \left[ \cos \frac{\phi_+ \tau}{2} - \tilde{S}_+ \right] - \cos \frac{\phi_- \tau}{2} \frac{\sin \left( \frac{\phi_- + \phi_+ \tau}{2} \right)}{\phi_- \tau} + \tilde{S}_+ \right\} \quad \text{(F.4a)}
\]

\[
M_6 = \frac{1}{\phi_- \tau} \left\{ \frac{1}{\phi_+ \tau} \left[ \cos \frac{\phi_- \tau}{2} - \tilde{S}_- \right] - \cos \frac{\phi_+ \tau}{2} \frac{\sin \left( \frac{\phi_- + \phi_+ \tau}{2} \right)}{\phi_+ \tau} + \tilde{S}_- \right\} \quad \text{(F.4b)}
\]
\[ M_3 = \frac{1}{\phi_{-\tau}} \left\{ \frac{\cos \frac{\phi_{-\tau}}{2}}{\phi_{-3\tau}} \right\} - \frac{\tilde{S}_-}{\phi_{-3\tau} (\phi_{-\tau} + \phi_{-3\tau})} - \cos \left( \frac{\phi_{-\tau} + \phi_{-3\tau} + \phi_{\tau}}{2} \right) \frac{\sin \frac{\phi_{-3\tau}}{2}}{\phi_{-3\tau} (\phi_{-\tau} + \phi_{-3\tau})} \right\} \] (F.4c)

\[ M_4 = \frac{1}{\phi_{+\tau}} \left\{ \frac{\cos \frac{\phi_{+\tau}}{2}}{\phi_{+3\tau}} \right\} - \frac{\tilde{S}_+}{\phi_{+3\tau} (\phi_{+\tau} + \phi_{+3\tau})} - \cos \left( \frac{\phi_{+\tau} + \phi_{+3\tau} + \phi_{\tau}}{2} \right) \frac{\sin \frac{\phi_{+3\tau}}{2}}{\phi_{+3\tau} (\phi_{+\tau} + \phi_{+3\tau})} \right\} \] (F.4d)

The two-photon transition probabilities are [13]

\[ rP(n + 2 \to n) = r(g\tau)^4 \left( 2 + 3n + n^2 \right) \frac{1}{(\phi_{+\tau})^2} \left[ \sin^2 \frac{(\phi_{+3\tau} + \phi_{+\tau})}{2} - 2 \sin \left( \frac{(\phi_{+3\tau} + \phi_{+\tau})}{2} \right) \sin \left( \frac{\phi_{+3\tau}}{2} \right) \frac{\cos \phi_{+\tau}}{2} \right] \] (F.5a)

\[ rP(n - 2 \to n) = r(g\tau)^4 \left( -n + n^2 \right) \frac{1}{(\phi_{-\tau})^2} \left[ \sin^2 \frac{(\phi_{-3\tau} + \phi_{-\tau})}{2} - 2 \sin \left( \frac{(\phi_{-3\tau} + \phi_{-\tau})}{2} \right) \sin \left( \frac{\phi_{-3\tau}}{2} \right) \frac{\cos \phi_{-\tau}}{2} \right]. \] (F.5b)


Bibliography


Bibliography


Statement of Authorship

Statement

I hereby confirm that this diploma thesis has been composed solely by myself, and I did not use other sources and resources than the acknowledged ones.

Erklärung

Ich erkläre, dass ich die Arbeit selbständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel verwendet habe.

Ulm, den 15. November 2011

(Enno Giese)