Theoretical Quantum Optics

Sheet 3 **SS 2015**
Date of issue: 02-05-2015 Discussion: 15-05-2015

Exercise 6 *Population and coherence in terms of the density matrix*

We consider the density matrix $\hat{\rho}$ and an orthogonal basis $|u_n\rangle$ $(n = 1, 2, ...)$ of the state space.

a) By using the Cauchy-Schwarz inequality, prove the relation

$$\rho_{nn}\rho_{mm} \quad \geq \quad |\rho_{nm}|^2 \tag{1}$$

for $\rho_{nm} \equiv \langle u_n| \hat{\rho} |u_m\rangle$. Inequality (1) reveals that for a density matrix $\hat{\rho}$, the coherence ($\rho_{nm} \neq 0$ for $n \neq m$) can only occur between the states $|u_n\rangle$ and $|u_m\rangle$ whose populations (ρ_{nn} and ρ_{mm}) are non-zero.

(1 point)

b) Moreover, prove that Inequality (1) results in the criteria

$$\mathrm{Tr}\rho^2 \leq 1$$

for the system to be in a pure ($\mathrm{Tr}\rho^2 = 1$) or in a mixed ($\mathrm{Tr}\rho^2 < 1$) state.

(1 point)

Exercise 7 *Density matrix for a continuous basis*

A bipartite system, consisting of the two subsystems A and B, is prepared in a pure state and characterized by the wave function $\psi(\alpha, \beta)$, where α and β are continuous variables of the subsystems A and B, accordingly. We introduce the reduced density matrix

$$\rho_A(\alpha, \alpha') \quad = \quad \int \mathrm{d}\beta \, \psi^*(\alpha, \beta)\psi(\alpha', \beta)$$

to describe only the subsystem A.

a) Find the condition for the function $\psi(\alpha, \beta)$, such that

- $\mathrm{Tr}(\rho_A) = 1$ is fulfilled.
- the reduced density matrix $\rho_A(\alpha, \alpha')$ describes a pure state.

(2 points)

b) We define the mean value $\left\langle \hat{f}_A \right\rangle$ of the operator $\hat{f}_A$, which only acts on the variable α of the subsystem A, as

$$\left\langle \hat{f}_A \right\rangle \equiv \int \mathrm{d}\alpha \int \mathrm{d}\beta \, \psi^*(\alpha,\beta) \hat{f}_A \psi(\alpha,\beta) \, .$$

Represent $\left\langle \hat{f}_A \right\rangle$ in terms of $\rho_A(\alpha,\alpha')$.

(1 point)

c) Write down the dynamical equation for $\rho_A(\alpha,\alpha',t)$.

(2 points)

Exercise 8 *The Bloch and pseudofield vectors*

a) From the Bloch equations without relaxation, prove that for a constant pseudofield vector $\vec{\Omega}$ the angle between the Bloch vector and $\vec{\Omega}$ remains constant. Discuss (suggest) the geometric interpretation of this result.

(1 point)

b) Let the pseudofield vector $\vec{\Omega}(t)$ be a slowly varying function of time t compared to the inverse Rabi frequency $\Omega_R^{-1}(t)$. Prove that if initially the Bloch vector is parallel to $\vec{\Omega}(t)$, it remains like that (approximately).

(2 points)