Theoretical Quantum Optics

Sheet 4 **SS 2017**

Date of issue: 09-05-2017 Discussion: 19-05-2017

Exercise 5 *Atom-light dressed states*

Within the rotating wave approximation, the dynamics of a two-level atom, consisting of the ground $|g\rangle$ and exited $|e\rangle$ states and interacting with an external and near-resonant electromagnetic field, is described by the Hamiltonian

$$\hat{H}_{RWA} = \frac{\hbar}{2}\begin{pmatrix} \Delta & -\Omega_0 \\ -\Omega_0 & -\Delta \end{pmatrix} .$$

a) Find the eigenvalues $E_\pm$ and the corresponding normalised eigenvectors $|\pm\rangle$ of the Hamiltonian $\hat{H}_{RWA}$. Proof the orthogonality of $|+\rangle$ and $|-\rangle$ states.

(2 points)

b) Sketch and discuss the behaviour of the energies $E_\pm(\Omega_0)$ as a function of Ω_0 for the cases of $\Delta = 0$ and of $\Delta \neq 0$.

(1 point)

c) By introducing $\cos\theta = \frac{\Delta}{\Omega_R}$ and $\sin\theta = \frac{\Omega_0}{\Omega_R}$ and presenting the Hamiltonian $\hat{H}_{RWA}$ in the form

$$\hat{H}_{RWA} = \frac{\hbar\Omega_R}{2}\begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{pmatrix} ,$$

with $\Omega_R = \sqrt{\Omega_0^2 + \Delta^2}$ being the Rabi frequency, find the real matrix $\hat{\mathcal{U}}(\theta)$ with $\det(\hat{\mathcal{U}}) = 1$, that transforms $\hat{H}_{RWA}$ into the diagonal form

$$\hat{H}'_{RWA} = \hat{\mathcal{U}}^{-1}(\theta)\hat{H}_{RWA}\hat{\mathcal{U}}(\theta) = \frac{\hbar\Omega_R}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

(2 points)

Exercise 6 *The Bloch-Siegert shift*

The Hamiltonian describing the system of an atom and a near-resonant light field has been introduced in the lecture and reads

$$\hat{H} = \frac{\hbar}{2}\begin{pmatrix} \Delta & -\Omega_0 \\ -\Omega_0 & -\Delta \end{pmatrix} - \frac{\hbar\Omega_0}{2}\left(e^{2i\omega_L t}\sigma_+ + e^{-2i\omega_L t}\sigma_-\right) \equiv \hat{H}_{RWA} + \hat{V}(t)\ ,$$

where $\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

a) By using the transformation matrix $\hat{\mathcal{U}}(\theta)$ defined in the Exercise 5, represent $\hat{H}$ in the basis of the dressed states $|\pm(\theta)\rangle$

$$\hat{H}' = \frac{\hbar\Omega_R}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{\hbar\Omega_0}{2}\left[e^{2i\omega_L t}\hat{\Sigma}_+(\theta) + e^{-2i\omega_L t}\hat{\Sigma}_-(\theta)\right] \equiv \hat{H}'_{RWA} + \hat{V}'(t)\ ,$$

where $\hat{\Sigma}_\pm(\theta) = \hat{\mathcal{U}}^{-1}(\theta)\sigma_\pm\hat{\mathcal{U}}(\theta)$.

(2 points)

b) Proof that the solution

$$\begin{pmatrix} c_+(t) \\ c_-(t) \end{pmatrix} = e^{-\frac{i}{\hbar}\hat{H}'_{RWA}t}\,\hat{U}(t)\begin{pmatrix} c_+(0) \\ c_-(0) \end{pmatrix}$$

of the Schrödinger equation

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} c_+(t) \\ c_-(t) \end{pmatrix} = \hat{H}'(t)\begin{pmatrix} c_+(t) \\ c_-(t) \end{pmatrix}$$

is given by the time-evolution operator $\hat{U}(t)$, presented in the form of the Magnus-Dyson expansion

$$\hat{U}(t) = \mathbb{1} - \frac{i}{\hbar}\int_0^t \hat{H}'_I(t')\,\mathrm{d}t' + \frac{(-i)^2}{\hbar^2}\int_0^t \mathrm{d}t'\int_0^{t'}\mathrm{d}t''\,\hat{H}'_I(t')\hat{H}'_I(t'') + \dots\ , \qquad (1)$$

where

$$\hat{H}'_I(t) = e^{\frac{i}{\hbar}\hat{H}'_{RWA}t}\,\hat{V}'(t)\,e^{-\frac{i}{\hbar}\hat{H}'_{RWA}t}\ .$$

(3 points)

c) Within the first-order approximation with respect to $\hat{H}'_I(t)$, find the probability of transition between the dressed states $|\pm\rangle$.

(2 points)

Hint: Find the coefficients $\langle\pm|\,\hat{U}(t)\,|+\rangle$ of the expansion

$$c_\pm(t) \quad = \quad e^{-\frac{i}{\hbar}E_\pm t}\left(\langle\pm|\,\hat{U}(t)\,|+\rangle\, c_+(0) + \langle\pm|\,\hat{U}(t)\,|-\rangle\, c_-(0)\right) \ ,$$

by taking into account only the second term in Eq. (1) for $\hat{U}(t)$.

d) Consider the third term of the Magnus-Dyson expansion for $\hat{U}(t)$, see Eq. (1), and proof that this term has a contribution, which is a linear function of time t, that is

$$\langle\pm|\,\hat{U}(t)\,|\pm\rangle \Rightarrow \frac{i}{\hbar}\delta E_\pm t$$

with

$$\delta E_\pm \quad = \quad \frac{\hbar^2\Omega_0^2}{4}\sum_{\nu=\pm}\left(\frac{\langle\pm|\,\hat{\Sigma}_+\,|\nu\rangle\,\langle\nu|\,\hat{\Sigma}_-\,|\pm\rangle}{E_\nu - E_\pm - 2\hbar\omega_L} + \frac{\langle\pm|\,\hat{\Sigma}_-\,|\nu\rangle\,\langle\nu|\,\hat{\Sigma}_+\,|\pm\rangle}{E_\nu - E_\pm + 2\hbar\omega_L}\right)$$

being the Bloch-Siegert shift and $E_\pm = \pm\frac{\hbar\Omega_R}{2}$. In particular, consider the case of $\Omega_R \ll \omega_L$ and $\Omega_0 \ll \omega_L$.

(5 points)