Theoretical Quantum Optics

Sheet 5 | **SS 2017**
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Exercise 7 *Population and coherence in terms of the density matrix*

We consider the density matrix $\hat{\rho}$ and an orthogonal basis $|u_n\rangle$ $(n = 1, 2, ...)$ of the state space.

a) By using the Cauchy-Schwarz inequality, prove the relation

$$\rho_{nn}\rho_{mm} \quad \geq \quad |\rho_{nm}|^2 \tag{1}$$

for $\rho_{nm} \equiv \langle u_n| \hat{\rho} |u_m\rangle$. Inequality (1) reveals that for a density matrix $\hat{\rho}$, the coherence ($\rho_{nm} \neq 0$ for $n \neq m$) can only occur between the states $|u_n\rangle$ and $|u_m\rangle$ whose populations (ρ_{nn} and ρ_{mm}) are non-zero.

(1 point)

b) Moreover, prove that Inequality (1) results in the criteria

$$\mathrm{Tr}\rho^2 \leq 1$$

for the system to be in a pure ($\mathrm{Tr}\rho^2 = 1$) or in a mixed ($\mathrm{Tr}\rho^2 < 1$) state.

(1 point)

Exercise 8 *Two spin-1/2 particles and Werner states*

We consider a system of two spin-1/2 particles. A basis of this bipartite system is given, for example, in terms of the Bell states

$$\begin{aligned}
|\Phi^+\rangle &= \frac{1}{\sqrt{2}} \left(|{\downarrow}\rangle_1 |{\downarrow}\rangle_2 + |{\uparrow}\rangle_1 |{\uparrow}\rangle_2 \right) \,, \\
|\Phi^-\rangle &= \frac{1}{\sqrt{2}} \left(|{\downarrow}\rangle_1 |{\downarrow}\rangle_2 - |{\uparrow}\rangle_1 |{\uparrow}\rangle_2 \right) \,, \\
|\Psi^+\rangle &= \frac{1}{\sqrt{2}} \left(|{\downarrow}\rangle_1 |{\uparrow}\rangle_2 + |{\uparrow}\rangle_1 |{\downarrow}\rangle_2 \right) \,, \\
|\Psi^-\rangle &= \frac{1}{\sqrt{2}} \left(|{\downarrow}\rangle_1 |{\uparrow}\rangle_2 - |{\uparrow}\rangle_1 |{\downarrow}\rangle_2 \right) \,,
\end{aligned}$$

where the spin-up state $|{\uparrow}\rangle_i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and the spin-down state $|{\downarrow}\rangle_i = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ form a basis of the state space for a single spin-1/2 particle i $(i = 1, 2)$.

a) Find the two-particle states corresponding to the total spin $S = 1$ (triplet) and the two-particle state corresponding to the total spin $S = 0$ (singlet) in terms of the Bell states.

(1 point)

b) Present the Bell states as matrix by evaluating the tensor products, for example $|{\downarrow}\rangle_1 |{\downarrow}\rangle_2 = |{\downarrow}\rangle_1 \otimes |{\downarrow}\rangle_2$.

(1 point)

c) We introduce a family of the Werner states

$$\hat{\rho}(\alpha) = \alpha |\Phi^+\rangle \langle\Phi^+| + \frac{1}{4}(1-\alpha)\mathbb{1}_4$$

characterized by a real parameter α. Represent the Werner states as a 4x4-matrix in the basis $\{|{\uparrow}\rangle_1 |{\uparrow}\rangle_2, |{\uparrow}\rangle_1 |{\downarrow}\rangle_2, |{\downarrow}\rangle_1 |{\uparrow}\rangle_2, |{\downarrow}\rangle_1 |{\downarrow}\rangle_2\}$.

(1 point)

d) Find the conditions for α, such that $\hat{\rho}(\alpha)$ is a density matrix.

(2 points)

e) Calculate the parameter α for which the Werner state describes a pure state.

(1 point)

f) Obtain the reduced density matrices $\hat{\rho}_1$ and $\hat{\rho}_2$ for the single particle states. Find the values of α, such that either $\hat{\rho}_1$ or $\hat{\rho}_2$ are not in a mixed state.

(2 points)

Exercise 9 *Density matrix for a continuous basis*

A bipartite system, consisting of the two subsystems A and B, is prepared in a pure state and characterized by the wave function $\psi(\alpha, \beta)$, where α and β are continuous variables of the subsystems A and B, accordingly. We introduce the reduced density matrix

$$\rho_A(\alpha, \alpha') = \int \mathrm{d}\beta \, \psi^*(\alpha, \beta)\psi(\alpha', \beta)$$

to describe only the subsystem A.

a) Find the condition for the function $\psi(\alpha, \beta)$, such that

- $\mathrm{Tr}(\rho_A) = 1$ is fulfilled.
- the reduced density matrix $\rho_A(\alpha, \alpha')$ describes a pure state.

(2 points)

b) We define the mean value $\left\langle \hat{f}_A \right\rangle$ of the operator $\hat{f}_A$, which only acts on the subsystem A, as

$$\left\langle \hat{f}_A \right\rangle \equiv \langle\psi| \, \hat{f}_A \, |\psi\rangle \ .$$

Represent $\left\langle \hat{f}_A \right\rangle$ in terms of $\rho_A(\alpha, \alpha')$.

(1 point)

c) Write down the dynamical equation for $\rho_A(\alpha, \alpha', t)$.

(2 points)