Moving mirror in the vacuum emits photons.[1]
Dynamical Casimir effect

- Parametric amplifier
- Experimental realization of the DCE
- Cavity
The static Casimir effect

- Two static mirrors in vacuum
- The vacuum is not empty! \[ \Delta E \Delta t \geq \frac{\hbar}{2} \]
- Mismatch of vacuum modes in space

\[ \downarrow \]

Attracting force between two metal plates
The dynamical Casimir effect (1)

- Moving mirror in vacuum
- Non-uniform acceleration
- Mismatch of vacuum modes in time

\[ \text{Moving mirror producing pair of photons.}[3] \]

\[ \text{Creation of photon pairs from the vacuum} \]
The Casimir effect

The dynamical Casimir effect (2)

Klein-Gordon equation for scalar field $\phi(x, t)$

$$\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} = 0$$

Another setup for the DCE

Cavity with moving mirror. Non-adiabatic adaption of the EM field to the boundary conditions if mirror velocity $v \lesssim c$.

Boundary conditions

$$\phi(x_m(t), t) = 0 \text{ and } \phi(l_0, t) = 0$$
Parametric amplifier

Classical example of a parametric amplifier (1)

Child swing as a simple pendulum

\[ \ddot{\vartheta} = -\frac{g}{l} \sin \vartheta \approx -\frac{g}{l} \vartheta \]

Now:

Sinusoidal modulation of the center of mass

Slightly varied effective length

\[ l(t) = l_0 - \Delta l \sin (2\omega_0 t) \]

where \( \omega_0 := \sqrt{\frac{g}{l_0}} \) and \( \Delta l \ll l_0 \)
Classical example of a parametric amplifier (2)

**Frequency \( \omega(t) \) for modulated effective length**

\[
\omega^2(t) = \frac{g}{l(t)} = \frac{g}{l_0 - \Delta l \sin (2\omega_0 t)} \approx \omega_0^2 \left( 1 + \epsilon \sin (2\omega_0 t) \right), \quad \epsilon := \Delta l / l_0
\]

Equation of motion for a parametric amplifier:

\[
\ddot{\vartheta} = -\omega^2(t) \vartheta
\]

Ansatz:

\[
\vartheta(t) = A(t) \cos (\omega_0 t) + B(t) \sin (\omega_0 t)
\]

Approximate solution:

\[
\vartheta(t) \approx \vartheta(0) \underbrace{e^{\frac{\epsilon \omega_0 t}{4}}}_{\text{exp. amplification}} \cos (\omega_0 t) + \frac{\dot{\vartheta}(0)}{\omega_0} \underbrace{e^{-\frac{\epsilon \omega_0 t}{4}}}_{\text{exp. suppression}} \sin (\omega_0 t)
\]
Parametric amplifier

“Quantum swinging child” (1)

QM 1D harmonic oscillator

\[ \hat{H}(t) = \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega^2 \hat{x}^2 \]

Introducing ladder operators:

\[ \hat{x} = \sqrt{\frac{\hbar}{2m\omega_0}} (\hat{a} + \hat{a}^\dagger) \quad \hat{p} = -i \sqrt{\frac{m\hbar \omega_0}{2}} (\hat{a} - \hat{a}^\dagger) \quad [a, a^\dagger] = 1 \]

QM parametric amplifier

\[ \hat{H}_{pa}(t) \approx \hbar \omega_0 \hat{a}^\dagger \hat{a} + \frac{\hbar \omega_0}{4} \epsilon \sin(2\omega_0 t) \left( \hat{a}^\dagger \hat{a}^\dagger + \hat{a} \hat{a} \right) \]
Particle number operator \( \hat{N} = \hat{a}^\dagger \hat{a} \)

**Initial state:** system in groundstate |0\rangle at \( t = 0 \) with \( \langle \hat{N} \rangle(0) = 0 \).

Ladder operators in **Heisenberg picture**:

\[
\frac{d}{dt} a^{(\dagger)} = \frac{i}{\hbar} [H_{pa}, a^{(\dagger)}]
\]

**Number of quanta in the system at time** \( t \)

\[
\langle \hat{N} \rangle(t) = \langle 0 | \hat{a}^{(\dagger)}(t) \hat{a}(t) | 0 \rangle = \sinh^2 \left( \frac{\omega_0 \epsilon t}{2} \right)
\]

The number of quanta depends on

- frequency \( \omega_0 = \sqrt{\frac{g}{l_0}} \) determined by geometric settings
- relative length change \( \epsilon = \frac{\Delta l}{l_0} \) determined by the amplitude \( \Delta l \)
- time \( t \)
Kirchhoff’s laws

Capacitor: \( Q = CV \implies I = C \frac{dV}{dt} \)

Inductance: \( V = -L \frac{dI}{dt} \)

\[
\frac{d^2V}{dt^2} = -\frac{1}{LC} V
\]

Parametric amplifier

Parametric amplifier with frequency: \( \omega(t) = \sqrt{\frac{1}{L(t)C}} \)

Remember swing: \( \omega(t) = \sqrt{\frac{g}{l(t)}} \)
Experimental realization

**DC SQUID - Tunable inductance**

Current through two JJs:

\[ I = I_c \left( \sin \phi_1 + \sin \phi_2 \right) \]

with \( I \ll I_c \)

Phase:

\[ \phi_1 - \phi_2 = 2\pi \frac{\Phi_{\text{ext}}}{\Phi_0} + 2\pi n \]

Voltage:

\[ V = \frac{\Phi_0}{2\pi} \frac{d}{dt} \left( \frac{\phi_1 + \phi_2}{2} \right) \]

**Circuit diagram of a DC-SQUID**

Rewrite in the form \( V = \frac{d}{dt} (LI) \)

⇒ **Tunable inductance**

\[ L(t) = \frac{\Phi_0}{4\pi I_C} \cdot \frac{1}{\cos \left( \pi \frac{\Phi_{\text{ext}}(t)}{\Phi_0} \right)} \]
Microwave resonator: analogue to two mirrors, where one is moving.
Discrete model for a stripline resonator.
Wave equation

\[
\frac{1}{v^2} \frac{d^2 \varphi}{dt^2} - \frac{d^2 \varphi}{dx^2} = 0
\]

inductance/length \( l \)

capacitance/length \( c \)

phase field \( \varphi(x, t) = \frac{2\pi}{\Phi_0} \int^t dt' V(x, t') \)

velocity \( v = \sqrt{\frac{1}{lc}} \)
Experimental realization

Stripline resonator (3)

Wave equation

\[
\frac{1}{v^2} \frac{d^2 \varphi}{dt^2} - \frac{d^2 \varphi}{dx^2} = 0
\]

inductance/length \( l \)

phase field \( \varphi(x, t) = \frac{2\pi}{\Phi_0} \int v dt' V(x, t') \)

velocity \( v = \sqrt{\frac{1}{lc}} \)

capacitance/length \( c \)

Boundary conditions

At \( x = 0 \):

\[
\varphi(0, t) - \frac{L(\Phi_{ext})}{l} \cdot \frac{\partial \varphi(0, t)}{\partial x} = 0
\]

\[
\Rightarrow \varphi \left( -\frac{L(\Phi_{ext})}{l}, t \right) = 0
\]

At \( x = l_0 \):

\[
I(l_0) = 0
\]

\[
\Rightarrow \frac{\partial \varphi(l_0, t)}{\partial x} = 0
\]

SQUID changes effective length of the cavity to \( l_0 + \frac{L(\Phi_{ext})}{l} \)
Experimental realization

Solving the wave equation (1)

Boundary conditions (without external flux $\Phi_{ext}$)

\[
\phi(0, t) = 0 \quad \text{and} \quad \frac{\partial \phi(l_0, t)}{\partial x} = 0
\]

Solutions

\[
\phi_n(x) = \sqrt{\frac{2}{l_0}} \sin \left( \left( n + \frac{1}{2} \right) \frac{\pi x}{l_0} \right)
\]
Experimental realization

Solving the wave equation (2)

Boundary conditions for external flux $\Phi_{ext}$

$$\phi(0, t) - \frac{L(\Phi_{ext})}{l} \frac{\partial \phi(0, t)}{\partial x} = 0$$

and

$$\frac{\partial \phi(l_0, t)}{\partial x} = 0$$

Ansatz

$$\phi(x, t) = \sum_n q_n(t) \phi_n(x)$$

Insert into wave equation (fundamental mode)

$$\ddot{q}_0(t) = -\omega_0^2 (1 + \epsilon \sin(2\omega_0 t)) q_0(t)$$

for an appropriate choice of $\Phi_{ext}(t)$

Parametric amplifier!!!
Again the parametric amplifier...

Recall:

Number of quanta in the system at time $t$

$$
\langle \hat{N} \rangle (t) = \langle 0 | \hat{a}^\dagger (t) \hat{a} (t) | 0 \rangle = \sinh^2 \left( \frac{\omega_0 \epsilon t}{2} \right)
$$

Properties of the system

- Typical velocities in a stripline resonator: $v = \frac{\Delta \omega_0}{\pi} \approx 10^7 \text{ m/s}$
- Resonance frequencies $\omega_0 \sim 2\pi \cdot 5$ GHz (microwaves).
- Number of photons $\langle \hat{N} \rangle (t)$ is limited by quality factor $Q$ of the resonator.
Transmission Stripline (1)

open coplanar waveguide (CPW), sinusoidally driven boundary at frequency $\omega_d$

Figures taken from [3].

Transmission Stripline (analogue to one mirror)
Production of pairs of correlated photons
frequency: $\omega_- + \omega_+ = \omega_d$

Spectrum of emitted photons.[3]

maximum effective velocity $v_e \approx 0.25 \cdot v_0$
$v_0$: speed of light in transmission line

Photons with frequencies of 4-6 GHz are generated (microwaves)
C. Wilson et. al, Nature 479 (2011)
Measurements with open Al waveguide terminated by a SQUID
The dynamical Casimir effect

- Virtual particles can be converted into real particles.
The dynamical Casimir effect

- Virtual particles can be converted into real particles.
- Parametric amplification occurs in many physical systems, e.g. dynamical Casimir effect in a cavity, Hawking radiation, Unruh effect,...
Summary

The dynamical Casimir effect

- Virtual particles can be converted into real particles.
- Parametric amplification occurs in many physical systems, e.g. dynamical Casimir effect in a cavity, Hawking radiation, Unruh effect,...
- SQUIDs and striplines can be used to observe these quantum electrodynamic effects.
THANK YOU FOR YOUR ATTENTION!


