COUNTING ELECTRICAL CHARGES: A PROBLEM OF THERMAL ESCAPE AND QUANTUM TUNNELING IN PRESENCE OF NON-GAUSSIAN NOISE

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Electrical noise produced when charges flow through a mesoscopic conductor is non-Gaussian, where higher order cumulants carry valuable information about the microscopic transport process. In these notes we briefly discuss the experimental situation and show that theoretical descriptions for the detection of electrical noise with Josephson junctions lead to generalizations of classical and quantum theories, respectively, for decay rates out of metastable states.

Keywords: Non-Gaussian noise in mesoscopic systems, thermal activation, quantum tunneling.

1. Introduction

Typically noise is an annoying phenomenon. This is particularly true for mesoscopic systems since they are inevitably in contact with external leads, substrates, gates etc., which only allow for control and manipulation of the system under investigation. However, this is not the whole story. In fact, noise may even carry information that cannot be obtained by standard measurements detecting mean values. The relevance of the shot noise power of the electrical current was already discovered by Schottky, who showed that one gains direct access to the effective charge carrying the current by observing the shot noise and the mean current. Based on this seed, within the last decade electrical noise has moved into the focus of research activities on electronic transport in nanostructures [1,2] since it provides information on microscopic mechanisms of the transport not available from the voltage dependence of the average current. Lately, attention has turned from the noise auto-correlation function (shot noise) to higher order cumulants of the current fluctuations characterizing non-Gaussian statistics [3,2]. While theoretical attempts to predict these cumulants for a variety of devices are quite numerous [2], experimental observation is hard because of small signals, large bandwidth detection, and strict filtering demands. A first pioneering measurement by Reulet *et al.* [4] of the third cumulant of the current noise from a tunnel junction has intensified efforts and several new proposals for experimental set-ups have been put forward very recently, some of which are based on Josephson junctions (JJ) as on-chip noise detectors [5].

In all experimental set-ups to measure higher order cumulants realized and proposed so far, heating is one of the major experimental obstacles [4,6]. Thus, experiments have primarily attempted to establish just the unspecified non-Gaussian nature of the noise or to measure the third cumulant (skewness). The latter one is particularly accessible since it can be discriminated from purely Gaussian noise due to its asymmetry, e.g. when inverting the current through the conductor. This way, very recently the skewness of current noise generated by a tunnel junction was measured in the adiabatic limit (low frequency noise) of the quantum tunneling regime of a JJ [7]. In the even more interesting regime of finite frequencies [8], first data have been obtained in Ref. [9], where the JJ stays in the classical domain of thermally activated switching. This latter situation requires a new theoretical framework generalizing the well-established Kramers' theory [10] to escape in presence of non-Gaussian noise [11]. We will present the main results in the first part of these notes.

The problem for a corresponding quantum theory is that electrical noise is produced by an environment in a steady state but far from equilibrium. A consistent general theory for quantum tunneling in such a situation is still elusive. What one could think about is to place the mesoscopic conductor in parallel to a current biased JJ in the zero voltage state [12]. Then, no net current flows through the sample prior to the read out and the generated electrical noise is equilibrium noise (with non-Gaussian cumulants though). Accordingly, on the one hand only even higher order cumulants exist so that in particular the fourth order cumulant (sharpness) becomes accessible that due to heating effects may always be hidden behind the second and third order ones. On the other hand, a generalization of the standard ImF approach [13], which is based on a proper evaluation of the partition function of the total system, is possible as we will briefly discuss in the second part.

2. Preliminaries

The complete statistics of current noise generated by a mesoscopic conductor can be gained from the generating functional

$$G[\chi] = \mathrm{e}^{-S_G[\chi]} = \langle \mathcal{T} \exp \left[\frac{i}{e} \int_{\mathcal{C}} dt I(t) \chi(t) \right] \rangle , \tag{1}$$

where $I(t) = e \int_0^t ds N(s)$ is the current operator, N the number of transferred charges, χ the counting field, and $\mathcal{T}$ the time ordering operator along the Kadanoff-Baym contour $\mathcal{C}$. Time correlation functions of arbitrary order of the current are determined from functional derivatives of $G[\chi]$, namely,

$$C_n(t_1, \ldots, t_n) = -(-ie)^n \, \partial^n S_G[\chi]/\partial\chi(t_1) \cdots \partial\chi(t_n)|_{\chi=0} \ , \tag{2}$$

where cumulants $C_n, n \geq 3$ display non-Gaussian properties of the noise. We remark that the functional $S_G[\chi]$ carries the full frequency dependence of all current cumulants and not just their time averaged zero frequency values usually studied in the field of full counting statistics [3].

3. Classical thermal activation

Let us consider the experimental situation summarized in Fig. 1, where a tunnel junction represents the noise generating sample and a JJ the detector. We assume the underlying stochastic processes to be classical, both for the dynamics of the detector and for the electrical noise originating from the mesoscopic conductor. Hence, the standard RSJ model applies, where now the total noise consists of the Johnson-Nyquist noise $\delta I_b = I_b - \langle I_b \rangle$, in the simplest case produced by a resistor R in parallel to the JJ biased by I_b, and of the weak stationary mesoscopic noise δI_m from the fluctuating part of the mesoscopic current $I_m = \langle I_m \rangle + \delta I_m$. If the phase φ of the JJ is initially trapped in one of the wells of the tilted washboard potential $U(\varphi) = -E_J \ \cos(\varphi) - (\langle I_b \rangle (\hbar/2e)\varphi$ (zero-voltage state), it may for sufficiently large $\langle I_b \rangle < I_0 = (2e/\hbar)E_J$ escape so that the JJ switches to a finite voltage state. Further, since the third cumulant vanishes in equilibrium due to time-reversal symmetry, experimentally, the mesoscopic conductor is at low temperatures driven far from equilibrium, where no fluctuation-dissipation theorem applies. Hence, the switching of the JJ can be visualized as the diffusive dynamics of a fictitious particle in a metastable well with non-Gaussian continuous fluctuations acting as *external* random driving force. In a refined version of this theory also back-action effects can be taken into account.

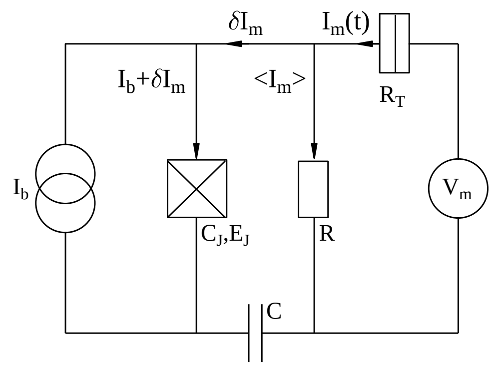

Fig. 1. Simplified scheme of the experiment reported in Ref. [9]. The high frequency ($f > 1/RC$) fluctuations δI_m of the current through a voltage-biased tunnel junction (tunnel resistance R_T) pass through a Josephson junction (capacitance C_J, Josephson energy E_J). The switching of the Josephson junction to the finite-voltage state depends on the sum of δI_m and of the pulsed current bias I_b. The switching probability of the junction during one pulse can be related to the current fluctuations.

According to the Kirchhoff rules and the Josephson relations we have for the currents entering the detector (cf. Fig. 1)

$$\langle I_b \rangle + \delta I_b + \delta I_m(t) = \frac{\hbar}{2e}\frac{\dot{\varphi}}{R} + I_0 \sin\varphi + C_J \frac{\hbar}{2e}\ddot{\varphi}\,, \tag{3}$$

where φ denotes the phase difference of the JJ. The two noisy forces are assumed to be Markovian, which is an accurate approximation since the experiment is operated in a regime, where the typical correlations times are much smaller than the typical time scales of the junction, e.g. the inverse of the plasma frequency Ω. This situation also ensures that we can work with continuous noise processes. The thermal noise δI_b has a vanishing mean and a standard δ-correlated second cumulant. The statistical properties of the stationary non-Gaussian noise δI_m are determined by the generating functional (1) with the current operator $I(t)$ replaced by the classical noise $\delta I_m(t)$. In particular, one has $\langle \delta I_m(t)\delta I_m(0)\rangle = F_2 e^2 \langle I_m \rangle \delta(t)$ and and $\langle \delta I_m(t_2)\delta I_m(t_1)\delta I_m(0)\rangle = F_3 e^3 \langle I_m\rangle \delta(t_2)\delta(t_1)$. The factors F_2 and F_3 denote the Fano-factors, which for a simple tunnel junction turn out to be $F_2 = F_3 = 1$. It is important to realize that the average mesoscopic current can be large so that the second moment can be comparable to or lager than that of the thermal noise, while the third moment is still small due to $e\Omega/I_0 \ll 1$. In fact, the Gaussian component of the mesoscopic noise effectively adds to the thermal noise to determine the effective temperature of the junction (heating). Equation (3) is a Langevin equation for a fictitious particle in the tilted Josephson potential $U(\varphi)$ in presence of thermal Gaussian and electrical non-Gaussian noise.

As usual in rate theory, for analytical treatments it is much more convenient to work with phase space probabilities rather than individual stochastic trajectories [10]. The general problem for non-Gaussian noise is then that the corresponding Fokker-Planck equation (FPE) based on a Kramers-Moyal expansion contains diffusion coefficients up to infinite order. The basic idea for weak non-Gaussian noise with a leading third cumulant is thus, to derive an effective, finite order Fokker-Planck equation. Based on a cumulant expansion of the noise generating functional (1) with the counting field proportional to the momentum derivative such a generalized FPE has been derived in Ref. [11]. It leads to a FPE with a momentum dependent diffusion term, where the momentum dependence is weighted by the third cumulant.

The thermal rate expression derived from the steady state solution of the standard FPE looks as $\Gamma = A\exp(-\beta U_b)$ and is dominated by the exponential (activation factor) being identical to the probability to reach the barrier top from the well bottom (energy difference U_b) by a thermal fluctuation [10]. Within the theoretical framework of the extended FPE an analytical expression for the exponent including leading corrections due to a third cumulant have been obtained [11]. The rate takes the form $\Gamma_\pm \propto \exp[-\beta U_b(1 \mp |g_3|)]$ with the correction g_3 due to the third cumulant such that Γ_+ corresponds to $\langle I_m \rangle > 0$ and Γ_- to $\langle I_m \rangle < 0$. The rate asymmetry $R_\Gamma = \Gamma_+/\Gamma_- - 1$ is found as being proportional to F_3 and strongly depends on the effective temperature, damping, and bias current. Hence, a measurement of R_Γ gives direct information about the third cumulant of the electrical noise. This has been discussed in detail in Ref. [9]. There also results of numerical simulations have been presented which are in agreement with the analytical expressions.

To complete this discussion, we remark that an analytical expression for the asymmetry of the escape rate in the limits of low and high friction has been also derived in Ref. [14] leading up to minor deviations to identical results.

4. Quantum tunneling

As mentioned above, a general theory for quantum tunneling in presence of steady state non-Gaussian noise has not been formulated yet. The idea is thus to place the mesoscopic conductor in parallel to a current biased JJ as depicted in the circuit diagram of Fig. 2. For a bias current I_b below the critical current I_0, the JJ is in its zero voltage state and the bias current flows as a supercurrent entirely through the JJ branch of the circuit. Consequently, no heating occurs in the conductor and the total system can easily be kept at low temperatures, where the decay of the zero voltage state occurs through Macroscopic Quantum Tunneling (MQT) [12]. The rate of this process depends with exponential sensitivity on the current fluctuations of the conductor so that the JJ acts as a noise detector.

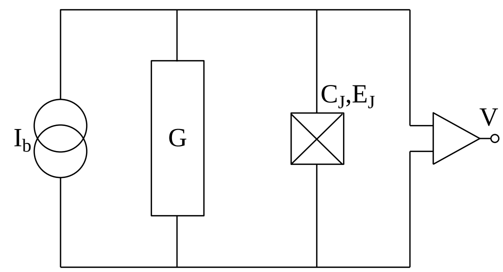

Fig. 2. Electrical circuit containing a mesoscopic conductor G in parallel to a JJ with capacitance C_J and coupling energy E_J biased by an external current I_b. The switching out of the zero voltage state of the JJ by MQT is detected as a voltage pulse V.

The MQT rate Γ can be calculated in the standard way [13,15] from the imaginary part of the free energy F, i.e., $\Gamma = (2/\hbar)\,\mathrm{Im}\{F\}$, where $F = -(1/\beta)\ln(Z)$ is related to the partition function $Z = \mathrm{Tr}\{\mathrm{e}^{-\beta H}\}$. In the path integral representation one has

$$Z = \int \mathcal{D}[\varphi]\mathrm{e}^{-S[\varphi]}\,,$$

which is a sum over all imaginary time paths with period $\hbar\beta$ where each paths is weighted by the dimensionless effective action of the JJ. In the present case, we have $S[\varphi] = S_{JJ}[\varphi] + S_G[\varphi/2]$, where the first term is the bare action of the JJ, i.e. the phase dynamics in the tilted Josephson potential, while the second one describes the influence of the environment. The factor of 2 in the argument of S_G arises from the fact that the voltage across the conductor equals the voltage $(\hbar/e)(\dot{\varphi}/2)$ across the JJ. In the standard theory only thermal Gaussian noise is present, which is described by an ohmic resistor in parallel to the JJ. The corresponding action $S_G[\varphi] \equiv S_R[\varphi]$ can be calculated exactly and gives the well-known Feynman-Vernon influence functional [15]. Now, if we consider a tunnel junction as environment generating stationary electrical non-Gaussian noise an exact expression for the action S_T is found as well [16], where the granularity of the charge appears as a periodic dependence on φ. The current generating functional thus determines a non-Gaussian influence functional for the system with the detector degree of freedom being the counting field.

In the MQT regime the partition function of the isolated JJ is dominated by the so-called bounce trajectory, an extremal $\delta S[\varphi] = 0$ periodic path in the inverted barrier potential. In the limit of vanishing temperature one finds an analytical solution of the bounce φ_B so that for zero temperature the bare rate coincides with the usual WKB result, namely, $\Gamma_0 \propto \exp[-(36/5)U_b/\hbar\Omega]$ with U_b being the barrier height and Ω the frequency for oscillations near the well bottom (plasma frequency). Following the theory of the effect of an electromagnetic environment on MQT [13], the partition function can now be calculated for arbitrary coupling between detector and conductor based on a numerical scheme developed in [15]. Analytical progress is made when the noise generating element has a dimensionless conductance $g_T \equiv \hbar/(2e^2 R_T) \ll I_0/2e\Omega$ so that the influence of the noise can be calculated by expanding about the unperturbed bounce which gives

$$\Gamma = \Gamma_0 \, \mathrm{e}^{-S_G[\varphi_B/2]} \, . \tag{4}$$

For a tunnel junction with tunnel resistance R_T, the correction $S_G[\varphi_B/2]$ can be represented as a series in even order cumulants, which is usually dominated by the second cumulant C_2 and the fourth cumulant C_4, see Ref. [12]. Note that this treatment still contains the full dynamics of detector and noise source.

For the on-chip detection circuit proposed here, the impact of the fourth order cumulant needs to be clearly discriminated from effects of purely Gaussian noise. This is achieved by considering the function $B(x) = -\ln[\Gamma(x)/\Gamma_0(x)]$ with the variable $x = (1-s^2)/s^2$ ($s = \langle I_b \rangle / I_0$) which allows to discriminate between weak Gaussian and non-Gaussian noise due to a qualitatively different scaling behavior with varying x. For purely Gaussian noise $B(x)$ results essentially in a straight line, while non-Gaussian noise displays a nonlinear behavior. Even more pronounced are the differences in the slopes $dB(x)/dx$, which saturate for larger x-values when only Gaussian noise is present, but strongly decrease with increasing x in presence of a nonlinear conductor. This scaling property is robust against additional Gaussian noise present in the wiring incorporated by an additional resistor with resistance $R \ll R_T$: it merely shifts $dB(x)/dx$ and thus does *not* spoil the scaling behavior originating from C_4 (cf. Ref. [12]).

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References

1. Y.M. Blanter and M. Büttiker, Phys. Rep. **336**, 1 (2000).
2. *Quantum Noise in Mesoscopic Physics*, edited by Y.V. Nazarov, NATO Science Series in Mathematics, Physics and Chemistry (Kluwer, Dordrecht, 2003).
3. L.S. Levitov, H.B. Lee, and G.B. Lesovik, J. Math. Phys. **37**, 4845 (1996).
4. B. Reulet, J. Senzier, and D.E. Prober, Phys. Rev. Lett. **91**, 196601 (2003).
5. J. Tobiska and Y.V. Nazarov, Phys. Rev. Lett. **93**, 106801 (2004); T.T. Heikkila *et al*, Phys. Rev. Lett. **93**, 247005 (2004); R.K. Lindell *et al.* , Phys. Rev. Lett. **93**, 197002 (2004); E.B. Sonin, Phys. Rev. B **70**, 140506(R) (2004); Yu. Bomze *et al*, Phys. Rev. Lett. **95**, 176601 (2005); J.P. Pekola, Phys. Rev. Lett. **93**, 206601 (2004); S. Gustavsson *et al*, Phys. Rev. Lett. **96**, 076605 (2006).
6. J.P. Pekola *et al.*, Phys. Rev. Lett. **95**, 197004 (2005).
7. A.V. Timofeev *et al.*, Phys. Rev. Lett. **98**, 207001 (2007).
8. K.E. Nagaev, S. Pilgram, M. Büttiker, Phys. Rev. Lett. **92**, 176804 (2004).
9. B. Huard *et al.*, Ann. Phys. 16, 736 (2007).
10. For a review see: P. Hänggi, P. Talkner, and M. Borkovec, Rev. Mod. Phys. **62**, 251 (1990).
11. J. Ankerhold, Phys. Rev. Lett. **98**, 036601 (2007).
12. J. Ankerhold and H. Grabert, Phys. Rev. Lett. **95**, 186601 (2005).
13. A.O. Caldeira and A.J. Leggett, Ann. Phys. (New York) **149**, 374 (1983).
14. E.V. Sukhorukov and A.N. Jordan, Phys. Rev. Lett. **98**, 136803 (2007).
15. H. Grabert, P. Olschowski, and U. Weiss, Phys. Rev. B **36**, 1931 (1987).
16. G. Schön and A.D. Zaikin, Phys. Rep. **198**, 237 (1990).