SOLUTION OF THE FRIEDMANN EQUATION DETERMINING THE TIME EVOLUTION, ACCELERATION AND THE AGE OF THE UNIVERSE

FRANK STEINER

ABSTRACT. The time evolution of the Universe from the Big Bang until today is described by General Relativity, i.e. by Einstein's gravitational field equations. Assuming the Cosmological Principle, the four-dimensional space-time manifold is given by a warped product $M^4 = \mathbb{R} \times_R M^3$ (Robertson-Walker metric with warp function $R(t)$). Here $R(t)$ is the cosmic scale factor which must be obtained from a solution of the Friedmann equation. In this talk the analytic solution of the Friedmann equation is presented in terms of the Weierstraß elliptic $\wp$-function. The solution – determining the time evolution and the observed accelerated expansion of the Universe – is discussed using astrophysical data. As one of the results, it is possible to compute the age of the Universe and the time evolution of the cosmic energy-/matter-densities.

1. The geometry of the Universe

The spatial geometry in Newtonian Physics, i.e. the three-dimensional configuration space is absolute, infinite, euclidean and static, i.e. independent of (absolute) time t. The spatial line element is then given by the euclidean metric $(x, y, z) \in \mathbb{R}^3$

$$dl^2 = (\text{infinitesimal spatial distance})^2 = dx^2 + dy^2 + dz^2 \qquad \text{(Pythagoras)}. \tag{1}$$

In Special Relativity we have to consider the four-dimensional space-time geometry with the line element

$$\begin{aligned} ds^2 &= (\text{infinitesimal four-dimensional distance between space-time events})^2 \\ &= c^2 dt^2 - dx^2 - dy^2 - dz^2 = c^2 \left[1 - \left(\frac{dl}{c\,dt}\right)^2\right] dt^2 = c^2 d\tau^2. \end{aligned} \tag{2}$$

(c denotes the velocity of light in vacuo.) This is the Minkowski metric valid in any given inertial system. An invariant quantity is the proper time $d\tau = \sqrt{1 - \left(\frac{v}{c}\right)^2} dt$ with velocity $v := \frac{dl}{dt}$, which describes the famous time dilatation. It is to be noted that the four-dimensional space-time continuum is still not dynamical in Special Relativity! This is generally valid for the whole of physics, as long as gravitation does not play a role.

The situation changes drastically when we consider gravitation which plays the dominant role in cosmology. Then the four-dimensional space-time becomes dynamical and the Minkowski metric (2) has to be replaced by the general Riemannian resp. Lorentzian metric

$$ds^2 = g_{\mu\nu}(x^\rho) dx^\mu dx^\nu, \quad x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z). \tag{3}$$

The symmetric metric tensor $g_{\mu\nu}$ plays now a double role: on the one hand it determines the four-dimensional geometry of space-time, and on the other hand it replaces the single scalar Newtonian gravitational potential by a tensor field which in Einstein's General Relativity plays the role of the gravitational field. Given the energy-momentum tensor $T^{\mu\nu}$ of matter, $g_{\mu\nu}$ has to be determined as a solution of Einstein's gravitational field equations (G is Newton's gravitational constant)

$$G^{\mu\nu} - \Lambda g^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu}. \tag{4}$$

This work is a shortened version of a talk delivered at the workshop on "Evolution Equations – the State of the Art", Reisensburg, Germany, December 2007, and was supported by Deutsche Forschungsgemeinschaft under contract no. DFG STE 241/16-1.

Here the left-hand side depending on $g_{\mu\nu}$ and its first and second partial derivatives (see eqs. (7) and (8) below) is purely geometrical. It expresses Einstein's deep insight that gravitation is a property of space-time geometry. In eq. (4) we have already taken into account the cosmological term proportional to the cosmological constant Λ, which will be discussed in more detail below. The right-hand side of eq. (4), the "source term", is proportional to the energy-momentum tensor $T^{\mu\nu}$ which is determined by the matter-/energy-distribution of the Universe. For a perfect relativistic "fluid" (for example radiation) it reads

$$T^{\mu\nu} = (\epsilon + p)\frac{u^\mu u^\nu}{c^2} - p g^{\mu\nu}, \tag{5}$$

where ϵ and p are functions of t alone and

$$\begin{aligned} \epsilon &= \text{energy density} \\ p &= \text{pressure} \\ u^\mu &= \frac{dx^\mu}{d\tau} = \text{four-velocity} \qquad \left(d\tau = \tfrac{1}{c} ds\right). \end{aligned} \tag{6}$$

The Einstein tensor $G^{\mu\nu}$ is given by

$$G^{\mu\nu} := R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \tag{7}$$

with

$$\begin{aligned} R_{\alpha\beta} &:= R^\mu_{\alpha\beta\mu} && \text{(Ricci tensor)} \\ R &:= g^{\mu\nu} R_{\mu\nu} && \text{(curvature scalar)} \\ R^\mu_{\alpha\beta\gamma} &:= \frac{\partial \Gamma^\mu_{\alpha\gamma}}{\partial x^\beta} - \frac{\partial \Gamma^\mu_{\alpha\beta}}{\partial x^\gamma} + \Gamma^\mu_{\sigma\beta}\Gamma^\sigma_{\gamma\alpha} - \Gamma^\mu_{\sigma\gamma}\Gamma^\sigma_{\beta\alpha} && \text{(Riemann tensor)} \\ \Gamma^\alpha_{\lambda\mu} &:= \frac{1}{2} g^{\alpha\nu}\left(\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\nu}\right). && \text{(Christoffel symbol)} \end{aligned} \tag{8}$$

The Einstein equations (4) are 10 coupled, nonlinear partial differential equations for which in the general case only a few exact solutions are known. A great simplification has been achieved by Einstein in 1917 [1] when he constructed for the first time a generally covariant model of the Universe. The so-called Einstein Universe marks the beginning of modern cosmology. Although the Einstein Universe is static and thus does not describe the expansion of the Universe observed more than 10 years later by Hubble, Einstein introduced in his paper [1] the Cosmological Principle which plays an important role in cosmology ever since. The Cosmological Principle states that the Universe is homogeneous and isotropic if we average over huge spatial distances ($\geq 100\,\text{Mpc}$; $1\,\text{pc} = 3.262\ldots$ light years $= 3.085\ldots \times 10^{16}\,\text{m}$). This assumption leads to an enormous symmetry reduction of the Einstein equations (4): the space-time metric $g_{\mu\nu}$ depends apart from the spatial curvature parameter K only on a single function of time, the so-called cosmic scale factor $R(t)$, and the energy-momentum tensor must have the perfect-fluid form (5). Mathematically speaking, the four-dimensional pseudo-Riemannian space-time manifold (M^4, g) factorizes, at least locally, as a warped product $M^4 = \mathbb{R} \times_R M^3$, where $\mathbb{R}$ describes cosmic time, M^3 the maximally symmetric three-dimensional subspace with Gaussian curvature $\mathcal{K} = \frac{K}{R^2(t)}$, and $R(t)$ an everywhere positive warp function. Here the dimensionless curvature parameter K is a constant, which by a suitable choice of units can be chosen to take the three values $K = 0, \pm 1$ corresponding to the three possible spatial curvatures: $K = 0$ (M^3 = three-dimensional euclidean space, the so-called "flat universe"), $K = 1$ (M^3 possesses constant positive curvature, the so-called "spherical" or "closed universe") and $K = -1$ (M^3 possesses constant negative curvature, the so-called "hyperbolic universe").

When Einstein was looking for a static solution, he realized that he had to modify his original equations by subtracting the term $-\Lambda g_{\mu\nu}$ in eq. (4), where Λ is the cosmological constant which he considered as a new constant of Nature. Choosing $\Lambda > 0$, he found that his equations lead to a static universe (R = constant) with positive curvature, and it was then natural for him to identify M^3 with the simplest spherical space S^3 such that space-time obtained the structure $M^4 = \mathbb{R} \times S^3$.

In 1922 and 1924 it was discovered by Alexander Friedmann [2, 3] that the generic solutions of the gravitational field equations, with and without Λ, are not static but rather lead to a time-dependent scale factor $R(t)$. After Friedmann's early death in 1925, the subject was taken up by Georges Lemaître [4, 5, 6, 7, 8] who predicted the expanding universe ($R(t)$ monotonically increasing) and developed the idea of a non-static, non-eternal universe, originating from a very hot initial state which he called "primeval atom", called Big Bang today.

It turns out that the most general solution to Einstein's equations for a homogeneous and isotropic universe (Friedmann-Lemaître universe) is given by the so-called Robertson-Walker (RW) metric ($i, j = 1, 2, 3$)

$$ds^2_{\mathrm{RW}} = c^2 dt^2 - R^2(t)\gamma_{ij} dx^i dx^j. \tag{9}$$

Here $R(t)$ is the cosmic scale factor and γ_{ij} the three-dimensional spatial metric of constant Gaussian curvature parameterized by the curvature parameter K. Furthermore, the energy-momentum tensor takes the simple form (5).

The simplest case is obtained for a flat euclidean universe ($K = 0$), where the four-dimensional Robertson-Walker metric takes the simple form ($\gamma_{ij} = \delta_{ij}$)

$$ds^2_{\mathrm{RW}} = c^2 dt^2 - R^2(t)(dx^2 + dy^2 + dz^2). \tag{10}$$

Using spherical coordinates $x = r \cos\varphi \sin\vartheta$, $y = r \sin\varphi \sin\vartheta$, $z = r \cos\vartheta$, the metric (10) reads

$$\begin{aligned} ds^2_{\mathrm{RW}} &= c^2 dt^2 - R^2(t)(dr^2 + r^2 d\Omega^2) \\ d\Omega^2 &:= d\vartheta^2 + \sin^2\vartheta \, d\varphi^2, \qquad d\Omega := \text{solid angle on } S^2. \end{aligned} \tag{11}$$

In the general case of constant (spatial Gaussian) curvature K the Robertson-Walker metric is given by

$$ds^2_{\mathrm{RW}} = c^2 dt^2 - R^2(t)\big(dr^2 + S^2_K(r) d\Omega^2\big) \tag{12}$$

with

$$S_K(r) := \begin{cases} r & (K = 0) \\ \sin r & (K = +1) \\ \sinh r & (K = -1). \end{cases} \tag{13}$$

For our following discussion it is convenient to work instead with the cosmic time t with the conformal time $\eta = \eta(t)$, $d\eta := \frac{c\,dt}{R(t)}$, and thus with the corresponding cosmic scale factor $a(\eta) := R(t(\eta))$. The cosmic time t can then be recovered from a knowledge of $a(\eta)$ via $t = t(\eta) = \frac{1}{c}\int_0^\eta a(\eta')d\eta'$. The Robertson-Walker metric (9) reads then

$$ds^2_{\mathrm{RW}} = a^2(\eta)\left[d\eta^2 - \gamma_{ij} dx^i dx^j\right]. \tag{14}$$

$R(t)$ resp. $a(\eta)$ is obtained by inserting the RW metric (9) resp. (14) into the Einstein eqs. (4) and using the energy-momentum tensor (5) leading to the (Einstein)-Friedmann eq. (21), to be discussed in Sect. 3, which has to be solved with the initial conditions at $t = 0$ resp. at $\eta = 0$ (Big Bang)

$$\begin{aligned} a(0) &= 0 \\ a'(0) &> 0 \end{aligned} \qquad \text{with } a'(\eta) := \frac{da}{d\eta}. \tag{15}$$

Note that the cosmic scale factor $R(t)$ resp. $a(\eta)$ has the dimension of a length, while the conformal time η is dimensionless. With this choice of units, the three-dimensional coordinates $x^1 = x$, $x^2 = y$, $x^3 = z$ in eqs. (9), (10) and (14) are dimensionless, as well as the "radial variable" r in eqs. (11), (12), (13). The spatial coordinates (x^1, x^2, x^3) resp. (r, ϑ, φ) form a comoving system, in the sense that typical galaxies have constant spatial coordinates.

2. The energy budget and the curvature of the Universe

Before we can insert the Ansatz (9) resp. (14) for the metric into the left-hand side of the Einstein equations (4), we have to specify the right-hand side of (4), i.e. the energy momentum tensor $T^{\mu\nu}$. For this purpose we have to know the matter-/energy-content of the Universe. It is convenient to use dimensionless energy parameters $\Omega_k := \frac{\epsilon_{k,0}}{\epsilon_{\text{crit}}}$, where $k = 0, 1, \ldots$ denotes the various energy components. $\epsilon_{k,0}$ is the energy density of the k^{th} component at the present epoch and ϵ_{crit} is the so-called critical energy density defined by

$$\epsilon_{\text{crit}} := \frac{3H_0^2 c^2}{8\pi G} = 1.88 \times 10^{-29} h^2 \ \frac{\text{g}}{\text{cm}^3}. \tag{16}$$

Here H_0 is the Hubble constant, i.e. the Hubble parameter $H(t)$ evaluated at the present epoch t_0

$$H(t) := \frac{\dot{R}(t)}{R(t)} \quad \text{and} \quad H_0 := H(t_0) = 100\, h \ \frac{\text{km}}{\text{s Mpc}}. \tag{17}$$

The Hubble constant H_0 is usually given in terms of the dimensionless parameter h, see eq. (17). Then the total energy density parameter is given by

$$\Omega_{\text{tot}} := \Omega_{\text{r}} + \Omega_{\text{m}} + \Omega_\Lambda =: \frac{\epsilon_{\text{tot}}}{\epsilon_{\text{crit}}}. \tag{18}$$

From various astrophysical observations, in particular of the cosmic microwave background (CMB) radiation and the luminosity distances of Type Ia supernovae, we know that the Universe consists today of at least four components (see Fig. 1): (i) a very small component Ω_{r} consisting of radiation (r) corresponding to the CMB with temperature 2.726 K, including three massless neutrinos; (ii) the so-called baryonic matter (b) component Ω_{b} which consists of the normal atomic and nuclear matter; (iii) the so-called cold dark matter (cdm) component Ω_{cdm} whose origin is unknown and which is called "cold" because its energy density ϵ_{cdm} behaves nonrelativistically corresponding to a pressureless fluid ($p_{\text{cdm}} = 0$, see eqs. (5) and (6)) and is called "dark matter" because it does not couple to light but only to gravity; (iv) the so-called "dark energy" component which can be identified with Einstein's cosmological constant Λ, see (4), corresponding to the energy density

$$\epsilon_\Lambda := \frac{\Lambda c^4}{8\pi G} \quad \text{resp.} \quad \Omega_\Lambda = \frac{\Lambda}{3} L_H^2. \tag{19}$$

Here $L_H := \frac{c}{H_0} \approx 9.25 \times h^{-1} \times 10^{25}\,\text{m} \approx 4.28\,\text{Gps}$ (for $h = 0.70$) denotes the Hubble length. Note that Λ has the dimension m^{-2} and thus Ω_Λ is dimensionless as it should be. Taking $\Omega_\Lambda = 0.72$ corresponding to the dark energy component at the present epoch, see Fig. 1, one obtains for the corresponding length the estimate $\Lambda^{-1/2} = (3\Omega_\Lambda)^{-1/2} L_H = 0.68\, L_H \approx 8.99 \times 10^{25}\,\text{m}$ (for $h = 0.70$). If the cosmological constant is a new constant of Nature as originally conceived by Einstein [1], it follows that $\Lambda^{-1/2}$ presents a new fundamental length scale which should play a role not only in cosmology but also e.g. in our solar system. Until now the distances probed in the solar system are, however, much smaller than 10^{25} m such that the influence of Λ on the motion of the planets has not been detected.

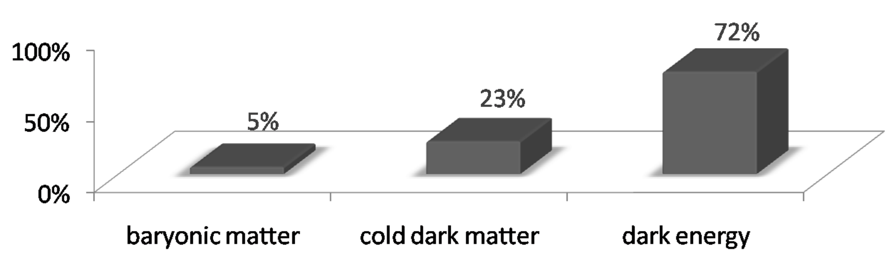

Figure 1. The energy budget of the Universe today. (The radiation component $\Omega_{\text{r}} = O(10^{-5})$ is not shown.)

$$\Omega_{\mathrm{tot}} =: 1 - \Omega_{\mathrm{curv}} = \begin{cases} > 0 & \text{for } K = +1 \\ = 0 & \text{for } K = 0 \\ < 0 & \text{for } K = -1 \end{cases}$$

WMAP only	0.99	$<$	Ω_{tot}	$<$	1.29	(95% CL)
			$\overline{\Omega}_{\mathrm{tot}}$	$=$	$1.099^{+0.100}_{-0.085}$	
			h	$=$	0.50 ± 0.14	
WMAP + HST	0.99	$<$	Ω_{tot}	$<$	1.05	(95% CL)
			$\overline{\Omega}_{\mathrm{tot}}$	$=$	$1.011^{+0.014}_{-0.015}$	
			h	$=$	$0.676^{+0.070}_{-0.068}$	
WMAP + SNall	1.00	$<$	Ω_{tot}	$<$	1.03	(95% CL)
			$\overline{\Omega}_{\mathrm{tot}}$	$=$	$1.0138^{+0.0098}_{-0.0099}$	
			h	$=$	0.653 ± 0.033	

TABLE 1. The curvature of the Universe determined from the WMAP-5yr-data [9] combined with other astronomical observations.

Since both the baryonic matter and the cold dark matter are pressureless, $p_{\mathrm{b}} = p_{\mathrm{cdm}} = 0$, the two components together make up the total nonrelativistic matter component $\Omega_{\mathrm{m}} := \Omega_{\mathrm{b}} + \Omega_{\mathrm{cdm}}$ of the Universe. While the nature of the baryonic matter is well established, the cold dark matter and dark energy components are very mysterious. However, there exist attractive theories for cold dark matter which identify it with heavy elementary particles which are predicted in certain supersymmetric theories generalizing the standard model of particle physics. One candidate is the so-called neutralino which may be discovered at the large Hadron Collider (LHC) at the *European Organization for Nuclear Research* CERN in Geneva. Even more mysterious is the dark energy component which was discovered by observations of supernovae only recently (around 1998) and which dominates the energy budget of the Universe at the present epoch, see Fig. 1. As already mentioned, one possibility – compatible with all astronomical observations – is to identify the dark energy with a positive cosmological constant Λ, see eq. (19). In this case, the dark energy density does not vary in time. There are, however, other models in which the dark energy is identified with a homogeneous scalar field $\phi = \phi(t)$, called quintessence, leading to a time- (or redshift-) dependent energy density.

In Table 1 we show the important cosmological parameters Ω_{tot} and h obtained from the most recent measurements of the NASA satellite mission WMAP (= Wilkinson Microwave Anisotropy Probe) [9] combined with other astronomical data.

Since the Einstein eqs. (4) are differential equations, they determine the local geometry but do not fix the global geometry of the Universe, i.e. its spatial curvature K and its topology. Nevertheless the curvature K can be determined from observations by measuring the total energy amount Ω_{tot} of the Universe using the equation

$$\Omega_{\mathrm{curv}} := -\frac{Kc^2}{(a_0 H_0)^2} = 1 - \Omega_{\mathrm{tot}}, \tag{20}$$

where $a_0 = a(\eta_0) = R(t_0)$ denotes the cosmic scale factor at the present epoch. (t_0 is the age of the Universe whose value is about 13.7 Gyr.) Eq. (20) is a direct consequence of the Friedmann eq. (21). If Ω_{tot}, eq. (18), is exactly one, we infer from eq. (20) $K \equiv 0$, i.e. that the spatial curvature of the Universe vanishes. On the other hand, if $\Omega_{\mathrm{tot}} \gtrless 1$, then the curvature takes the values $K = \pm 1$ corresponding to positive or negative curvature of the Universe. The measured values for Ω_{tot} given in Table 1 show that Ω_{tot} is very near to one pointing to an almost flat Universe. Taking the mean value $\overline{\Omega}_{\mathrm{tot}}$ of Ω_{tot} at its face value, we conclude that the Universe probably possesses positive curvature.

Since at present the true value of the curvature is not known, we shall leave the curvature parameter K as a free parameter in our following discussion.

3. The Friedmann equations

Inserting the Robertson-Walker metric (9) and the energy-momentum tensor (5) into the Einstein equations (4), one arrives at the Friedmann equations

$$\dot{R}^2 + Kc^2 = \frac{8\pi G}{3c^2}\epsilon_{\text{tot}} R^2 + \frac{\Lambda}{3}c^2 R^2 \tag{21}$$

$$\ddot{R} = -\frac{4\pi G}{3c^2}(\epsilon_{\text{tot}} + 3p_{\text{tot}})R + \frac{\Lambda}{3}c^2 R. \tag{22}$$

Here ϵ_{tot} and p_{tot} stand for the total energy density resp. total pressure by adding up the various components. In addition there is the equation of energy conservation

$$\dot{\epsilon}_k = -3(\epsilon_k + p_k)\frac{\dot{R}}{R}, \tag{23}$$

which follows from the vanishing of the covariant divergence of $T^{\mu\nu}$. Since the three equations (21), (22), (23) are not independent, they are not sufficient to determine the three functions $R(t)$, $\epsilon_{\text{tot}}(t)$ and $p_{\text{tot}}(t)$. We therefore require a fourth relation which expresses the pressure p_k of the k^{th} component in terms of its corresponding energy density ϵ_k. It turns out that the relevant components in cosmology are well described by the equation of state

$$p_k = w_k \epsilon_k, \tag{24}$$

where e.g. $k = \mathrm{r}, \mathrm{m}, \Lambda$ with

$$\begin{aligned} w_{\mathrm{r}} &= \tfrac{1}{3} \\ w_{\mathrm{m}} &= 0 \\ w_{\Lambda} &= -1. \end{aligned} \tag{25}$$

Thus the pressure p_k is a linear function of the energy density ϵ_k with a constant parameter w_k.

Using the equation of state (24) with $w_k = $ constant, one finds the following solution of the eq. (23) for energy conservation in terms of the scale factor $R(t)$

$$\epsilon_k(t) = \epsilon_{k,0}\left(\frac{R_0}{R(t)}\right)^{3(1+w_k)}. \tag{26}$$

Inserting eqs. (24) and (26) into the Friedmann equation (21), we obtain a nonlinear ordinary differential equation for the cosmic scale factor $R(t)$. This equation has then to be solved with the initial conditions at the Big Bang $t = 0$

$$R(0) = 0, \ \dot{R}(0) > 0. \tag{27}$$

In the following, we shall use as time-variable the conformal time η. Then the Friedmann equation (21) can be rewritten in the compact form ($a(0) = 0$, $a'(0) > 0$)

$$a'^2(\eta) = \left(\frac{da}{d\eta}\right)^2 = \left(\frac{H_0}{c}\right)^2 \sum_{k=0}^{4} \Omega_k a_0^k a^{4-k}(\eta). \tag{28}$$

Here we have assumed that the energy-/matter-content of the Universe is made up of five components, i.e. we work with the following five-component-model

$p_k = w_k \epsilon_k$; $\quad k = 0, 1, 2, 3, 4$

$k = 4$: radiation

$k = 3$: matter (=baryonic + cold dark matter)

$k = 2$: cosmic strings + curvature

$$\begin{aligned} k=1: &\quad \text{quintessence (domain walls)} \\ k=0: &\quad \text{cosmological constant} \end{aligned}$$

with

(29)
$$w_k = \frac{k-3}{3}.$$

Using (26) we obtain for the time–time component of the energy-momentum tensor

$$T^0{}_0 = \sum_{k=0}^{4} \epsilon_{k,0} \left(\frac{a_0}{a}\right)^k$$

$$\begin{cases} \epsilon_{k,0} &= \text{present energy densities} \\ a_0 &= a(\eta_0) = \text{present cosmic scale factor} \\ \eta_0 &= \text{present time} = \text{age of universe} \end{cases}$$

(30)
$$\Omega_k := \frac{\epsilon_{k,0}}{\epsilon_{\text{crit}}}; \qquad k = 0, 1, 3, 4.$$

In eq. (28) the term with $k = 2$ takes into account cosmic strings (s) and the curvature term, i.e. $\Omega_2 := \Omega_s + \Omega_{\text{curv}}$. Note that the scale factor a_0 at the present epoch is in the non-flat case completely determined by the total energy density Ω_{tot} and the Hubble length L_H (see eq. (20))

(31)
$$a_0 = \frac{L_H}{\sqrt{|1-\Omega_{\text{tot}}|}}.$$

4. Exact solution for the cosmic scale factor in terms of the Weierstrass elliptic ℘-function

Solving the Friedmann equation (28) is equivalent to computing the integral

(32)
$$\eta = \frac{c}{H_0} \int_0^{a(\eta)} \frac{da}{\left[\sum_{k=0}^{4} \Omega_k a_0^k a^{4-k}\right]^{\frac{1}{2}}},$$

i.e. $\eta = F(a(\eta))$, where $F(z)$ is an elliptic integral. Actually, what is required is the inverse function $a(\eta) = F^{-1}(\eta)$, which leads us to an elliptic function! We shall now show that the exact solution can be given in closed form in terms of the Weierstraß elliptic ℘-function defined by the series [10, 11, 12]

(33)
$$\wp(z) = \wp(z|\omega_1, \omega_2) := \frac{1}{z^2} + \sum_{m,n}{}' \left[\frac{1}{(z-\Omega_{mn})^2} - \frac{1}{\Omega_{mn}^2}\right].$$

Here the summation extends over all integer values of m and n, simultaneous zero values of m and n excepted with

(34)
$$\Omega_{mn} := 2m\omega_1 + 2n\omega_2; \quad \Im m\left(\frac{\omega_2}{\omega_1}\right) \neq 0.$$

$\wp(z)$ is doubly-periodic

(35)
$$\begin{aligned} \wp(z+2\omega_1) &= \wp(z) \\ \wp(z+2\omega_2) &= \wp(z) \end{aligned}$$

with periods $2\omega_1$ and $2\omega_2$, is even

(36)
$$\wp(z) = \wp(-z)$$

and is analytic throughout the whole z-plane except at the points Ω_{mn}, where it has double poles. In fact, $\wp(z)$ is an elliptic function of order 2. Defining the invariants

$$g_2 := 60 \sum_{m,n}{}' \frac{1}{\Omega_{mn}^4}$$
$$g_3 := 140 \sum_{m,n}{}' \frac{1}{\Omega_{mn}^6}, \tag{37}$$

we may also write $\wp(z) = \wp(z; g_2, g_3)$. The Weierstraß $\wp$-function satisfies the differential equation

$$\wp'^2(z) = 4\wp^3(z) - g_2\wp(z) - g_3. \tag{38}$$

At $z = 0$ it possesses the Laurent expansion

$$\wp(z) = \frac{1}{z^2} + \sum_{k=2}^{\infty} c_k z^{2k-2}, \tag{39}$$

where the coefficients c_k are obtained from $c_2 = \frac{g_2}{20}$, $c_3 = \frac{g_3}{28}$ and the recurrence relation

$$c_k = \frac{3}{(2k+1)(k-3)} \sum_{m=2}^{k-2} c_m c_{k-m}, \qquad k \geq 4. \tag{40}$$

Consider the integral

$$z = \int_w^\infty \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}} = z(w) \tag{41}$$

which determines z in terms of w. This integral is called an elliptic integral in Weierstrassian form. On differentiation and comparing with the differential equation (38), one infers $w = \wp(z)$. The result is sometimes used to write the equation (41) in the form

$$z = \int_{\wp(z)}^\infty \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}}. \tag{42}$$

Comparing (41) with eq. (32) derived from the Friedmann equation (28), we observe that (32) is not in Weierstrassian form, since it involves for $\Lambda \neq 0$ a fourth-order instead of a third-order polynomial. Luckily enough, it turns out that the general situation has been treated in 1865 in an Inaugural-dissertation at Berlin by Biermann [13] who ascribed it to Weierstraß.

Theorem 1 (Biermann-Weierstraß, 1865).
Let

$$A_0 x^4 + 4A_1 x^3 + 6A_2 x^2 + 4A_3 x + A_4 \equiv f(x) \tag{43}$$

be any quartic polynomial which has no repeated factors; and let its invariants be

$$g_2 \equiv A_0 A_4 - 4A_1 A_3 + 3A_2^2$$
$$g_3 \equiv A_0 A_2 A_4 + 2A_1 A_2 A_3 - A_2^3 - A_0 A_3^2 - A_1^2 A_4. \tag{44}$$

Let

$$z(w) = \int_\alpha^w \frac{dt}{\sqrt{f(t)}}, \tag{45}$$

where α is any constant, not necessarily a zero of $f(x)$, then

$$w(z) = \alpha + \frac{\frac{1}{2}f'(\alpha)\{\wp(z) - \frac{1}{24}f''(\alpha)\} + \frac{1}{24}f(\alpha)f'''(\alpha) - \sqrt{f(\alpha)}\wp'(z)}{2\{\wp(z) - \frac{1}{24}f''(\alpha)\}^2 - \frac{1}{48}f(\alpha)f^{(4)}(\alpha)}, \tag{46}$$

the function $\wp(z) = \wp(z; g_2, g_3)$ being formed with the invariants (44) of the quartic $f(x)$.

Theorem 2 (Biermann-Weierstraß, 1865).
With the notation of Theorem 1:

$$\wp(z) = \frac{\sqrt{f(w)f(\alpha)} + f(\alpha)}{2(w-\alpha)^2} + \frac{f'(\alpha)}{4(w-\alpha)} + \frac{f''(\alpha)}{24} \tag{47}$$

$$\wp'(z) = -\left\{\frac{f(w)}{(w-\alpha)^3} - \frac{f'(w)}{4(w-\alpha)^2}\right\}\sqrt{f(\alpha)} - \left\{\frac{f(\alpha)}{(w-\alpha)^3} + \frac{f'(\alpha)}{4(w-\alpha)^2}\right\}\sqrt{f(w)}. \tag{48}$$

In citing Theorem 1 and 2 we follow [10], p. 454, correcting however the sign in the last term in the nominator in eq. (46).
A comparison between (45) and (32) shows that we can apply Theorem 1 by choosing $\alpha = 0$, $w = a(\eta)$ and $z = \eta$. The invariants g_2, g_3 (44) are then given by

$$\begin{aligned} g_2 &:= \frac{K^2}{12} + 4C - 2AB + D(3D-K) \\ g_3 &:= \frac{K}{216} - \frac{8KC + A^2\Lambda}{12} - \frac{KAB}{6} - A^2B^2\hat{\eta}^2 \\ &\quad + D\Big(AB - \frac{K^2}{12} + \frac{KD}{2} - D^2 + 4C\Big) \end{aligned} \tag{49}$$

in terms of the cosmological parameters

$$\begin{aligned} A &:= \frac{1}{2}\Omega_{\rm m}\frac{H_0^2 a_0^3}{c^2} && \text{matter} \\ B &:= \frac{1}{4}\Omega_\phi\frac{H_0^2 a_0}{c^2} && \text{quintessence } (\phi) \text{ (domain walls)} \\ C &:= \frac{1}{12}A^2\hat{\eta}^2\Lambda && \text{cosmological constant} \\ D &:= \frac{1}{6}\Omega_{\rm s}\frac{H_0^2 a_0^2}{c^2} && \text{cosmic strings (s)} \\ \hat{\eta} &:= 2\frac{\sqrt{\Omega_{\rm r}}}{\Omega_{\rm m}}\frac{c}{H_0 a_0} && \text{radiation.} \end{aligned} \tag{50}$$

Here we have included in addition to matter (= baryonic + cold dark matter), radiation and a cosmological constant also an energy component due to strings (s) with equation of state $w_{\rm s} = -\frac{1}{3}$ and quintessence (ϕ) resp. domain walls with $w_\phi = -\frac{2}{3}$. We then obtain from Theorem 1 the following Theorem for the cosmic scale factor:

Theorem 3.
Let the matter-/energy-density of the Universe consist of five components corresponding to radiation, matter, cosmic strings, quintessence (domain walls), and a cosmological constant, i.e.

$$\Omega_{\rm tot} = \Omega_{\rm r} + \Omega_{\rm m} + \Omega_{\rm s} + \Omega_\phi + \Omega_\Lambda. \tag{51}$$

Furthermore, let the invariants g_2, g_3 be given by (49) defined in terms of the cosmological parameters A, B, C, D and $\hat{\eta}$, see eqs. (50). Then the exact solution to the Friedmann equation (21) resp. (28) is given by

$$a(\eta) = \frac{A}{2}\,\frac{\wp(\eta) - \hat{\eta}\wp'(\eta) + (AB\hat{\eta}^2 + \frac{K}{12} - \frac{D}{2})}{\big(\wp(\eta) + \frac{K}{12} - \frac{D}{2}\big)^2 - C}, \tag{52}$$

where $\wp(\eta) = \wp(\eta; g_2, g_3)$, $0 \le \eta \le \eta_\infty \le \infty$. Here η_∞ denotes the conformal time corresponding to infinite cosmic time ($t \to \infty$), i.e. $a(\eta_\infty) = \infty$.

The limiting conformal time η_∞ can be calculated from the integral (32) in the limit $a(\eta) \to \infty$, and in the case of non-vanishing curvature, i.e. for $\Omega_{\text{tot}} \neq 1, K \neq 0$ from the integral

$$\eta_\infty = \sqrt{|1-\Omega_{\text{tot}}|} \int_0^\infty \frac{dx}{\sqrt{\sum_{k=0}^4 \Omega_k x^k}}. \tag{53}$$

Note that $\eta_\infty < \infty$ in the case of a non-vanishing cosmological constant or, if $\Lambda \equiv 0$, for a nonvanishing (general) quintessence component with $w < -\frac{1}{3}$. An alternative and very efficient way to compute η_∞ in the general case is provided by

Theorem 4.
With the notation of Theorem 3:

$$\wp(\eta_\infty) = -\frac{K}{12} + \sqrt{C} + \frac{D}{2} \tag{54}$$

$$\wp'(\eta_\infty) = -AB\hat{\eta} - \frac{A}{2}\sqrt{\frac{\Lambda}{3}}. \tag{55}$$

To derive (54) and (55) we have used eqs. (47) and (48) in Theorem 2 with $\alpha = 0$, $z = \eta$ and have considered the limit $w = \frac{a(\eta)}{a_0} \to \infty$. The formula (52) for the cosmic scale factor $a(\eta)$ together with the formula (54) has been first derived in [14] for the special case of negative curvature and without a cosmic string component, i.e. for $K = -1$ and $D = 0$.

5. The time evolution and the age of the Universe

From (52) one derives the following behaviour at small conformal time, i.e. near the Big Bang ($\eta = 0$)

$$a(\eta) = A\hat{\eta}\eta + \frac{A}{2}\eta^2 + O(\eta^3) \tag{56}$$

giving $a'(0) = A\hat{\eta} = \frac{a_0^2}{L_H}\sqrt{\Omega_{\text{r}}}$. It follows that the time evolution of the Universe immediately after the Big Bang is dominated by radiation and is determined by the parameter $\hat{\eta}$ (see eq. (50)) which is approximately given by $\hat{\eta} \approx (1+\sqrt{2})\ \eta_{\text{eq}}$, where the subscript "eq" marks the epoch of matter-radiation equality with $a_{\text{eq}} := a(\eta_{\text{eq}}) = (\Omega_{\text{r}}/\Omega_{\text{m}})a_0$. Converting from η to cosmic time t, we obtain for the cosmic scale factor ($t \to 0$)

$$R(t) = R_0(4\Omega_{\text{r}})^{1/4}\sqrt{\frac{t}{t_H}} + O(t^{2/3}). \tag{57}$$

Here $t_H := \frac{1}{H_0} \approx 9.78 \times h^{-1}$ Gyr denotes the Hubble time. If the radiation component is neglected ($\hat{\eta} = 0$), as often done in the literature, one obtains a different behaviour

$$R(t) = R_0\left(\frac{9}{4}\Omega_{\text{m}}\right)^{1/3}\left(\frac{t}{t_H}\right)^{2/3} + O(t), \tag{58}$$

which clearly shows that in this case the early time evolution of the Universe is dominated by the matter component determined by the parameter A (see eq. (50)).

Of particular interest is the time evolution of the Universe at late times $t \to \infty$ which for $\Lambda > 0$ amounts to the limit $\eta \to \eta_\infty < \infty$. Expanding the scale factor (52) at $\eta = \eta_\infty$ and using eqs. (54) and (55), one obtains the following Laurent expansion

$$a(\eta) = \frac{\sqrt{3/\Lambda}}{\eta_\infty - \eta} - \frac{3}{\Lambda}B + O(\eta - \eta_\infty), \tag{59}$$

which shows that $a(\eta)$ possesses a simple pole at $\eta = \eta_\infty$. This implies that for $\Lambda > 0$ the scale factor increases indefinitely and thus the Universe expands forever, i.e. there is no stage of contraction to a "Big Crunch". Converting from η to cosmic time t, one obtains for $t \to \infty$

$$R(t) = O(e^{c\sqrt{\frac{\Lambda}{3}}t}), \tag{60}$$

which shows that the Universe is not only expanding, but that the expansion occurs with an exponential rate with time constant $t_\Lambda := \frac{1}{c}\sqrt{\frac{3}{\Lambda}} = \frac{t_H}{\sqrt{\Omega_\Lambda}}$. The result (60) reveals the famous accelerated expansion of the Universe discovered recently which requires as an explanation a dominating dark energy component which in this model we have identified with Einstein's cosmological constant Λ corresponding to the energy density (19). The exponential behaviour (60) was first obtained in the case of the so-called de Sitter universe for which $T^{\mu\nu} \equiv 0$ ("vacuum solution" of the Einstein eqs. (4)) but $\Lambda > 0$.

At present, most cosmologists assume the validity of the so-called ΛCDM model, which is a special case of our five component model, where the cosmic string and quintessence (domain wall) contributions are set equal to zero, i.e. $B = D = 0$. Then $a(\eta)$ possesses still a single pole at $\eta = \eta_\infty$ with the residue $\sqrt{\frac{3}{\Lambda}}$ as in eq. (59) and thus $R(t)$ shows again the accelerated expansion (60). The time evolution near the Big Bang is then given by

$$\begin{aligned} a(\eta) \;=\; & A\Big\{\hat{\eta}\eta + \frac{\eta^2}{2} - K\hat{\eta}\frac{\eta^3}{3!} - K\frac{\eta^4}{4!} + \hat{\eta}\left(K^2 + 48C\right)\frac{\eta^5}{5!} \\ & + (K^2 + 288C)\frac{\eta^6}{6!} + \hat{\eta}(-K - 528KC + 60A^2\Lambda)\frac{\eta^7}{7!} + O(\eta^8)\Big\}. \end{aligned} \tag{61}$$

One observes that the cosmological constant Λ, which appears also in the parameter C (see eq. (50)), influences the small time behaviour only at $O(\eta^5)$ and higher, whereas the curvature parameter K appears already in the $O(\eta^3)$-term.

As an alternative to the ΛCDM model we consider a special version of quintessence where the dark energy is identified with a homogeneous scalar field $\phi = \phi(t)$ with the constant equation of state $w_\phi = -2/3$. (Obviously, in this case the dark energy plays the same role as domain walls). The scale factor for this particular quintessence model is then obtained from (52) by setting $C = D = 0$, and one obtains the simple result

$$a(\eta) = \frac{1}{B}\wp(\eta - \eta_\infty) + \frac{K}{12B}. \tag{62}$$

The double pole of the Weierstraß $\wp$-function leads then to the Laurent expansion

$$a(\eta) = \frac{1/B}{(\eta - \eta_\infty)^2} + \frac{K}{12B} + O\big((\eta - \eta_\infty)^2\big). \tag{63}$$

Due to the double pole at $\eta = \eta_\infty$, one obtains the asymptotic power-law expansion

$$R(t) = B(ct)^2 + \ldots \quad \text{for } t \to \infty, \tag{64}$$

which corresponds again to an accelerated expansion of the Universe, but a much weaker one than in the case of a cosmological constant (compare with eq. (60)).

In order to derive $R(t) = a(\eta(t))$ from formula (52), one requires the function $\eta = \eta(t)$ which in turn is defined as the inverse function of $t = t(\eta)$. In the general case, there is no explicit expression known for $\eta = \eta(t)$ or $t = t(\eta)$. For the special quintessence model discussed above, there is however the following explicit relation

$$t = t(\eta) = \frac{1}{Bc}\left[\zeta(\eta_\infty - \eta) + \frac{K}{12}\eta - \zeta(\eta_\infty)\right] \tag{65}$$

in terms of the Weierstraß ζ-function. The function $\zeta(z) = \zeta(z; g_2, g_3)$ is defined by

$$\zeta(z) = \frac{1}{z} + \sum_{m,n}{}' \left[\frac{1}{z - \Omega_{mn}} + \frac{1}{\Omega_{mn}} + \frac{z}{\Omega_{mn}^2}\right]. \tag{66}$$

It is an odd function, $\zeta(-z) = -\zeta(z)$, and satisfies the important relation $\zeta'(z) = -\wp(z)$ (compare (66) with (33)). Note that $\zeta(z)$ is not an elliptic function.

As already mentioned, in many discussions of cosmology the radiation component is neglected, because it is very small at times larger than $t_{\rm eq}$, the time of matter-radiation equality. In this case the expression (52) for the scale factor simplifies considerably by setting $\hat{\eta} = C = 0$:

$$a(\eta) = \frac{A/2}{\wp(\eta) - \wp(\eta_\infty)} \tag{67}$$

with $\wp(\eta_\infty) = -\frac{K}{12} + \frac{D}{2}$. A Taylor expansion of $\wp(\eta)$ at $\eta = \eta_\infty$ leads with $\wp'(\eta_\infty) = -\frac{A}{2}\sqrt{\frac{\Lambda}{3}}$ for$\Lambda > 0$ to the same leading pole term as in eq. (59) and thus to the accelerated expansion (60) of the Universe. This is in accordance with the fact that a non-vanishing positive cosmological constant is always dominating at late times causing an exponential expansion, i.e. that the universe approaches for $\Lambda > 0$ a de Sitter universe.

In the fashionable ΛCDM model one considers (if radiation is neglected) only a two-component model consisting of matter and a cosmological constant, i.e. $\Omega_{\rm s} = \Omega_\phi = 0$ and $\Omega_{\rm tot} = \Omega_{\rm m} + \Omega_\Lambda$. In this case eq. (67) still holds with $\wp(\eta) = \wp(\eta; g_2, g_3)$ however with the simple expressions

$$g_2 = \frac{K^2}{12}, \quad g_3 = \frac{K}{216} - \frac{A^2\Lambda}{12} \tag{68}$$

for the invariants (see eq. (49) for $\hat{\eta} = B = C = D = 0$).

In the case of a flat ΛCDM model, $K = 0$, the cosmic scale factor $R(t)$ can be directly obtained in terms of an elementary function by integrating the Friedmann eq. (21):

$$R(t) = R_0 \left(\frac{\Omega_{\rm m}}{\Omega_\Lambda}\right)^{1/3} \sinh^{2/3}\left(\frac{3}{2}\frac{t}{t_\Lambda}\right). \tag{69}$$

It is seen that this formula nicely interpolates between the small t-behaviour (58) valid in the matter-dominated epoch and the asymptotic behaviour (60) for large times corresponding to a de Sitter universe. For the age of the Universe one obtains from (69) the closed expression valid for the flat ΛCDM model

$$t_0 = \frac{2}{3} t_\Lambda \operatorname{arsinh}\left(\frac{\Omega_\Lambda}{\Omega_{\rm m}}\right)^{1/2} = \frac{2}{3} t_H \frac{1}{\sqrt{\Omega_\Lambda}} \ln\left(\sqrt{\frac{\Omega_\Lambda}{\Omega_{\rm m}}} + \sqrt{1 + \frac{\Omega_\Lambda}{\Omega_{\rm m}}}\right). \tag{70}$$

With $\Omega_{\rm m} = 0.28$, $\Omega_\Lambda = 0.72$, $h = 0.70$ this gives $t_0 \approx 13.73\,{\rm Gyr}$, in excellent agreement with the most recent results from WMAP [9].

The formula (67) simplifies even further, if a pure matter model is considered, in which case the only free parameters are A and the curvature parameter K. Due to the simple expression for the invariants (49) in this case, the scale factor can be expressed for all curvature values in terms of elementary functions.

For a flat ($K = 0$) matter-dominated Universe, the famous Einstein-de Sitter universe [15], we obtain $g_2 = g_3 = 0$ and thus from (39) $\wp(\eta; 0, 0) = \frac{1}{\eta^2}$, which gives for the scale factor the simple power law $a(\eta) = \frac{A}{2}\eta^2$ leading to $t = t(\eta) = (A/6c)\eta^3$ and thus

$$R(t) = R_0 \left(\frac{3}{2}\frac{t}{t_H}\right)^{2/3}. \tag{71}$$

(Note that in this case holds $\Omega_{\rm tot} = \Omega_{\rm m} = 1$, $A = a_0^3/2t_H^2$.) Since $R_0 = R(t_0)$, we obtain from (71) for the age of the Universe $t_0 = \frac{2}{3}t_H \approx 9.31\,{\rm Gyr}$ if we use $h = 0.70$ for the reduced Hubble constant (see eq. (17) and Table 1). Since the present value for the age of the Universe is $t_0 \approx 13.7\,{\rm Gyr}$, we infer that the pure matter model in a flat Universe is excluded.

In the case of a negatively curved ($K = -1$) matter-dominated Universe we obtain $g_2 = \frac{1}{12}$ and $g_3 = -\frac{1}{216}$ which leads to $\wp(\eta) = \wp(\eta_\infty) + (4\sinh^2(\frac{\eta}{2}))^{-1}$ and thus to

$$a(\eta) = \frac{a_0}{2}\frac{\Omega_{\rm m}}{1 - \Omega_{\rm m}}(\cosh\eta - 1). \tag{72}$$

(Note that in this case holds $\Omega_{\rm tot} = \Omega_{\rm m} < 1$ and $\eta_\infty = \infty$.) Integrating the expression (72) gives

$$t(\eta) = \frac{t_H}{2}\,\frac{\Omega_{\rm m}}{(1-\Omega_{\rm m})^{3/2}}(\sinh\eta - \eta), \tag{73}$$

and thus the two eqs. (72) and (73) provide a parametric solution for the cosmic scale factor $R(t)$.

In the case of a positively curved ($K = +1$) matter-dominated Universe we obtain $g_2 = \frac{1}{12}$, $g_3 = \frac{1}{216}$ which leads to the parametric solution

$$\begin{aligned} a(\eta) &= \frac{a_0}{2}\,\frac{\Omega_{\rm m}}{\Omega_{\rm m}-1}(1-\cos\eta) \\ t(\eta) &= \frac{t_H}{2}\,\frac{\Omega_{\rm m}}{(\Omega_{\rm m}-1)^{3/2}}(\eta - \sin\eta). \end{aligned} \tag{74}$$

(Note that in this case holds $\Omega_{\rm tot} = \Omega_{\rm m} > 1$.) The solution (74) is very special, since it has the form of a cycloid, which implies that the Universe starts from a Big Bang at $\eta = 0$ but does not expand forever, rather the expansion slows down and the Universe recollapses for $\eta > \pi$ until it reaches at $\eta = 2\pi$ a Big Crunch.

As a final example, we would like to consider a two-component model consisting only of radiation and matter which describes our Universe correctly at early times until the epoch where the dark energy component starts to be appreciable. With $\Omega_{\rm tot} = \Omega_{\rm r} + \Omega_{\rm m}$ and setting $B = C = D = 0$, we obtain from (52)

$$a(\eta) = \frac{A}{2}\left[\frac{1}{\wp(\eta) + \frac{K}{12}} - \frac{\hat{\eta}\wp'(\eta)}{(\wp(\eta)+\frac{K}{12})^2}\right], \tag{75}$$

where the invariants are given by $g_2 = \frac{K^2}{12}$, $g_3 = \frac{K}{216}$. Specializing only to the case of negative curvature, $K = -1$, $\Omega_{\rm tot} < 1$, we obtain (see eq. (72)) for the scale factor

$$a(\eta) = A\left[\hat{\eta}\sinh\eta + \cosh\eta - 1\right]. \tag{76}$$

Let us mention that the simple results (71)–(74) and (76) do not require the introduction of the Weierstraß $\wp$-function but can be directly obtained from a solution to the Friedmann equation. In fact, they have been known since a long time. It is quite remarkable, however, that there exists the explicit formula (52) in the general case of the five component model considered here.

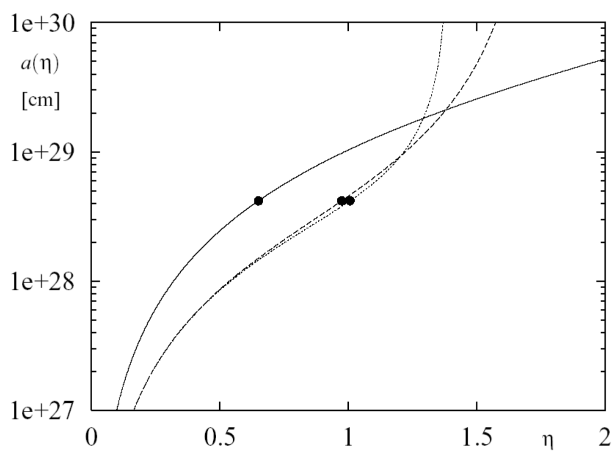

FIGURE 2. The cosmic scale factor $a(\eta)$ is shown as a function of conformal time η for three nearly flat hyperbolic universes as explained in the text. (Taken from ref.[14].)

In Fig. 2 we show as an illustration the cosmic scale factor $a(\eta)$ as a function of conformal time η for three different nearly flat hyperbolic universes ($K = -1$) for $\Omega_{\rm tot} = 0.90$ and reduced Hubble parameter $h = 0.70$. The full curve corresponds to a two-component model consisting of radiation and matter only, i.e. $\Omega_{\mathrm{m}} = 0.90$. The scale factor for this model is given by eq. (76). For the present scale factor $a_0 = a(\eta_0)$, indicated by a dot in Fig. 2, one obtains $a_0 \approx 4.18 \times 10^{28}$ cm. Note that for this model $\eta_\infty = \infty$ holds and that $a(\eta)$ describes an expanding but decelerating universe. The dashed curve represents a three-component model consisting of radiation, matter and a dark-energy component with constant equation of state $w_\phi = -\frac{2}{3}$ corresponding to a particular quintessence model with $\Omega_{\mathrm{m}} = 0.30$, $\Omega_\phi = 0.60$. The scale factor for the model is given by eq. (62). In Fig. 2 one clearly sees the approach to the double pole at $\eta_\infty \approx 1.74$ in accordance with eq. (63). For the age of the Universe one obtains $t_0 \approx 12.4$ Gyr. The dotted curve in Fig. 2 shows $a(\eta)$ for a three-component model consisting of radiation, matter and a cosmological constant with $\Omega_{\mathrm{m}} = 0.30$, $\Omega_\Lambda = 0.60$. The scale factor for this ΛCDM model is given by eq. (52) with $B = D = 0$ and $K = -1$. One observes a steep rise to the pole at $\eta_\infty \approx 1.39$, see eq. (59), corresponding to the exponential time evolution eq. (60) at late times indicating the accelerated expansion of the Universe. For the age of the Universe one obtains in this model $t_0 \approx 13.1$ Gyr. Note that a smaller Hubble constant leads to larger values for the age t_0. (For details, we refer the reader to [14].) Very similar curves are obtained in the case of flat or positively curved models. In the case of the fashionable flat ΛCDM model one obtains for the age of the Universe values between 13.67 and 13.81 Gyr depending on the various sets of cosmological parameters given by WMAP [9]. For a non-flat ΛCDM model with $\Omega_{\mathrm{m}} = 0.258$, $\Omega_\Lambda = 0.757$, i.e. $\Omega_{\mathrm{tot}} = 1.015$ corresponding to a small positive curvature, and $h = 0.719$ one obtains for the age of the Universe $t_0 \approx 13.529$ Gyr. All these values for t_0 are compatible with e.g. the limit given by globular cluster ages.

6. The time evolution of the cosmic energy-/matter-densities

Knowing the expansion rate of the Universe in terms of the cosmic scale factor $a(\eta)$, we are able to give explicit formulae for the time evolution of the energy densities of the various components making up the energy budget of the Universe. In Sect. 2 we have introduced the dimensionless energy densities Ω_k at the present epoch, $\Omega_k := \Omega_k(\eta_0)$, in terms of the critical energy density $\epsilon_{\mathrm{crit}} := \epsilon_{\mathrm{crit}}(\eta_0)$ defined in (16) in terms of the Hubble constant H_0. Generalizing eq. (16) to the time-dependent critical energy density

$$\epsilon_{\mathrm{crit}}(\eta) := \frac{3H^2(\eta)c^2}{8\pi G}, \tag{77}$$

where $H(\eta) := a'(\eta)/a^2(\eta) = \dot{R}(t(\eta))/R(t(\eta))$ denotes the Hubble parameter as a function of conformal time η, we are led to define the time-dependent energy density parameter

$$\Omega_{\mathrm{x}}(\eta) := \frac{\epsilon_{\mathrm{x}}(\eta)}{\epsilon_{\mathrm{crit}}(\eta)}. \tag{78}$$

Here $\mathrm{x} = 0, 1, 2, \ldots$ refers to the various energy-/matter-components. (Note that in this section we have changed the notation from k to x, see eqs. (24)–(30)). Rewriting the time-dependent energy density in terms of η,

$$\epsilon_{\mathrm{x}}(\eta) = \epsilon_{\mathrm{x}}(\eta_0)\left(\frac{a_0}{a(\eta)}\right)^{\mathrm{x}}, \tag{79}$$

one derives for the time dependence of the density parameter $\Omega_{\mathrm{x}}(\eta)$ of the x^{th} energy component [16]

$$\Omega_{\mathrm{x}}(\eta) = \frac{y^{4-\mathrm{x}}}{g(y)}\Omega_{\mathrm{x}}. \tag{80}$$

Here we have defined the dimensionless scale factor $y := \frac{a(\eta)}{a_0} = \frac{1}{1+z}$, where z is the redshift, and the dimensionless function

$$g(y) := \sum_{\mathrm{x}=0}^{4} \Omega_{\mathrm{x}} y^{4-\mathrm{x}}. \tag{81}$$

Note that $g(1) = \Omega_{\text{tot}} + \Omega_{\text{curv}} = 1$, see eq. (20), since $\Omega_2 := \Omega_{\text{s}} + \Omega_{\text{curv}}$. The general relation (80) determines the time evolution of the density parameters $\Omega_{\text{x}}(\eta)$ in terms of their present values $\Omega_{\text{x}} = \Omega_{\text{x}}(\eta_0)$. In addition one obtains for the time dependence of the total energy density $\Omega_{\text{tot}}(\eta)$

$$\Omega_{\text{tot}}(\eta) = 1 + \frac{y^2}{g(y)}(\Omega_{\text{tot}} - 1) \tag{82}$$

satisfying $\Omega_{\text{tot}}(0) = 1$.

To analyse the evolution of the energy densities at very early times shortly after the Big Bang, i.e. in the radiation-dominated epoch, we use the short-time expansion (56) of the scale factor to derive from eq. (80) for $\eta \to 0$

$$\begin{aligned} \Omega_{\text{r}}(\eta) &= 1 - 2\frac{\eta}{\hat{\eta}} + O(\eta^2) \\ \Omega_{\text{m}}(\eta) &= 2\frac{\eta}{\hat{\eta}} + O(\eta^2) \\ \Omega_{\text{s}}(\eta) &= O(\eta^2) \\ \Omega_{\phi}(\eta) &= O(\eta^3) \\ \Omega_{\Lambda}(\eta) &= O(\eta^4). \end{aligned} \tag{83}$$

For the total energy density follows from eq. (82)

$$\Omega_{\text{tot}}(\eta) = \begin{cases} 1 & \forall \eta \geq 0 \quad (K = 0) \\ 1 + K\eta^2 + \dots & (K = \pm 1). \end{cases} \tag{84}$$

The crucial point to observe is that the string-, quintessence- (domain wall-) and Λ-components are at early times suppressed, the more the smaller the equation of state ($w_{\text{x}} \leq -\frac{1}{3}$) is. This is of great importance, since there exists the strong constraint $\Omega_{\text{x}}(\eta_{\text{BBN}}) \ll 10^{-5}$ coming from the Big Bang nucleosynthesis (BBN).

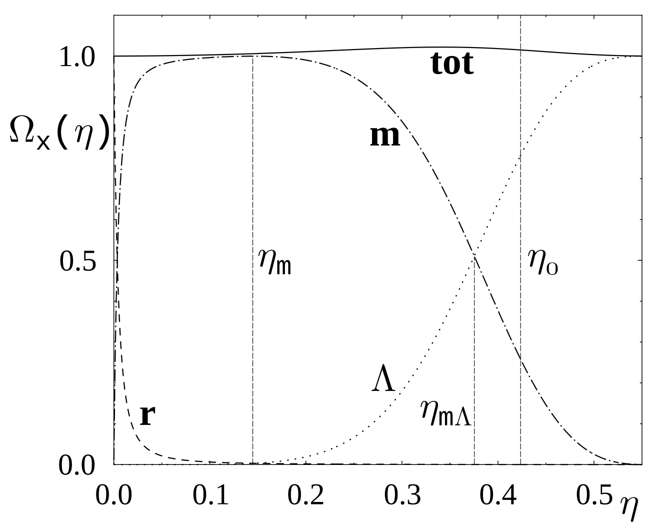

FIGURE 3. The dimensionless energy-/matter-densities $\Omega_{\text{x}}(\eta)$ are shown as a function of conformal time η for a nearly flat universe with positive curvature as explained in the text. Here the dark energy is identified with Einstein's cosmological constant Λ.

At late times, $\eta \to \eta_\infty$ resp. $z \to -1$, we obtain for $\Lambda > 0$

$$
\begin{aligned}
\Omega_{\rm r}(z) &= \frac{\Omega_{\rm r}}{\Omega_\Lambda}(1+z)^4 + \ldots \to 0 \\
\Omega_{\rm m}(z) &= \frac{\Omega_{\rm m}}{\Omega_\Lambda}(1+z)^3 + \ldots \to 0 \\
\Omega_{\rm s}(z) &= \frac{\Omega_{\rm s}}{\Omega_\Lambda}(1+z)^2 + \ldots \to 0 \\
\Omega_\phi(z) &= \frac{\Omega_\phi}{\Omega_\Lambda}(1+z) + \ldots \to 0 \\
\Omega_\Lambda(z) &= 1 - \frac{\Omega_\phi}{\Omega_\Lambda}(1+z) + \ldots \to 1 \\
\Omega_{\rm tot}(z) &= 1 + K\frac{|\Omega_{\rm tot}-1|}{\Omega_\Lambda}(1+z)^2 + \ldots \to 1.
\end{aligned} \tag{85}
$$

We see that in this case, where the dark energy is described by Einstein's cosmological constant, all energy-/matter-densities vanish asymptotically at late times apart from the Λ-contribution $\Omega_\Lambda(z)$ which approaches one in this limit. Furthermore, one obtains for $\Lambda > 0$ that the universe becomes asymptotically flat.

In Fig. 3 we show the energy-/matter-densities $\Omega_{\rm x}(\eta)$ as a function of conformal time η for a ΛCDM model with positive curvature. Here we used the following cosmological parameters: $\Omega_{\rm r} = 4.183 \times h^{-2} \times 10^{-5}$, $\Omega_{\rm m} = 0.258$, $\Omega_\Lambda = 0.757$, $h = 0.719$ in full agreement with the WMAP data [9]. It is to be noted that this model describes an almost flat universe, since $\Omega_{\rm tot}$ having the value $\Omega_{\rm tot} = 1.015$ exceeds the critical value one by 1.5% only.

Fig. 3 shows nicely the dominance of the radiation component (r) immediately after the Big Bang (as derived in eqs. (83)), but it is seen that this situation changes quickly due to the sharp rise of the matter component (m) which develops a relatively long plateau with maximum value at $\eta = \eta_{\rm m} \approx 0.145$. The dark energy component Λ behaving like $O(\eta^4)$ (see eqs. (83)) is negligible during a long time period until it increases and becomes equal to the (decreasing) matter component at $\eta = \eta_{{\rm m}\Lambda} \approx 0.375$. At the present epoch, $\eta = \eta_0 \approx 0.424$, corresponding to an age of the Universe of $t_0 \approx 13.53\,$Gyr, the Λ-component dominates already as a consequence of the input parameters ($\Omega_{\rm m}$, Ω_Λ) deduced from astrophysical observations. The total density parameter $\Omega_{\rm tot}(\eta)$ starts at one, increases slightly until it reaches a maximum at $\eta \approx 0.35$ and then approaches asymptotically one from above (see eq. (84)) due to the assumed positive curvature.

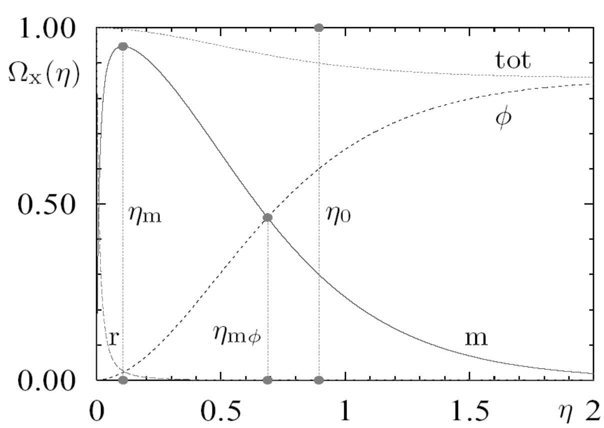

FIGURE 4. The dimensionless energy-/matter-densities $\Omega_{\rm x}(\eta)$ are shown as a function of conformal time η for a nearly flat hyperbolic universe as explained in the text. (Taken from ref.[16].) Here the dark energy is identified with a quintessence field ϕ with $w_\phi \equiv w_{\rm s} = -\frac{1}{3}$.

In Fig. 4 we show the energy-/matter-densities for a rather extreme model which consists of radiation, matter and a particular quintessence component with equation of state $w_\phi = -\frac{1}{3}$ and possesses negative curvature. For this model the quintessence potential $V(\phi)$ is explicitly known and reads [16]

$$V(\phi) = \frac{V_0}{[\tilde{\eta}\sinh(\phi/\phi_0) + \cosh(\phi/\phi_0) - 1]^2} \tag{86}$$

with $\tilde{\eta} = \frac{2}{\Omega_\mathrm{m}}\sqrt{\Omega_\mathrm{r}(\Omega_\phi + \Omega_\mathrm{curv})}$, $V_0 = \frac{8}{3}\epsilon_\mathrm{crit}\Omega_\phi\frac{(\Omega_\phi+\Omega_\mathrm{curv})^2}{\Omega_\mathrm{m}}$ and $\phi_0 = \frac{m_\mathrm{P}}{2\sqrt{\pi}}\sqrt{\frac{\Omega_\phi}{\Omega_\phi+\Omega_\mathrm{curv}}}$. (Here $m_\mathrm{P} := \sqrt{\frac{\hbar c}{G}} \approx 1.22 \times 10^{19}\,\mathrm{GeV}/c^2 \approx 2.18 \times 10^{-5}\,\mathrm{g}$ denotes the Planck mass.) The quintessence field $\phi(\eta)$ leads to an energy-momentum tensor $T_\phi^{\mu\nu}$ which has the perfect fluid form (5) with

$$\begin{aligned} \epsilon_\phi(\eta) &:= \frac{1}{2a^2}\phi'^2 + V(\phi) \\ p_\phi(\eta) &:= \frac{1}{2a^2}\phi'^2 - V(\phi). \end{aligned} \tag{87}$$

In this model the time evolution of $\Omega_\mathrm{x}(\eta)$ at late times is given by

$$\begin{aligned} \Omega_\mathrm{r}(z) &= \frac{\Omega_\mathrm{r}}{\Omega_\phi + \Omega_\mathrm{curv}}(1+z)^2 + \ldots \to 0 \\ \Omega_\mathrm{m}(z) &= \frac{\Omega_\mathrm{m}}{\Omega_\phi + \Omega_\mathrm{curv}}(1+z) + \ldots \to 0 \\ \Omega_\phi(z) &\cong \Omega_\mathrm{tot}(z) = 1 - \frac{\Omega_\mathrm{curv}}{\Omega_\phi + \Omega_\mathrm{curv}} + \ldots < 1. \end{aligned} \tag{88}$$

In Fig. 4 we plot $\Omega_\mathrm{x}(\eta)$ using the cosmological parameters $\Omega_\mathrm{m} = 0.30$, $\Omega_\phi = 0.60$, i.e. $\Omega_\mathrm{tot} = 0.90$ resp. $\Omega_\mathrm{curv} = 0.10$ which leads asymptotically to $\Omega_\mathrm{tot}(z = -1) = \frac{6}{7} \approx 0.86$. In contrast to the model discussed in Fig. 3, the matter component does not develop a plateau, and furthermore, the quintessence component starts to be appreciable rather early. The other values shown in Fig. 4 are [16]: $\eta_\mathrm{eq} = 0.0087$, $\eta_\mathrm{m} = 0.106$ resp. $z_m = \sqrt{\frac{\Omega_\phi+\Omega_\mathrm{curv}}{\Omega_\mathrm{r}}} - 1 \approx 83$, which gives the redshift at which the matter component reaches its maximum. Furthermore, $\eta_{\mathrm{m}\phi} = 0.687$, $\eta_0 = 0.8938$ corresponding to an age of the universe of $t_0 = 12.16\,\mathrm{Gyr}$.

I would like to thank Ralf Aurich, Holger S. Janzer and Sven Lustig for our fruitful collaboration during the last years and for providing various numerical results.

References

[1] A. Einstein: *Kosmologische Betrachtungen zur allgemeinen Relativitätstheorie*, Sitzungsberichte der Preußischen Akademie der Wissenschaften (1917), 142–152.
[2] A. Friedmann: *Über die Krümmung des Raumes*, Zeitschrift für Physik **10**(1922), 377–386.
[3] A. Friedmann: *Über die Möglichkeit einer Welt mit konstanter negativer Krümmung*, Zeitschrift für Physik **21**(1924), 326–332.
[4] G. Lemaître: *Note on De Sitter's Universe*, Journal of Mathematical Physics **4** (1925), 188–192.
[5] G. Lemaître: *Un Univers homogène de masse constante et de rayon croissant, rendant compte de la vitesse radiale des nébuleuses extra-galactiques*, Annales de Société scientifique de Bruxelles **47** (1927), 49–56.
[6] G. Lemaître: *The beginning of the World from the point of view of quantum theory*, Nature **127** (1931), 706.
[7] G. Lemaître: *The cosmological constant*, in: Paul A. Schilpp (ed.) *Albert Einstein: Philosopher - Scientist*, The Library of Living Philosophers vol. VII, Open Court, La Salle (1949).
[8] G. Lemaître: *The primeval atom hypothesis and the problem of the clusters of galaxies*, in: R. Stoops (ed.)*La Structure et l'Évolution de l'Univers*, Rapports et discussions du 11e Conseil de physique, Institut Internationale de Physique Solvay (1958), 1–31, Coudenberg, Bruxelles.
[9] E. Komatsu *et al.: Five-year Wilkinson Microwave Anisotropy Probe (WMAP) observations: cosmological interpretation*, Astrophysical Journal Supplement Series (to appear), arXiv: 0803.0547v2 [astro-ph].
[10] E.T. Whittaker and G.N. Watson: *A Course of Modern Analysis*, Cambridge University Press, Fourth Edition (reprinted 1973).
[11] F. Tricomi: *Elliptische Funktionen*, Akademische Verlagsgesellschaft, Leipzig (1948).

[12] M. Abramowitz and I.A. Stegun: *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover Publications, New York (1972).

[13] G.G.A. Biermann: *Problemata Quaedam Mechanica Functionum Ellipticarum Ope Soluta*, Dissertatio Inauguralis Quam Consensu et Auctoritate Amplissimi Philosophorum Ordinis in Alma Literarum Universitate Friderica Guilelma Berolinensi, Typis Caroli Schultze, Berlin (1865).

[14] R. Aurich and F. Steiner: *The cosmic microwave background for a nearly flat compact hyperbolic universe*, Monthly Notices of the Royal Astronomical Society **323** (2001), 1016–1024.

[15] A. Einstein and W. de Sitter: *On the relation between the expansion and the mean density of the universe*, National Academy of sciences, Proceedings **18** (1932), 213–214.

[16] R. Aurich and F. Steiner: *Quintessence with a constant equation of state in hyperbolic universes*, Physical Review **D67** (2003), 123511-1–123511-16.

Frank Steiner, Institut für Theoretische Physik, Universität Ulm, Albert-Einstein-Allee 11, 89081 Ulm, Germany

E-mail address: frank.steiner@uni-ulm.de