Computational Biomechanics 2017

Lecture 2:

Basic Mechanics 2

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Contents

Detailed Schedule Summer 2017

SW	Day	Date	Topic	Lecturer
01	Mo	17 Apr	- Holiday -	-
02	Mo	24 Apr	L01: Intro to Biomechanics; Mech 1: Statics	Ulli
03	Mo	01 May	- Holiday -	-
04	Mo	08 May	L02: Mech 2: Elastostatics; Mat. Props. Biol. Tissues	Ulli

Mechanical Basics

1.3 Variables, Dimensions and Units

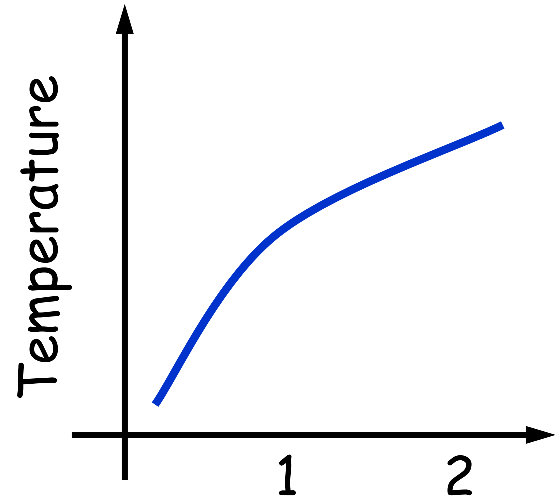
Standard: ISO 31, DIN 1313

Variable = Number $\cdot$ Unit

Length L = $2 \cdot \mathrm{m}$ = $2\ \mathrm{m}$

{Variable} = Number

[Variable] = Unit



~~Length L [m]~~

Length L / m

Length L in m →

Three mechanical SI-Units:

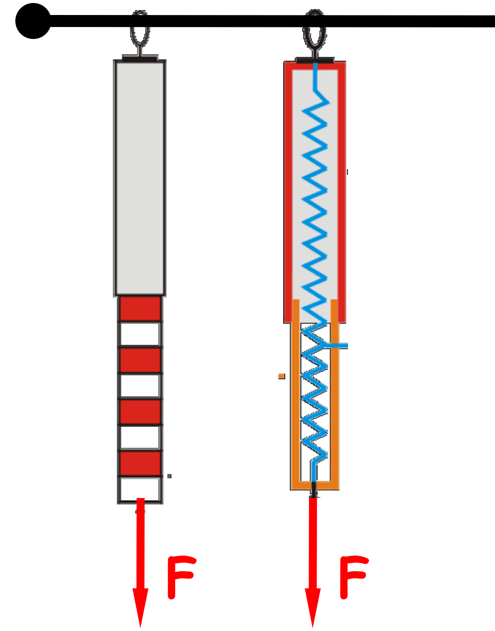
m (Meter)

kg (Kilogram)

s (Seconds)

2 STATICS OF RIGID BODIES

2.1 Force



- We all believe to know what a *force* is.
- But, *force* is an invention not a discovery!
- … it can not be measured directly.

Newton's 2nd Law [Axiom]:

Force = Mass times Acceleration or $F = m \cdot a$

Note to Remember:

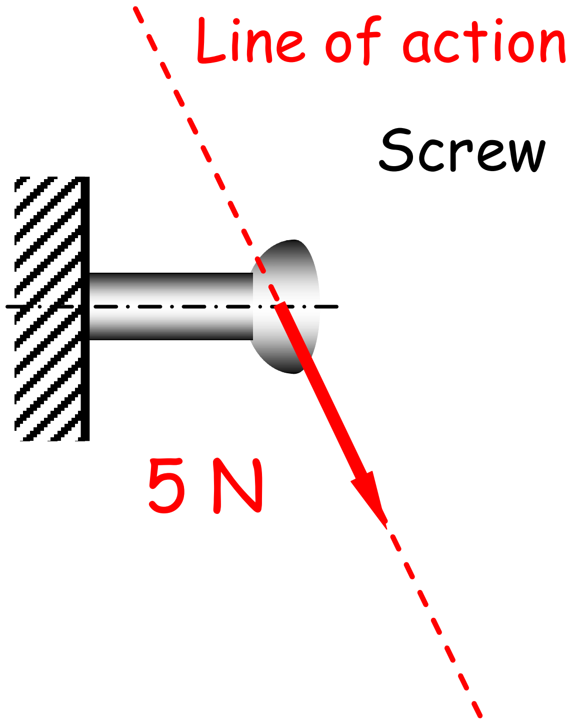
„A force is the cause of acceleration or deformation of a body"

Representation of Forces

... with arrows

Forces are Vectors with

- **Magnitude**
- **Direction**
- **Sense of Direction**



Units of Force

Newton

N = kg·m/s^2

$F_G = m \cdot g = 0{,}1\ \mathrm{kg} \cdot 9{,}81\ \mathrm{m/s^2}$

$= 0{,}981\ \mathrm{kg\ m/s^2}$

$\approx 1\ \mathrm{N}$

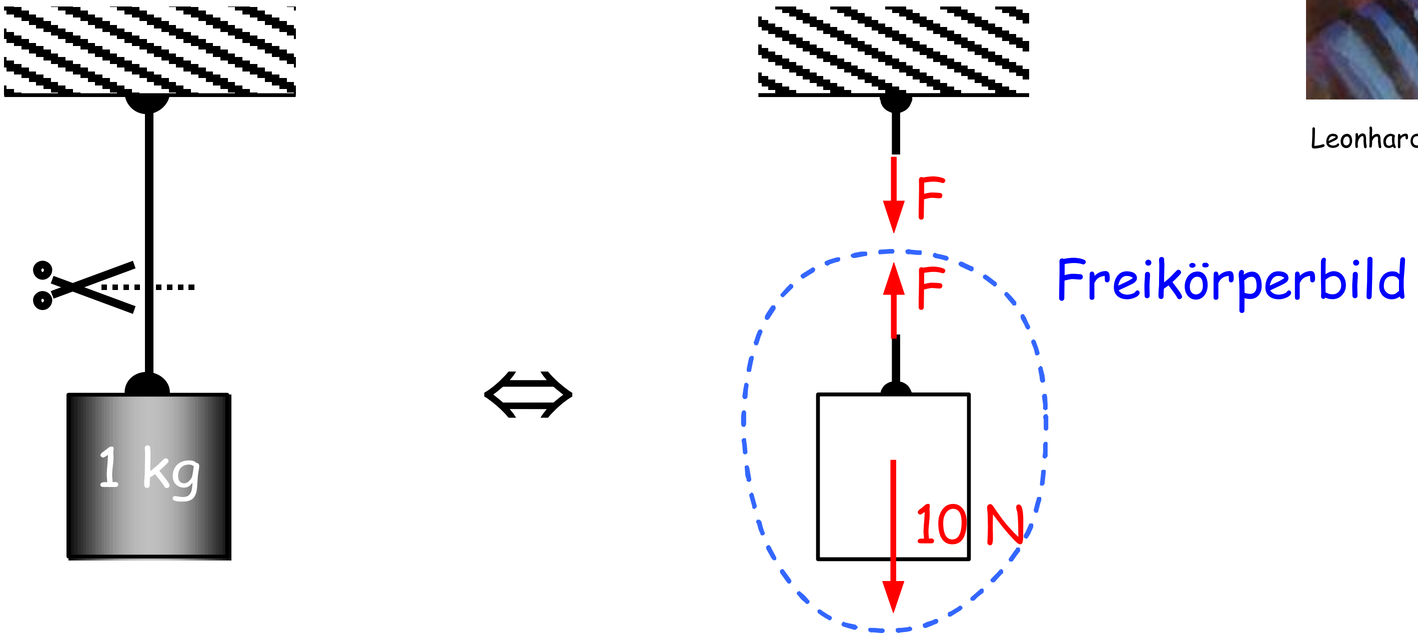
Note to Remember:

1 Newton ≈ Weight of a bar of chocolate (100 g)

2.2 Method of Sections (Euler) [Schnittprinzip]

Free-Body Diagramm (FBD) [Freikörper-Bild]


Leonhard Euler, 1707 - 1783

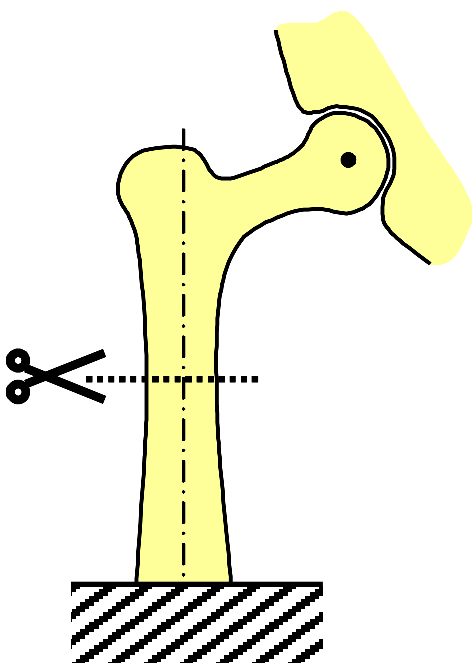


Note to Remember:

First, cut the system, then include forces and moments.

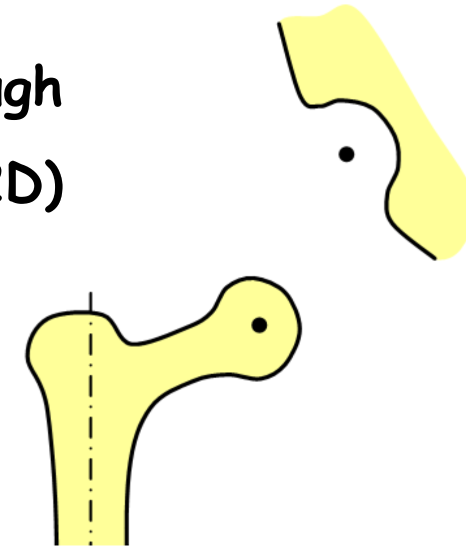
Free-body diagram = <u>completely</u> isolated part.

2.2 Method of Sections

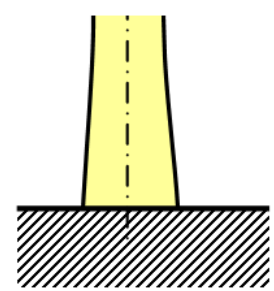


⇔

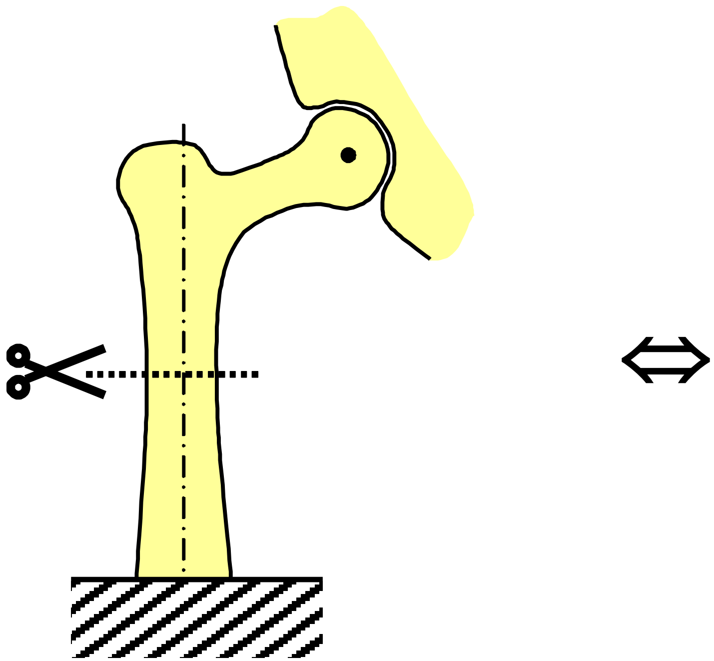
Cut through joint (2D)



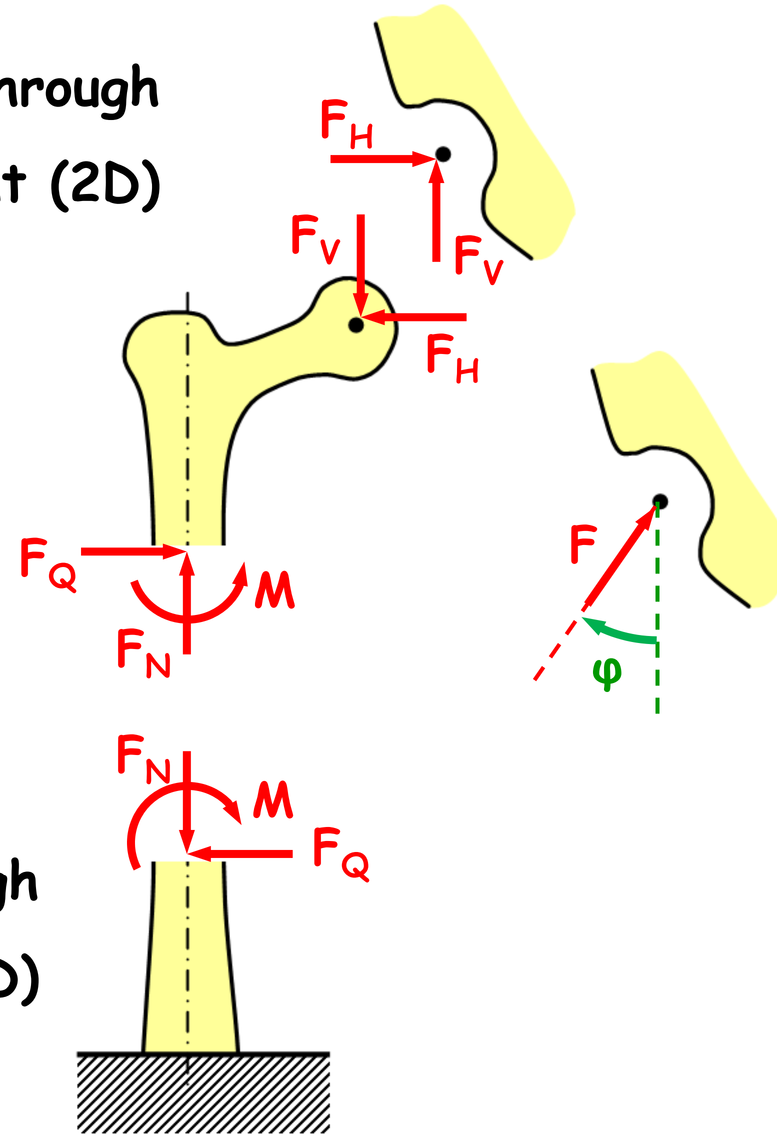
Cut through bone (2D)



2.2 Method of Sections



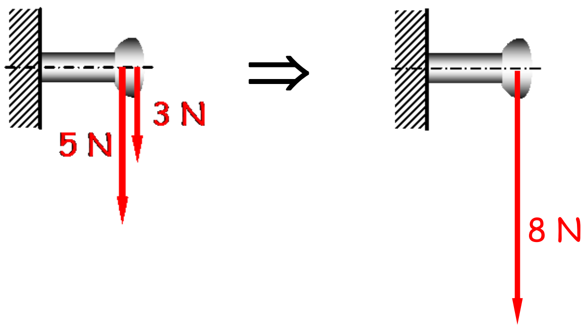
Cut through joint (2D)



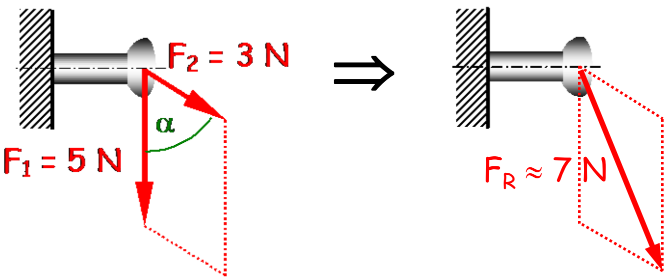
Cut through bone (2D)

2.3 Combining and Decomposing Forces

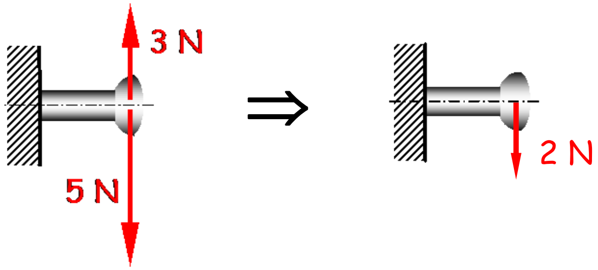
Summation of Magnitudes



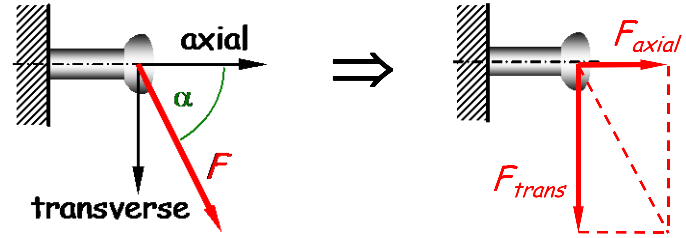
Vector Addition



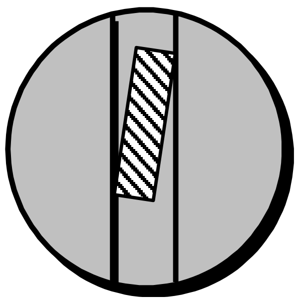
Subtraction of Magnitudes

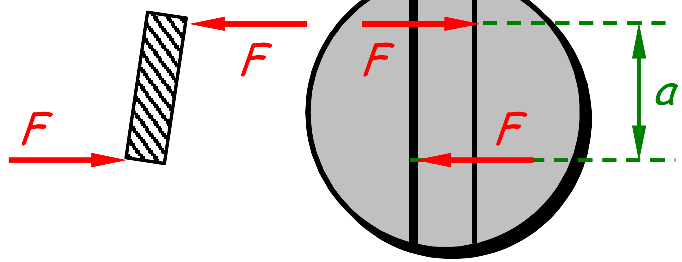


Decomposition into Components

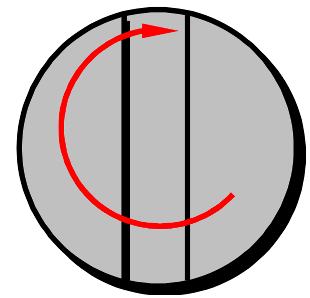


2.4 The Moment [Das Moment]





Force Couples (F, a)



Moment M

Note to remember:

The moment $M = F \cdot a$ is equivalent to a force couple (F, a).

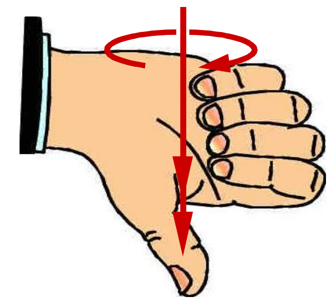
A moment is the cause for angular acceleration or angular deformation (Torsion, Bending) of a body.

Units for Moment

Newton-Meter

$N \cdot m = kg \cdot m^2/s^2$

Rechte-Hand-Regel:

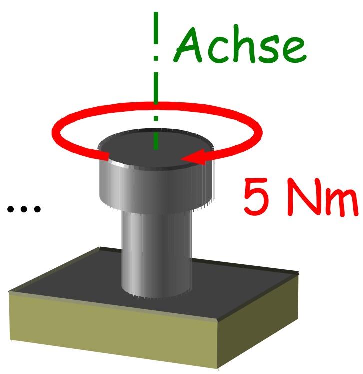


Representation of Moments

... with rotation arrows or double arrows

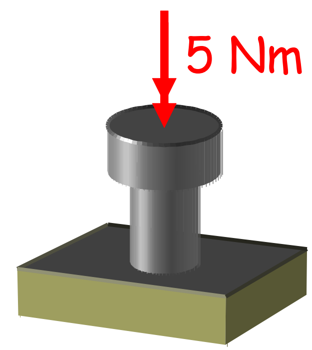
Moments are Vectors with ...

- **Magnitude**
- **Direction**
- **Sense of Direction**



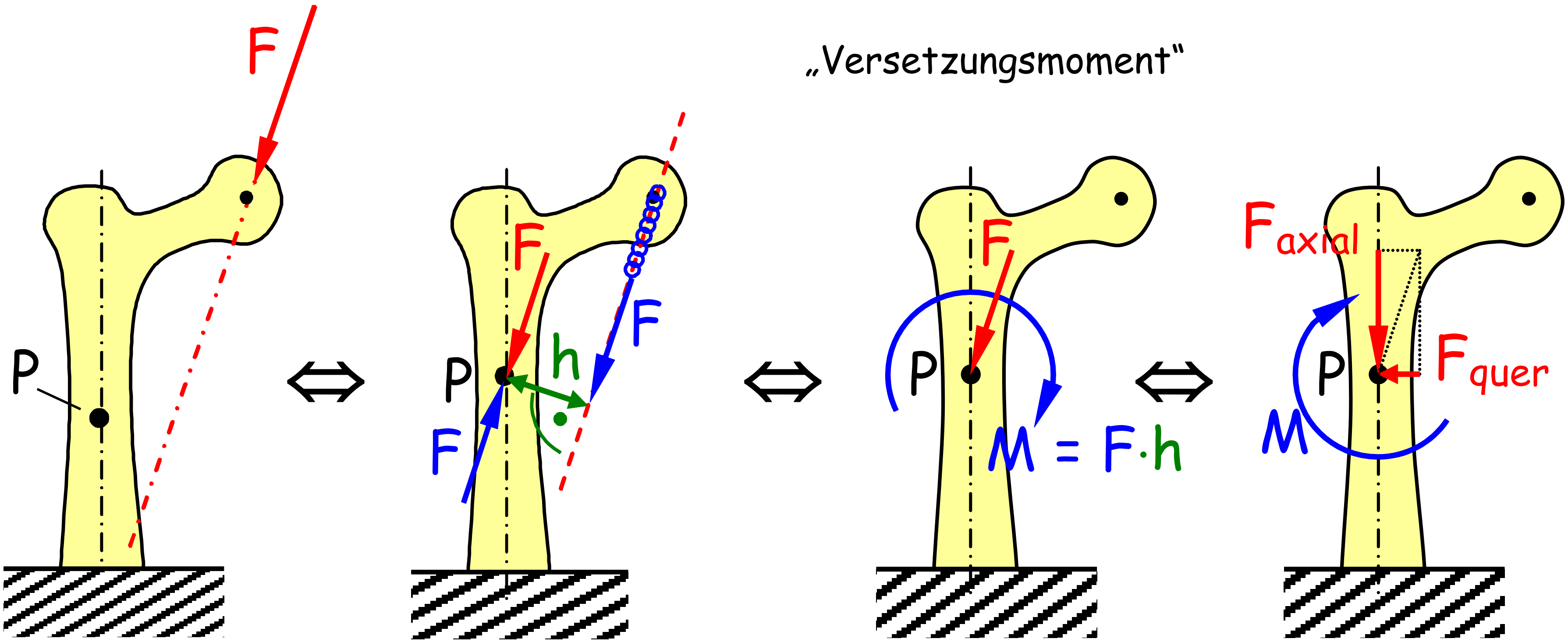
Drehpfeil

oder



Doppelpfeil

2.5 Moment of a Force about a Point [Versetzungsmoment]



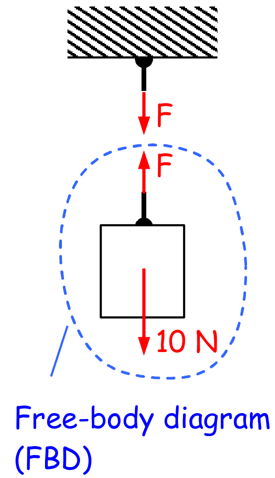
Note to Remember:

Moment = Force times lever-arm

2.7 Static Equilibrium

Important:

Free-body diagram (FBD) first, then equilibrium!

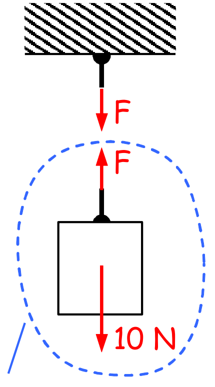


For 2D Problems max. 3 equations for each FBD:

The sum of all forces in *x*-direction equals zero:	$F_{1,x} + F_{2,x} + \ldots \overset{!}{=} 0$
The sum of all forces in *y*-direction equals zero:	$F_{1,y} + F_{2,y} + \ldots \overset{!}{=} 0$
The sum of Moments with respect to P equals zero:	$M_{1,z}^{P} + M_{2,z}^{P} + \ldots \overset{!}{=} 0$

(For 3D Problems max. 6 equations for each FBD)

2.7 Static Equilibrium



Free-body diagram (FBD)

Important:

Free-body diagram (FBD) first, then equilibrium!

3 equations of equilibrium for each FBD in 2D:

$$\text{Sum of all forces in x-direction: } \quad F_{1,x} + F_{2,x} + \ldots \overset{!}{=} 0,$$

$$\text{Sum of all forces in y-direction: } \quad F_{1,y} + F_{2,y} + \ldots \overset{!}{=} 0,$$

$$\text{Sum of all moments w.resp.to P: } \quad M_{1,z}^{P} + M_{2,z}^{P} + \ldots \overset{!}{=} 0.$$

- Force EEs can be substituted by moment EEs
- 3 moment reference points should not lie on one line

6 equlibrium equations for one FBD in 3D:

$$\text{Summe aller Kräfte in x-Richtung:} \quad \sum_i F_{ix} \overset{!}{=} 0,$$

$$\text{Summe aller Kräfte in y-Richtung:} \quad \sum_i F_{iy} \overset{!}{=} 0,$$

$$\text{Summe aller Kräfte in z-Richtung:} \quad \sum_i F_{iz} \overset{!}{=} 0,$$

$$\text{Summe aller Momente um x-Achse bezüglich Punkt P:} \quad \sum_i M_{ix}^{P} \overset{!}{=} 0.$$

$$\text{Summe aller Momente um y-Achse bezüglich Punkt Q:} \quad \sum_i M_{iy}^{Q} \overset{!}{=} 0.$$

$$\text{Summe aller Momente um z-Achse bezüglich Punkt R:} \quad \sum_i M_{iz}^{R} \overset{!}{=} 0.$$

- Force EEs can be substituted by moment EEs
- Max. 2 moment axis parallel to each other
- Determinant of coef. matrix not zero

2.8 Recipe for Solving Problems in Statics

Step 1: Model building. Generate a simplified replacement model (diagram with geometry, forces, constraints).

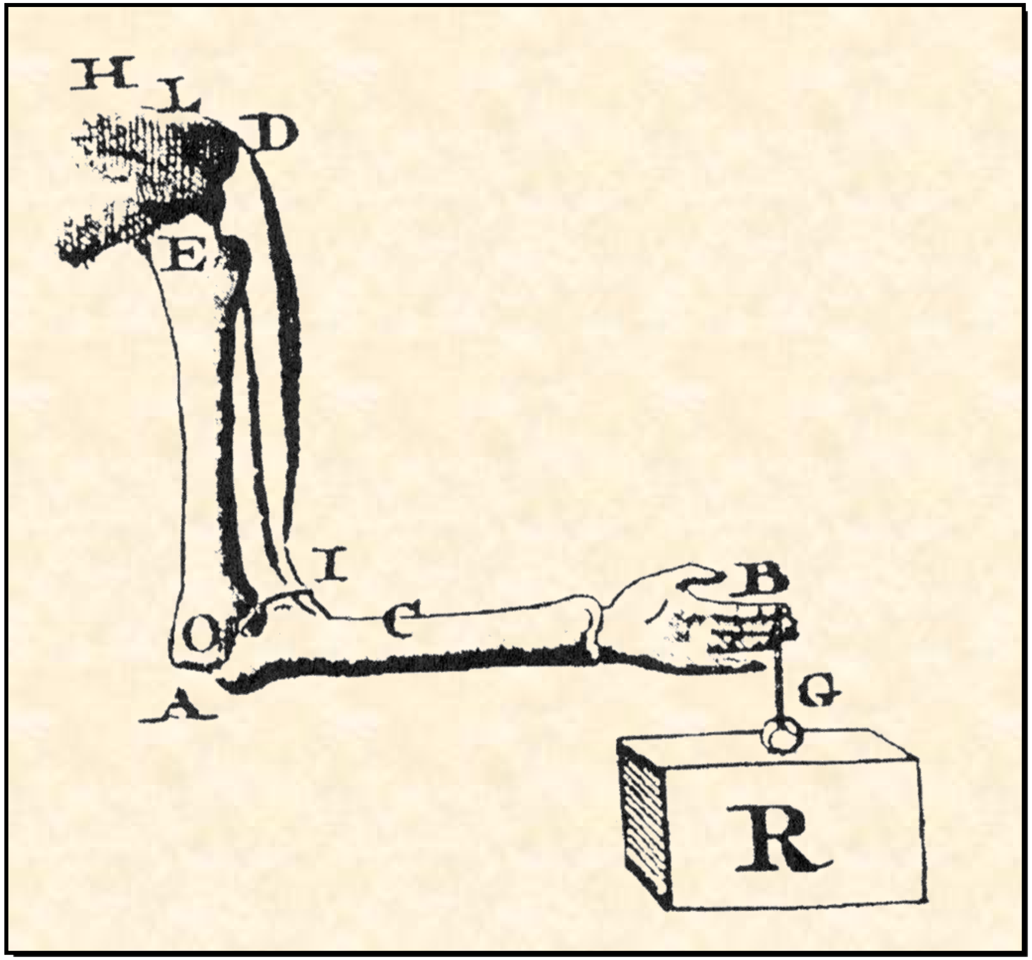
Step 2: Cutting, Free-body diagram. Cut system and develop free-body diagrams. Include forces and moments at cut, as well as weight.

Step 3: Equilibrium equations. Write the force- and moment equilibrium equations (<u>only</u> for free-body diagrams).

Step 4: Solve the equations. One can only solve for as many unknowns as equations, at most.

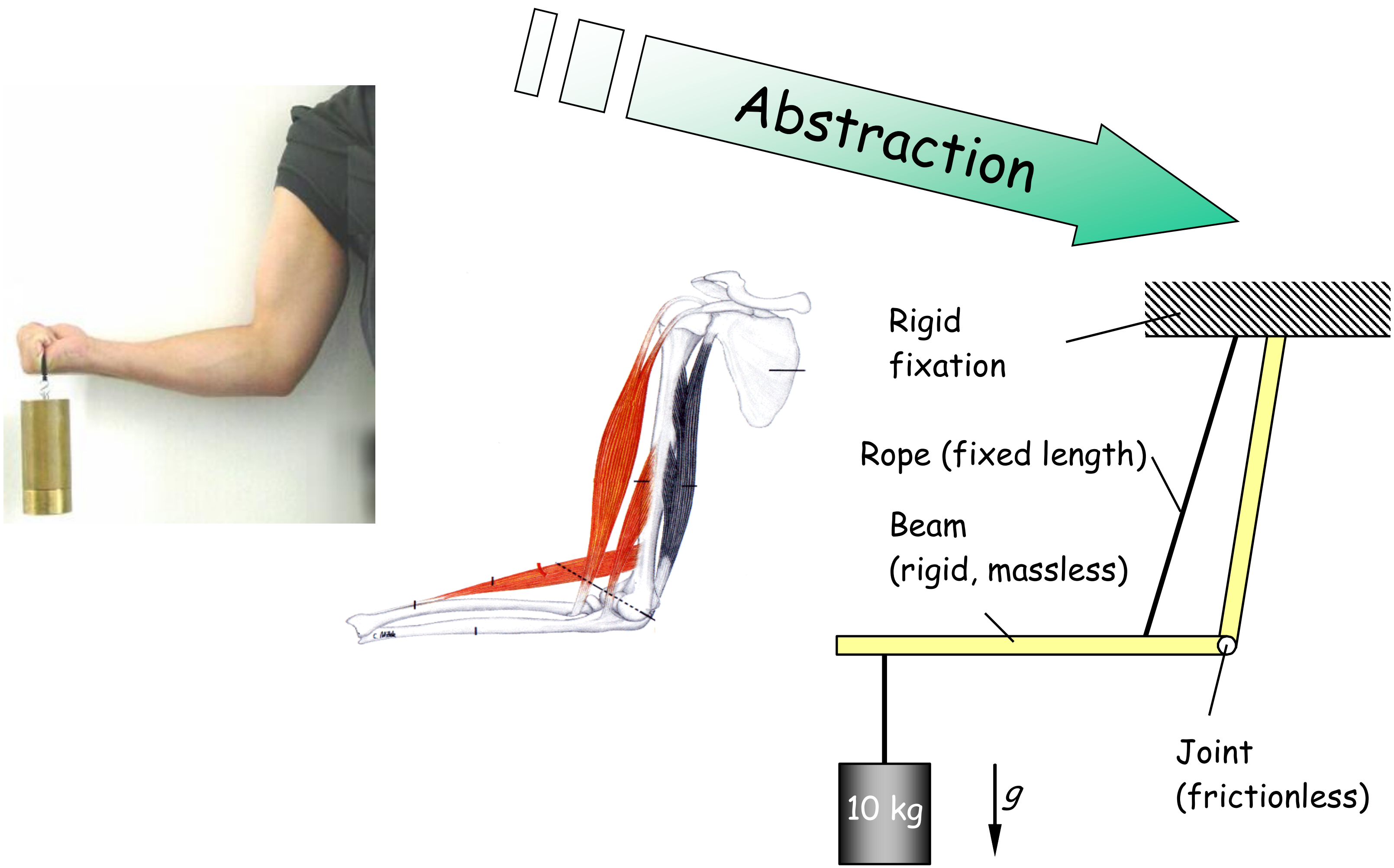
Step 5: Display results, explain, confirm with experimental comparisons. Are the results reasonable?

2.9 Classical Example: „Biceps Force"

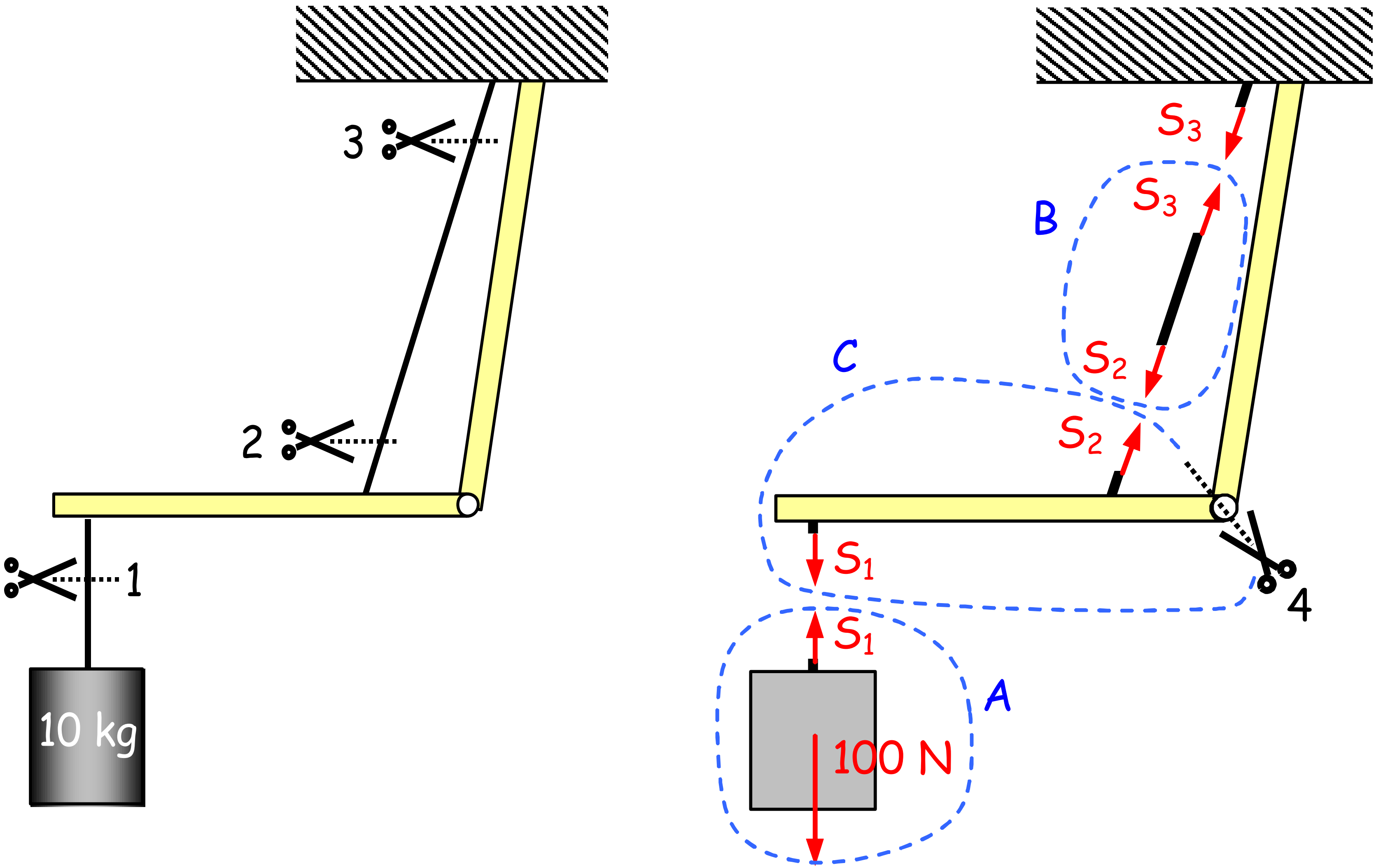


From:
„De Motu Animalium"
G.A. BORELLI
(1608-1679)

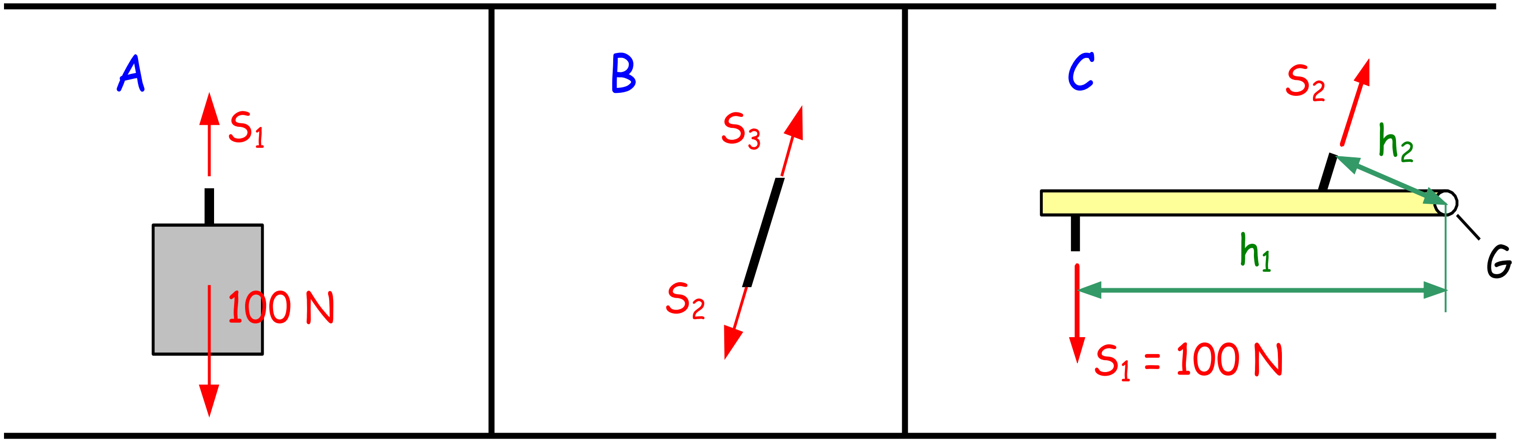
Step 1: Model building



Schritt 2: Schneiden und Freikörperbilder



More to Step 2: Cutting and Free-Body Diagrams



Step 3 and 4: Equilibrium and Solving the Equations

Sum of all forces in vertical direction = 0	Sum of all forces in "rope" direction = 0	Sum of all moments with respect to Point G = 0
$100\ \mathrm{N} + (-S_1) \overset{!}{=} 0$ $\Rightarrow \underline{\underline{S_1 = 100\ \mathrm{N}}}$	$S_2 + (-S_3) \overset{!}{=} 0$ $\Rightarrow S_3 = S_2$	$-S_1 \cdot h_1 + S_2 \cdot h_2 \overset{!}{=} 0$ $-100\ \mathrm{N} \cdot 35\ \mathrm{cm} + S_2 \cdot 5\ \mathrm{cm} \overset{!}{=} 0$ $\Rightarrow S_2 = 100\ \mathrm{N} \cdot \frac{35\ \mathrm{cm}}{5\ \mathrm{cm}} = \underline{\underline{700\ \mathrm{N}}}$

ELASTOSTATICS

3.1 Stresses

... to account for the loading of the material !

Fotos: Lutz Dürselen

Note to Remember:

Stress = „smeared" force

Stress = Force per Area or $\sigma = F/A$

(Analogy: „Nutella bread teast ")

Units of Stress

Pascal: $1\,\text{Pa} = 1\,\text{N/m}^2$

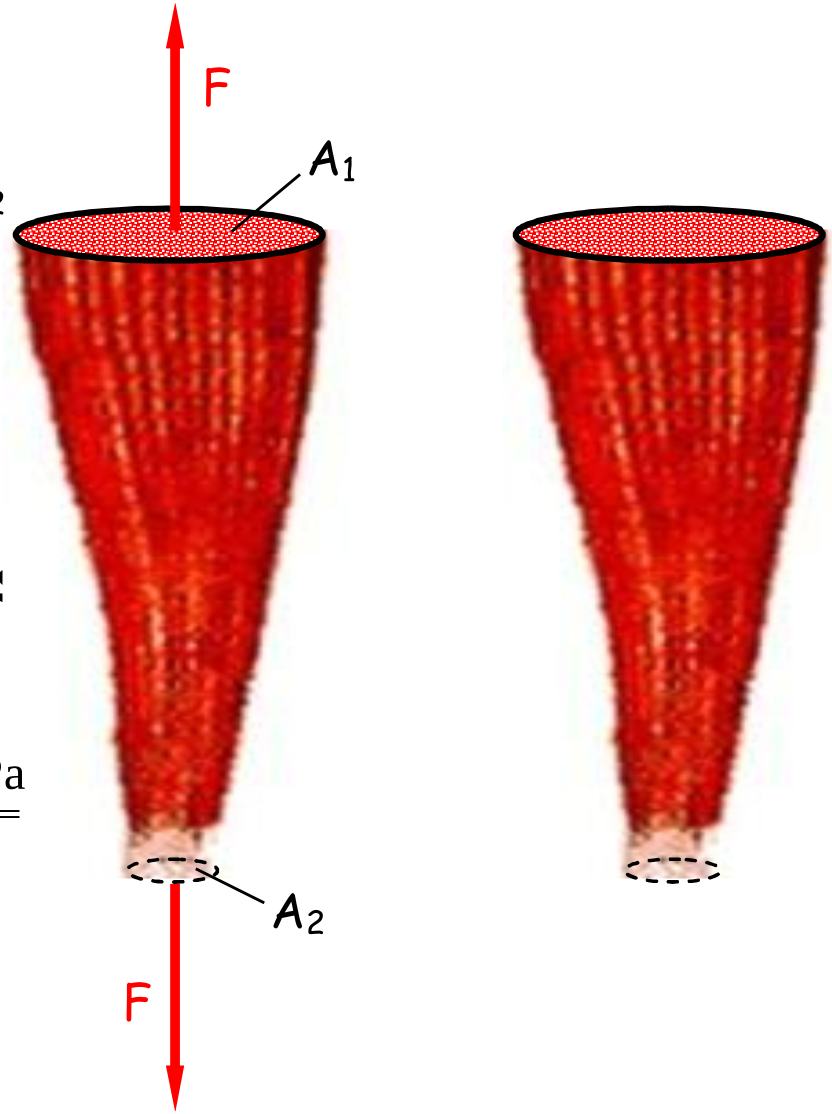
Mega-Pascal: $1\,\text{MPa} = 1\,\text{N/mm}^2$

3.2 Example:

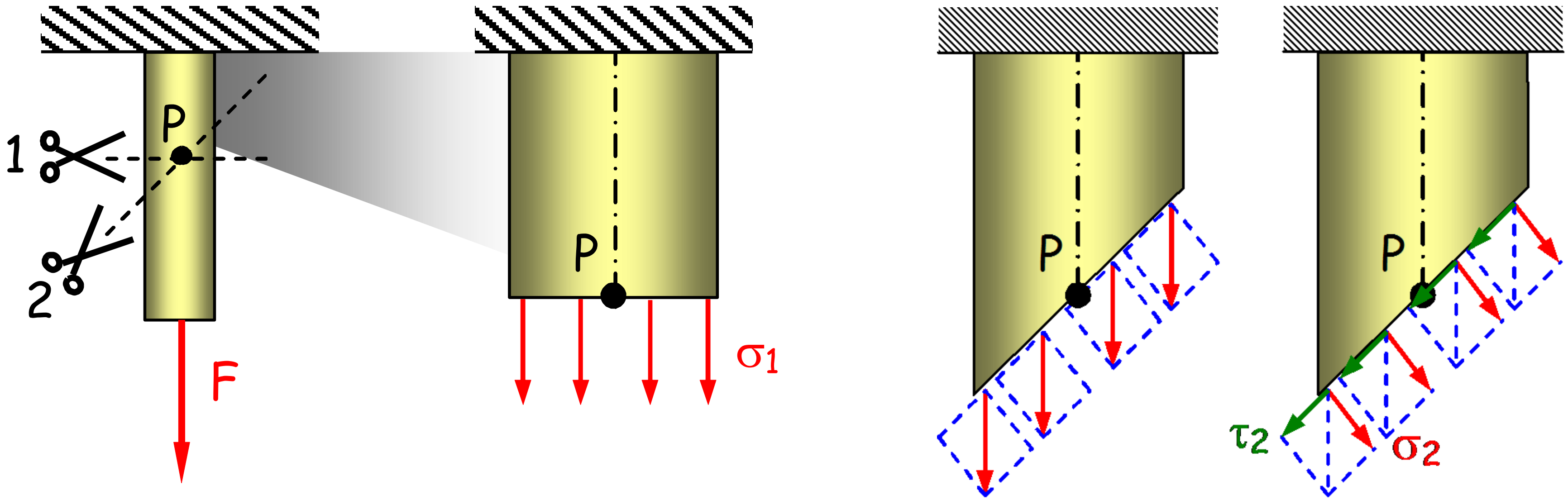
"Tensile stress in Muscle:

$$\sigma_1 = \frac{F}{A_1} = \frac{700\,\text{N}}{7000\,\text{mm}^2} = 0{,}1\frac{\text{N}}{\text{mm}^2} = \underline{\underline{0{,}1\,\text{MPa}}}$$

$$\sigma_2 = \frac{F}{A_2} = \frac{700\,\text{N}}{70\,\text{mm}^2} = 10\frac{\text{N}}{\text{mm}^2} = \underline{\underline{10\,\text{MPa}}}$$



3.3 Normal and Shear Stresses



Tensile bar

Cut 1:
Normal stress σ_1

Cut 2:
Normal stress σ_2
Shear stress τ_2

Note to Remember:

First, you must choose a point and a **cut** through the point, **then** you can specify (type of) **stresses** at this point in the body.

Normal stresses (tensile and compressive stress) are oriented perpendicular to the cut-surface.

Shear stresses lie tangential to the cut-surface.

General (3D) Stress State: Stress Tensor

... in one point of the body: **How much numbers do we need?**

- 3 stress components in one cut (normal str., 2x shear str.)

  times

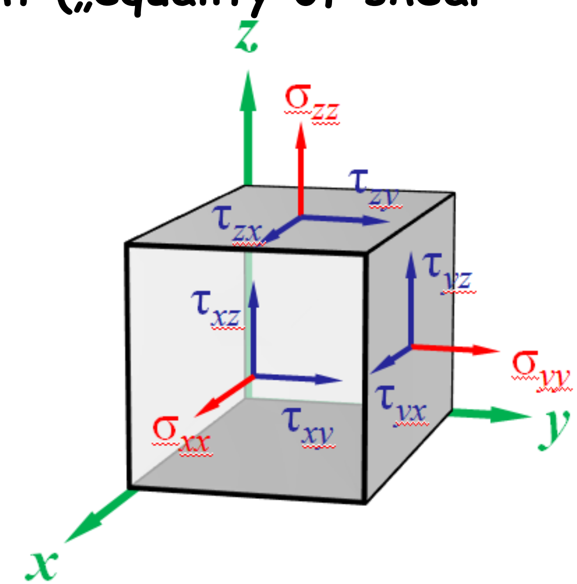
- 3 cuts (e.g. frontal, sagittal, transversal)

  results in

- 9 stress components for the full stress state in the point.
- But only 6 components are linear independent („equality of shear stresses“)

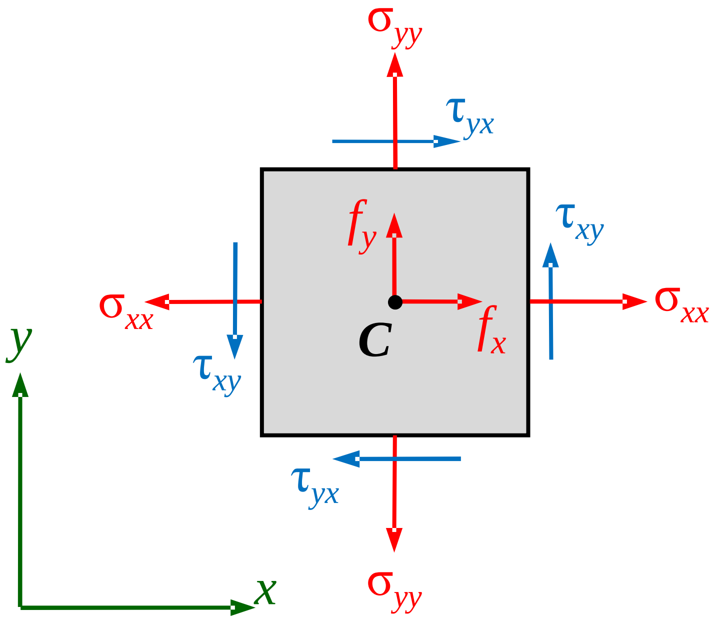
„Stress Tensor“

$$\underline{\underline{\sigma}} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$



Symmetry of the Stress Tensor

Boltzmann Continua: Only volume forces (f_x und f_y), no volume moments assumed
→ „Equality of corresponding shear stresses"

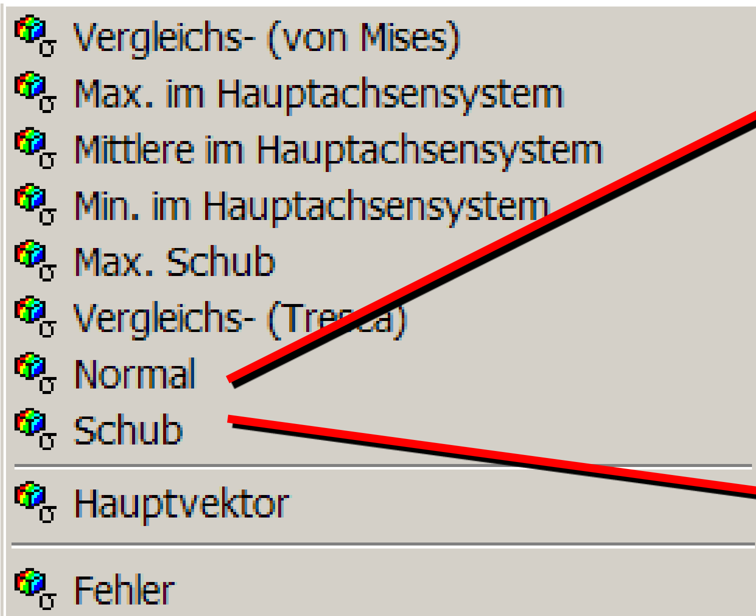


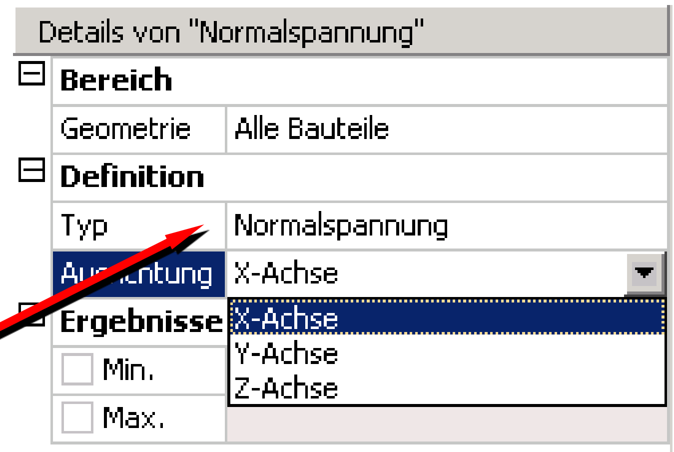
$$\underline{\underline{\sigma}} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ . & \sigma_{yy} & \tau_{yz} \\ \mathrm{sym} & . & \sigma_{zz} \end{bmatrix}$$

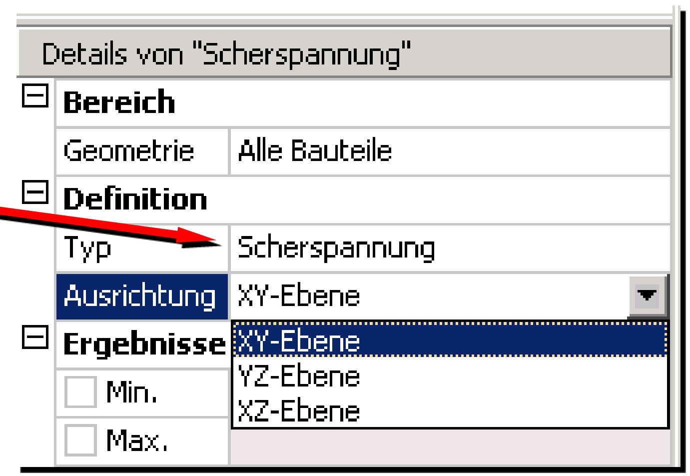
$$\sum M^{(C)} = 2 \cdot \underbrace{\tau_{xy} \Delta y \Delta z}_{\text{Kraft}} \cdot \underbrace{\tfrac{1}{2} \Delta x}_{\text{Hebelarm}} - 2 \cdot \underbrace{\tau_{yx} \Delta x \Delta z}_{\text{Kraft}} \cdot \underbrace{\tfrac{1}{2} \Delta y}_{\text{Hebelarm}} = 0.$$

General 3D Stress State

6 Components → 6 Pictures

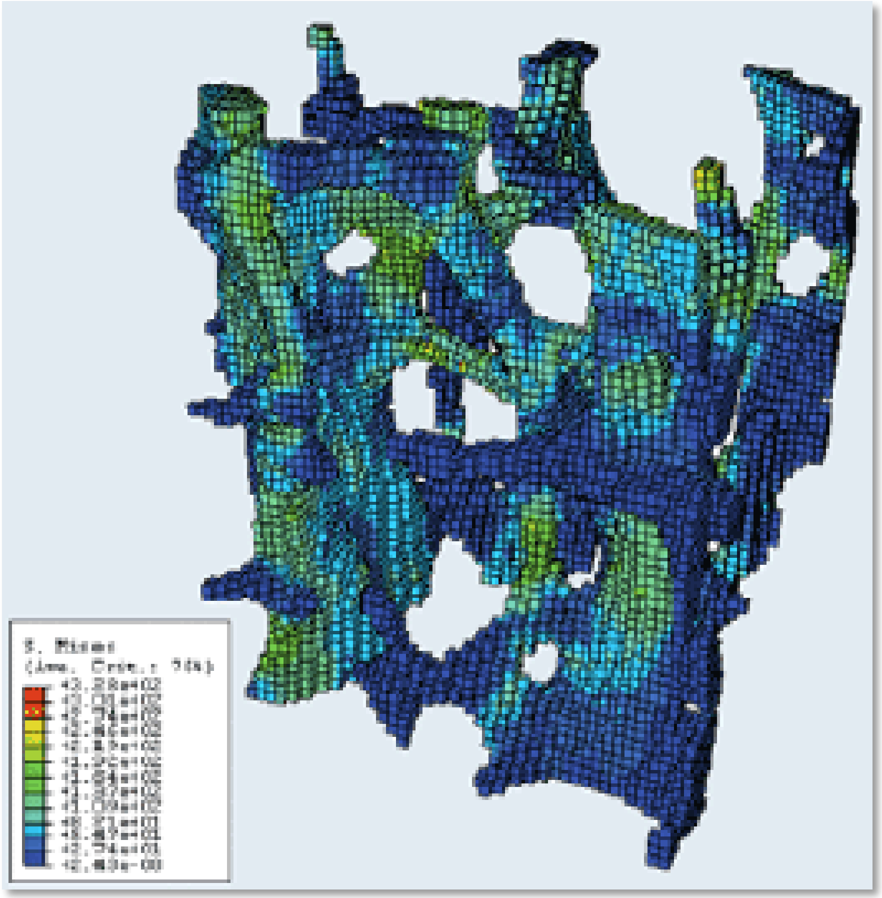






Problem:

- How to produce nice Pictures?
- Which component should I use?
- Do I need 6 pictures at the same time?



So called „Invariants“ are „smart mixtures“ of the components

- Vergleichs- (von Mises)
- Max. im Hauptachsensystem
- Mittlere im Hauptachsensystem
- Min. im Hauptachsensystem
- Max. Schub
- Vergleichs- (Tresca)
- Normal
- Schub

$$\sigma_{Mises} = \sqrt{\sigma_{xx}{}^2 + \sigma_{yy}{}^2 + \sigma_{zz}{}^2 - \sigma_{xx}\,\sigma_{yy} - \sigma_{xx}\,\sigma_{zz} - \sigma_{yy}\,\sigma_{zz} + 3\,\tau_{xy}{}^2 + 3\,\tau_{xz}{}^2 + 3\,\tau_{yz}{}^2}$$

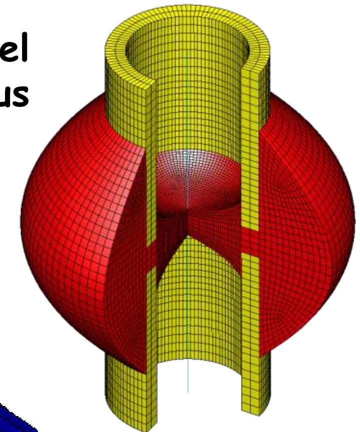
3.4 Strains

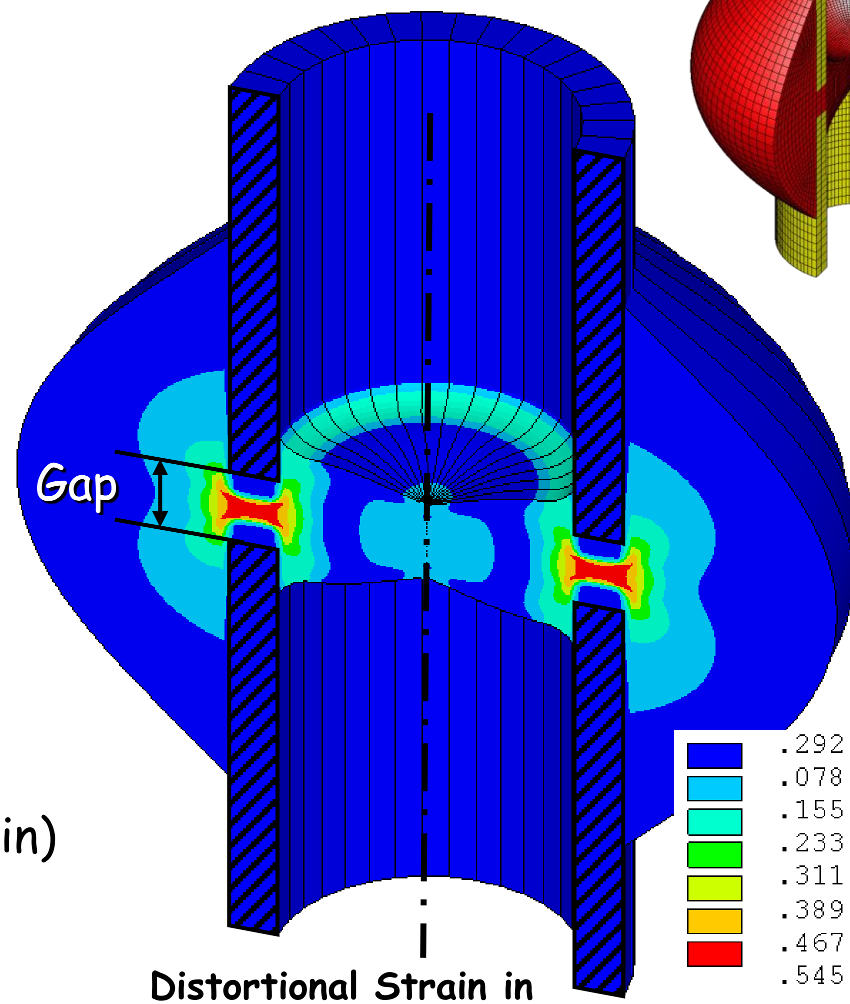
- Global, (external) strains

$$\varepsilon := \frac{\text{Change in length}}{\text{Original length}} = \frac{\Delta L}{L_0}$$

- Local, (internal) strains

Finite element model
of the fracture callus





Distortional Strain in
a fracture callus

Units of Strain

without a unit

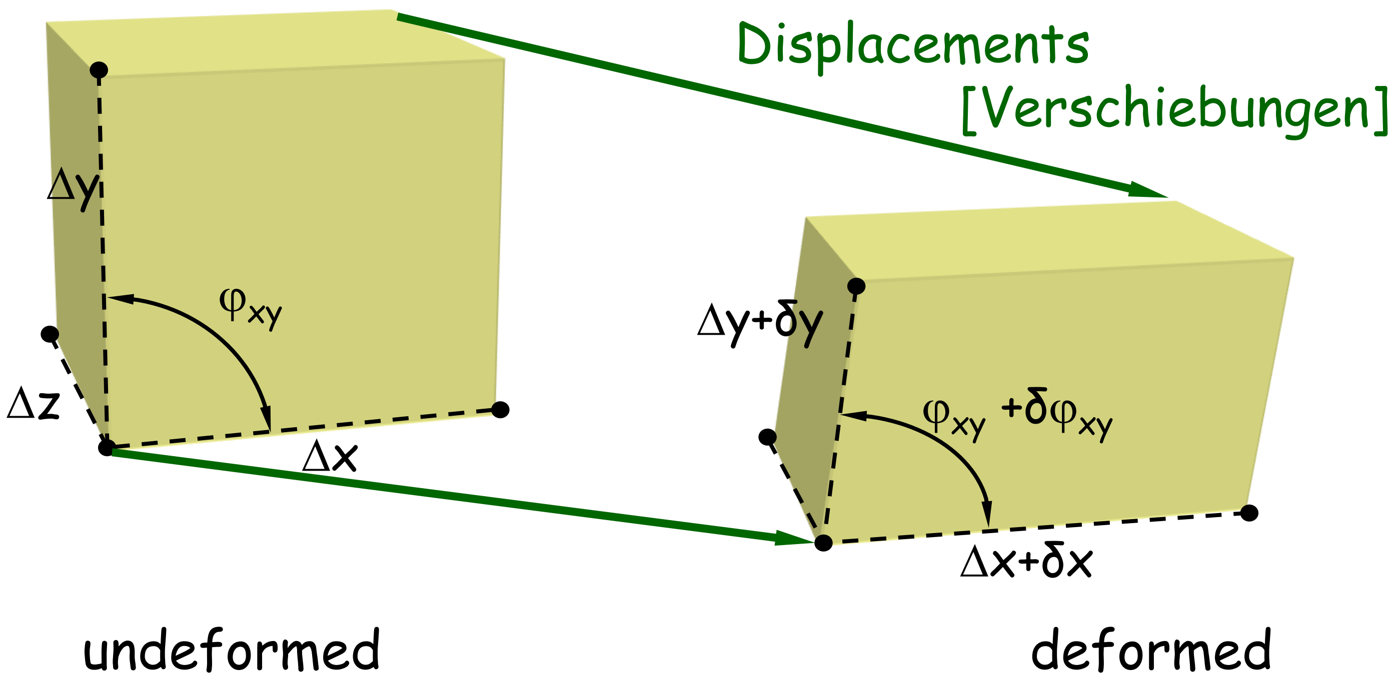
1

1/100 = %

1/1.000.000 = με (micro strain)

= 0,1 %

3D Local Strain State: Strain Tensor



3D Local Strain State: Strain Tensor

Definition:

$$\varepsilon_{xx} = \lim_{x_0 \to 0} \frac{\Delta x}{x_0}, \quad \varepsilon_{yy} = \lim_{y_0 \to 0} \frac{\Delta y}{y_0}, \quad \varepsilon_{zz} = \lim_{z_0 \to 0} \frac{\Delta z}{z_0}$$

$$\varepsilon_{xy} = \frac{1}{2} \cdot \Delta\gamma, \quad \varepsilon_{xz} = \frac{1}{2} \cdot \Delta\beta, \quad \varepsilon_{yz} = \frac{1}{2} \cdot \Delta\alpha$$

Universal Strain Definition:

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \quad i, j = \{x, y, z\}$$

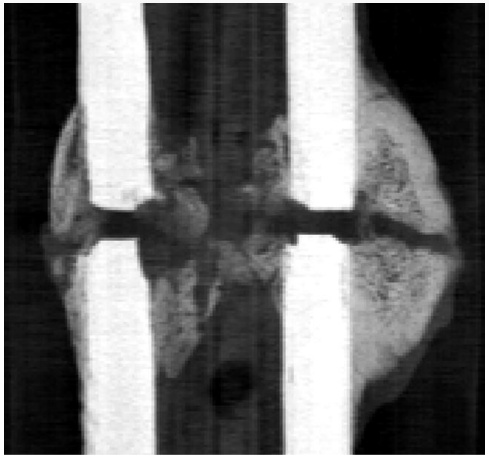
$$\underline{\underline{\varepsilon}} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{zz} \end{bmatrix}$$

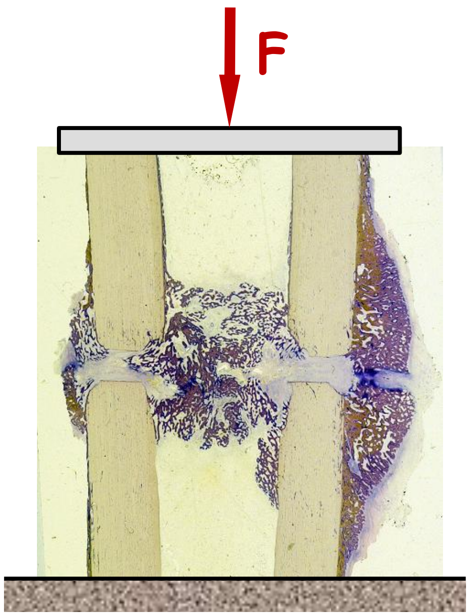
Strain Tensor "

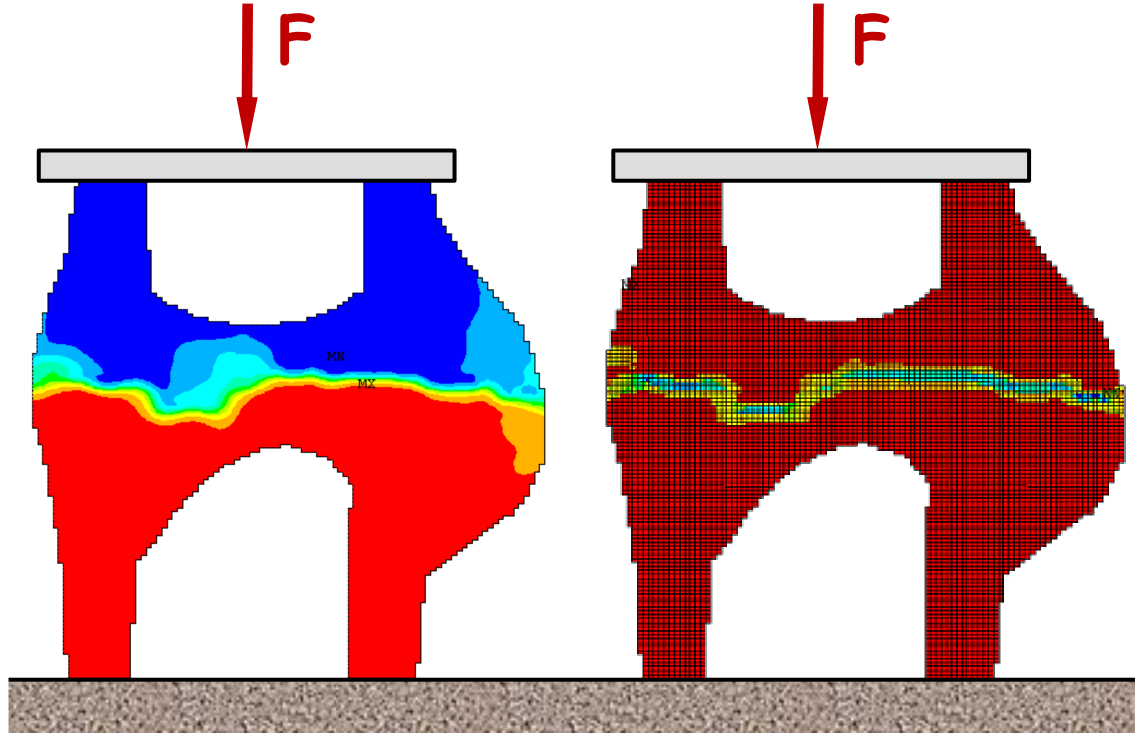
Note to Remember:

Strain is relative change in length (and shape)

Displacement vs. Strain







Displacement u_x

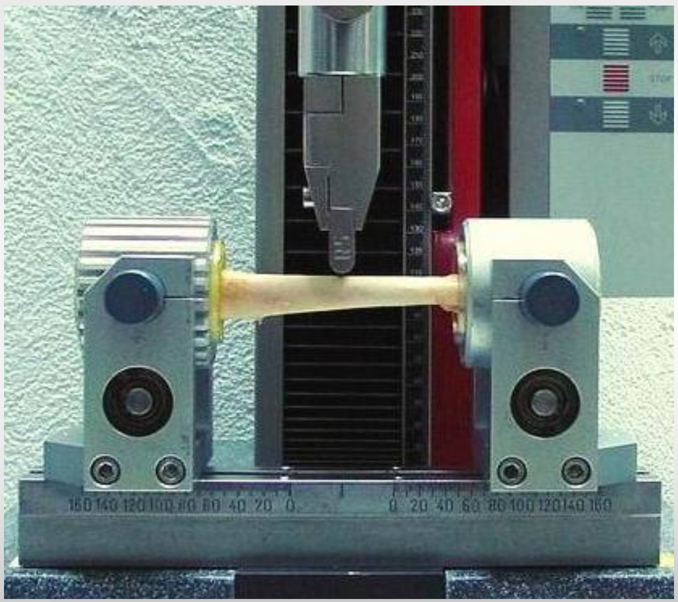
Strain, eps_xx

Material Laws
for Biological Tissues

Characterizing Mechanical Properties

Structural Properties

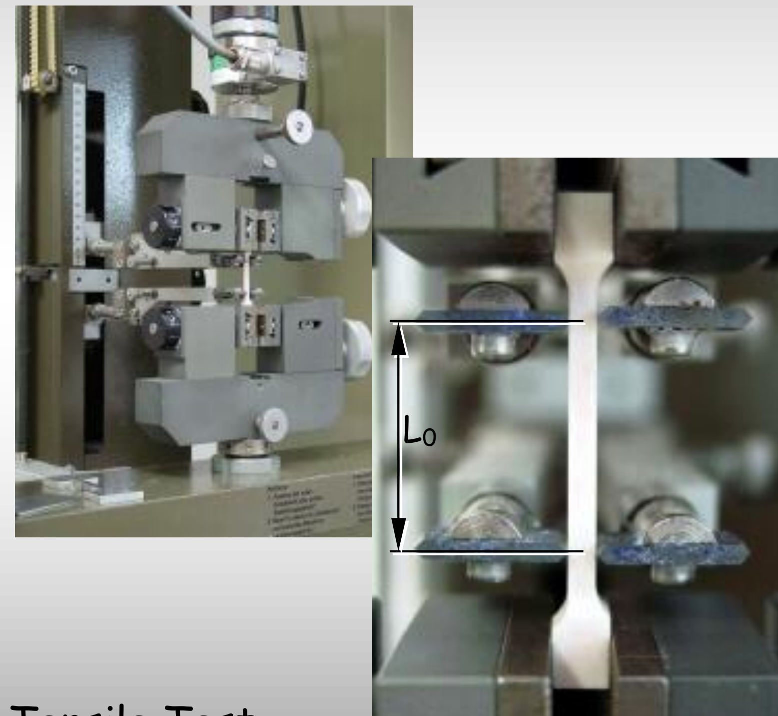
(Organ Properties)



3-Point Bending
of a
bone specimen

Material Properties

(Tissue Properties)

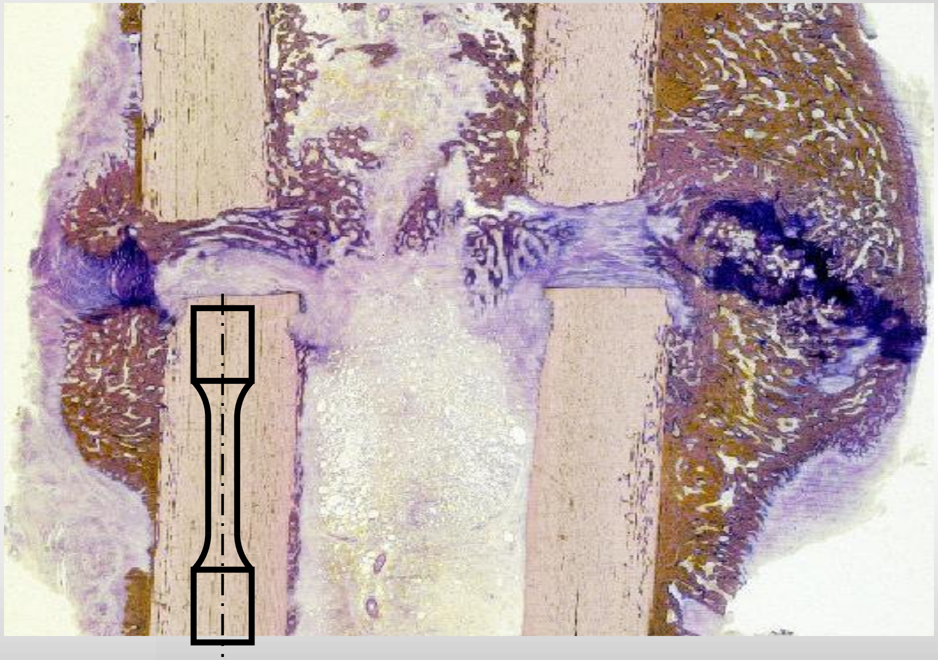


Tensile Test
of a
standardized Specimen

Mechanical Properties

	Organ Properties	Tissue Properties
Elastic	Different stiffnesses: load/deformation	Young's Modulus Poisson's Ratio Shear Modulus Compressive Modulus
Ultimate	Ultimate forces, moments, displacements: tension, compression, bending, torsion	Ultimate stresses, strains: tensile, compressive, shear at fracture at end of elastic region, at start of yielding

Example: Fracture Callus



Haematoma Granulation tissue Fibrous tissue	0,01 ... 3 MPa
Cartilage	0,4 … 500 MPa
Woven bone	400 … 6000 MPa
Cortical bone	10 ... 20 GPa

Yamada 1970; Hori & Lewis 1982; Davy & Connolly 1982; Proctor et al. 1989; Mente & Lewis 1994; Rho et al. 1995; Augat et al. 1998; Tanck et al. 1999

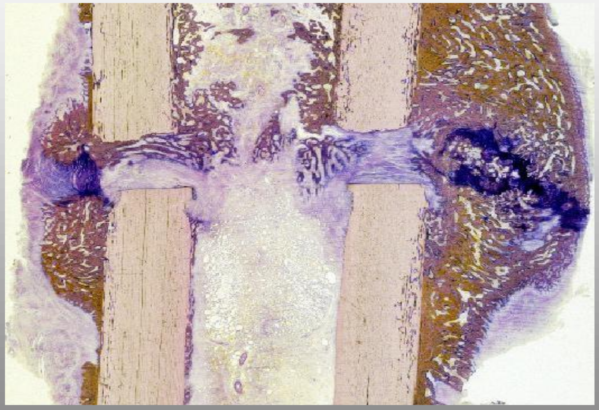
Overview

Simplest Material Law:

- Linear-elastic, isotropic (Hook's Law)

More complex Laws:

- Nonlinear
- Hyperelastic (nonlinear elastic + large deformations)
- Viscoelastic
- Non-elastic, plastic, yielding, hardening
- Anisotropic
- Fatigue, recovery
- Remodeling, healing



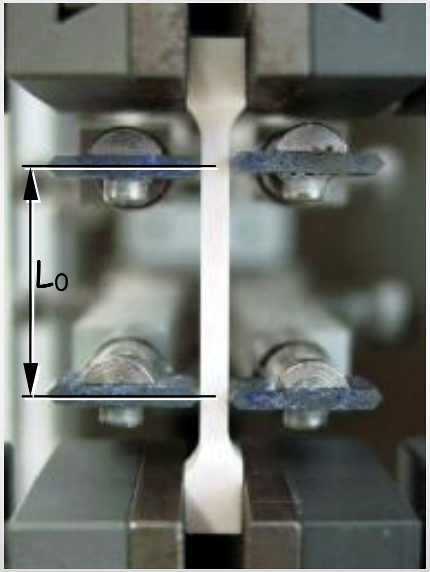
Overview

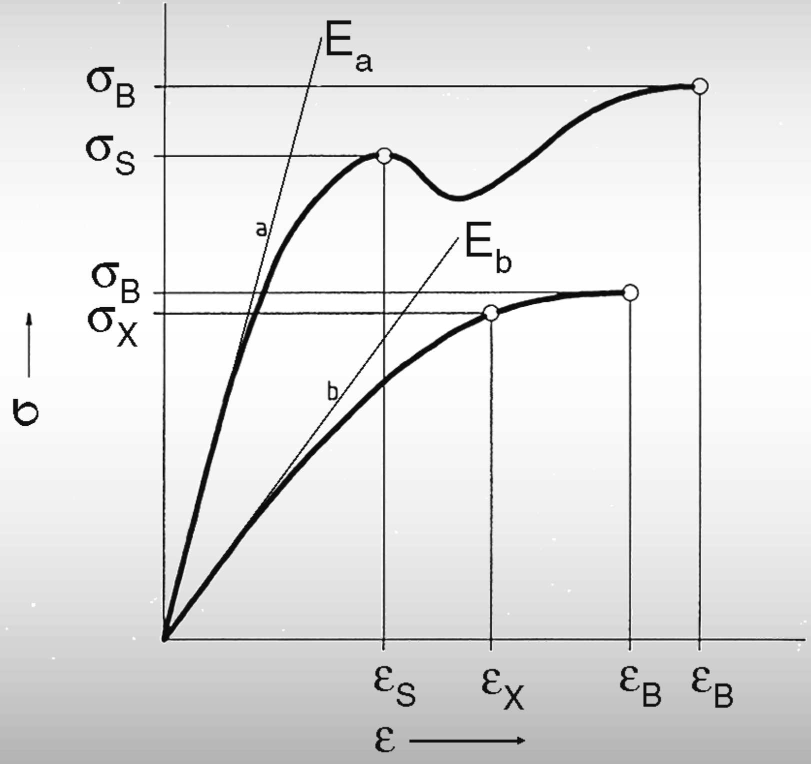
In principal:

Biological Tissues know all of these bad things.

They often combine the bad things

Stress-strain curve





σ_B	*Ultimate stress*
σ_S	*Yield point*
ε_B	*Ultimate strain*
ε_S	*Strain at yoild point*
E	Young's modulus

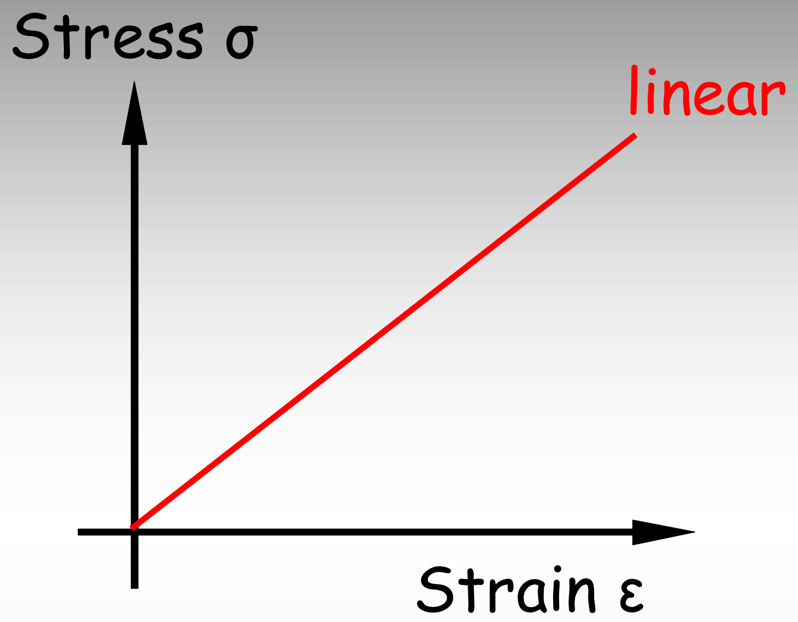
Linear Elastic Law

Linear stress-strain relation

$$\sigma = E \cdot \varepsilon$$

$$\underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{E}}}} \cdot \underline{\underline{\varepsilon}}$$

$$\sigma_{ij} = E_{ijkl} \cdot \varepsilon_{kl} \qquad (81)$$



- Equality of shear stresses (Boltzmann Continua) and strains (36)
- Reciprocity Theorem from Maxwell* → fully anisotropic (21)
- Orthotropic (9)
- Transverse Isotropic (5)
- Isotropy (2)

*) also known as Betti's theorem or Maxwell-Betti reciprocal work theorem

Linear Elastic Law

$$\underline{\sigma} = \underline{\underline{E}} \cdot \underline{\varepsilon}$$

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \frac{E}{(1+\nu)\cdot(1-2\nu)} \cdot \begin{bmatrix} (1-\nu) & \nu & \nu & 0 & 0 & 0 \\ & (1-\nu) & \nu & 0 & 0 & 0 \\ & & (1-\nu) & 0 & 0 & 0 \\ & & & \frac{(1-2\nu)}{2} & 0 & 0 \\ & & & & \frac{(1-2\nu)}{2} & 0 \\ \mathrm{sym} & & & & & \frac{(1-2\nu)}{2} \end{bmatrix} \cdot \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix}$$

E - Young's modulus

ν - Poisson's ratio (0 ... 0.5)

G - Shear modulus

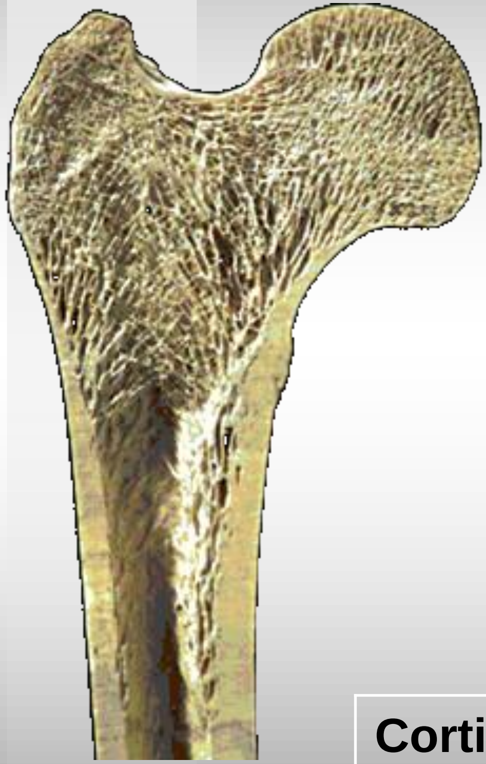
K - Bulk modulus

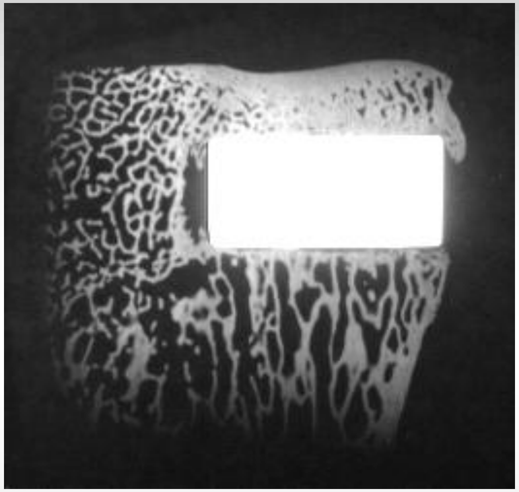
μ, λ - Lame Constants

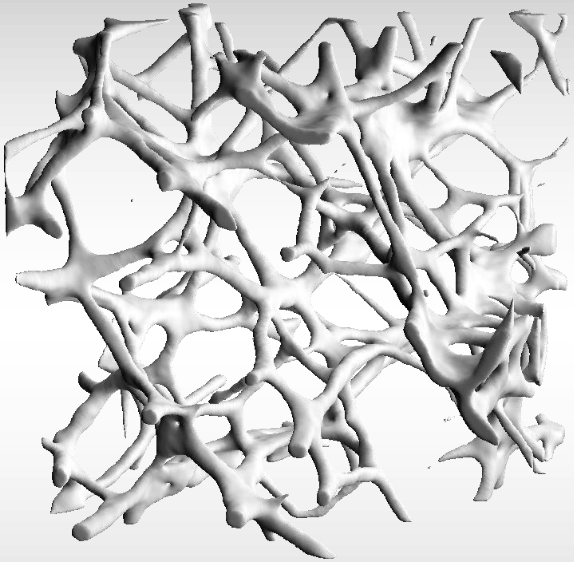
← 2 of these

Anisotropic Properties

→ Bone tissue







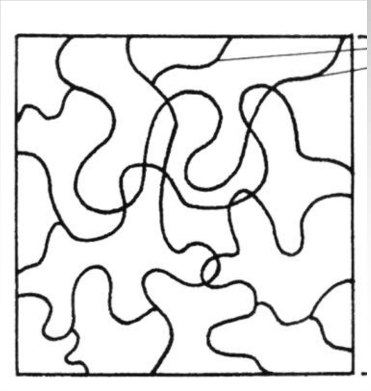
Corticale bone	Transverse isotropic (5)	Like Wood
Trabecular bone	Orthotropic (9)	3D Grid

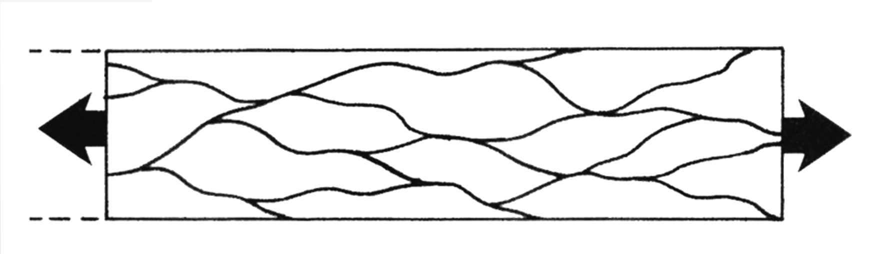
Anisotropic Properties

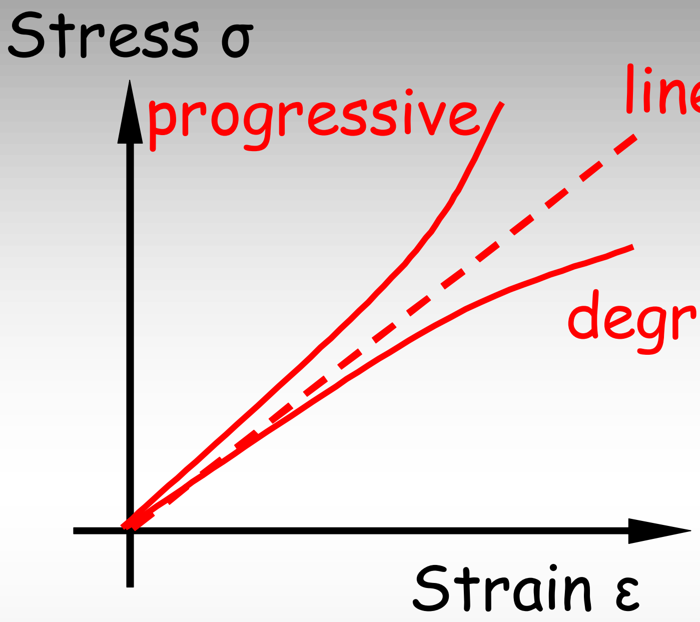
Material	E Moduli in MPa	Strength in MPa	Fracture strain in %
Spongy bone			
Vertebra	60 (male) 35 (female)	4,6 (male) 2,7 (female)	6
prox. Femur	240	2.7	2.8
Tibia	450	5...10	2
Bovine	200...2000	10	1,7...3,8
Ovine	400...1500	15	
Cortical bone			
Longitudinal	17000	200	2,5
Transversal	11500	130	

Nonlinear Elastic and Hyperelastic Law

→ soft, fibrous tissues







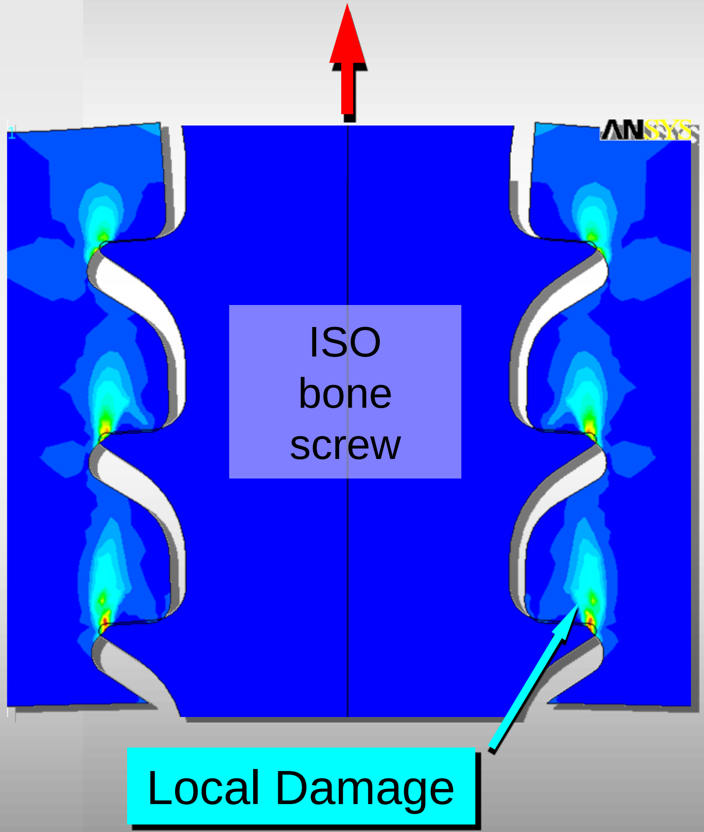
Hyperelastic Laws: Nonlinear + large deformations
SED-stress relation, (Mooney-Rivlin, Neo-Hookean, ...)

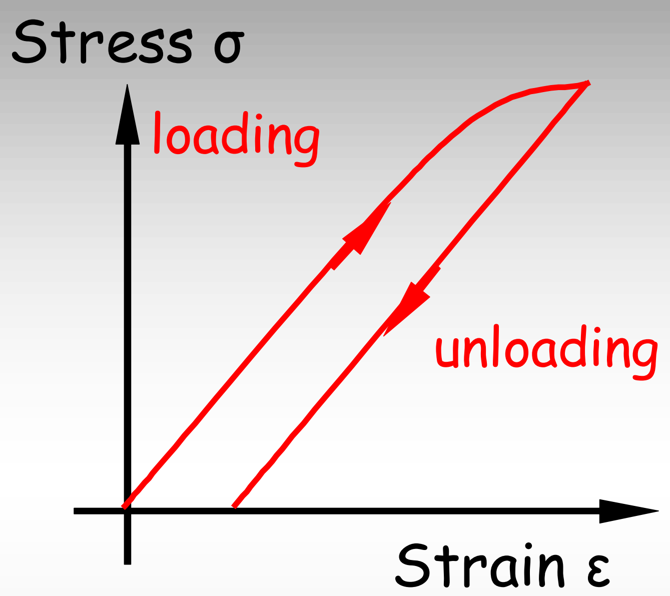
Properties for Ligaments

Properties	Value
Tensile strength	20 – 50 MPa
Shear strenght	1 kPa (fast loading 0.04mm/h) 740 kPa (slow loading 420mm/h)
Rupture strain	2 m/m (Shear) 0,1 – 0,5 m/m (Tension)
Rupture force	1500 N (Finger) 100 - 400 N (Ankle) 1200 N (ACL)
Stiffness	100 - 700 N/mm (ACL)
E Modulus, tension	100 – 1000 MPa

Non-elastic = plastic

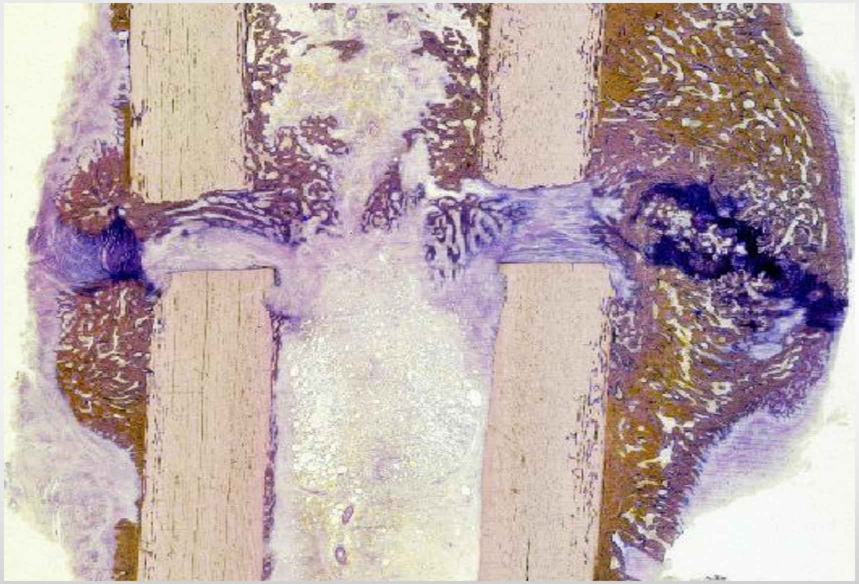
→ Bone strength

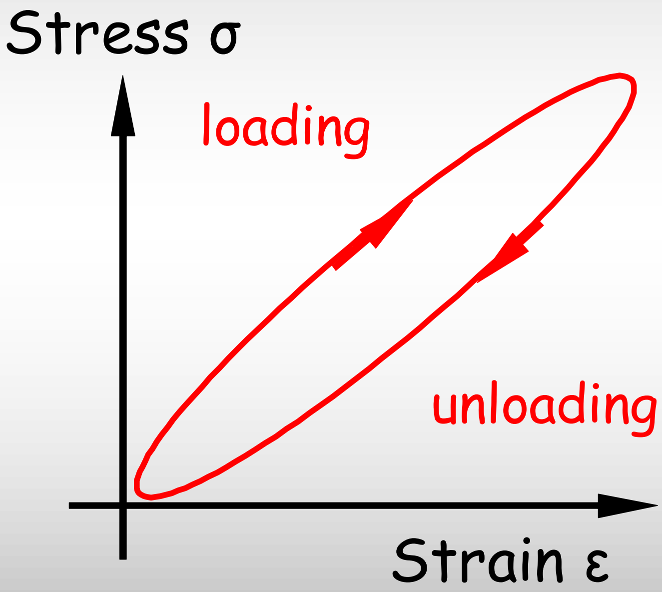




Viscoelastic Laws (Time Dependent)

→ Cartilage, Fibrous Tissue, Bone



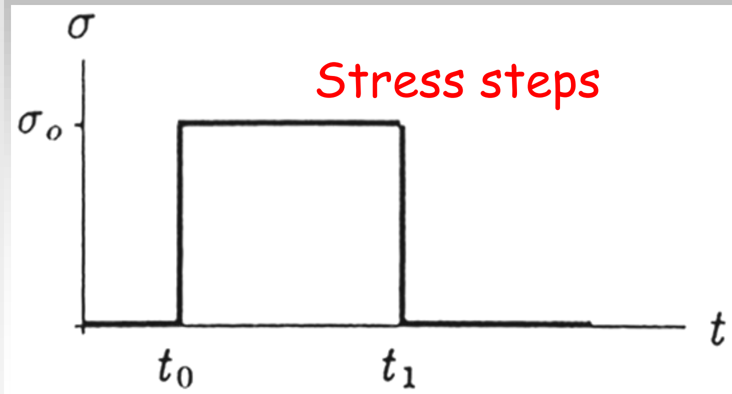


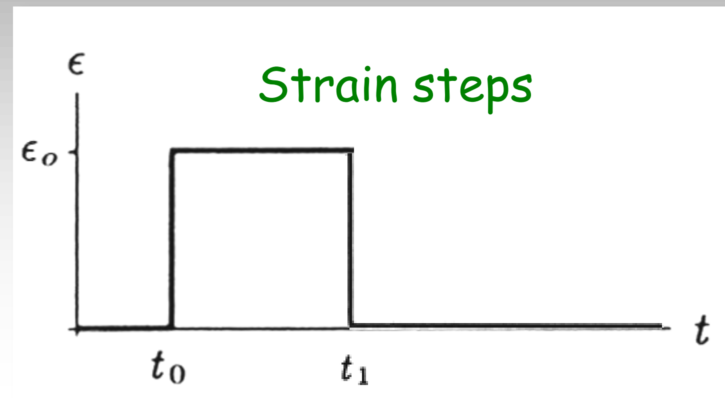
Effects of Viscoelastic Materials

Retardation (Creep)

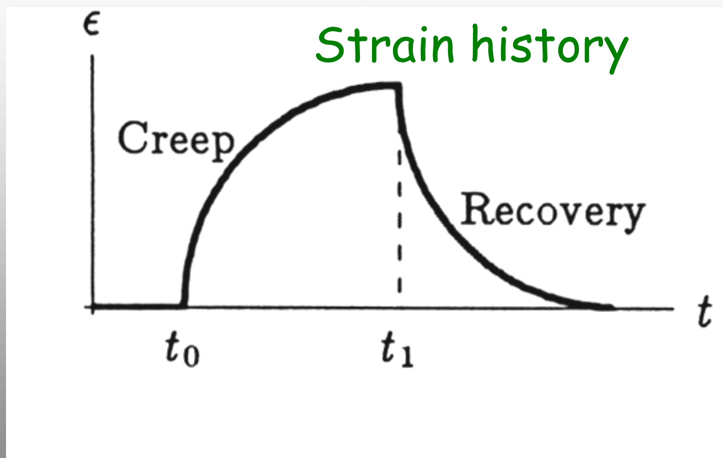
Relaxation

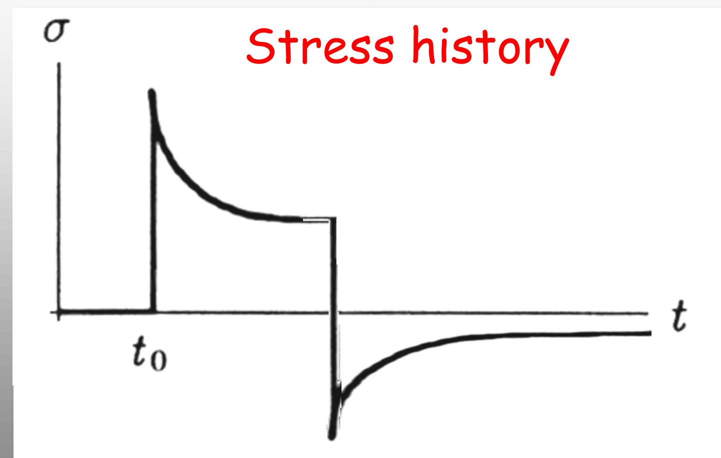
Applied



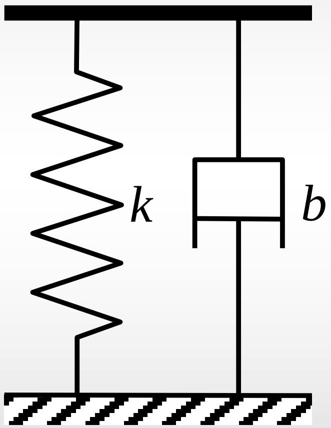


Measured

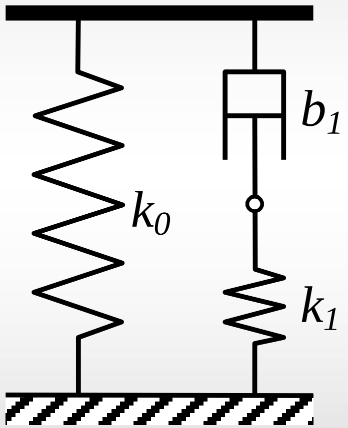




Viscoelastic Laws (Time Dependent)



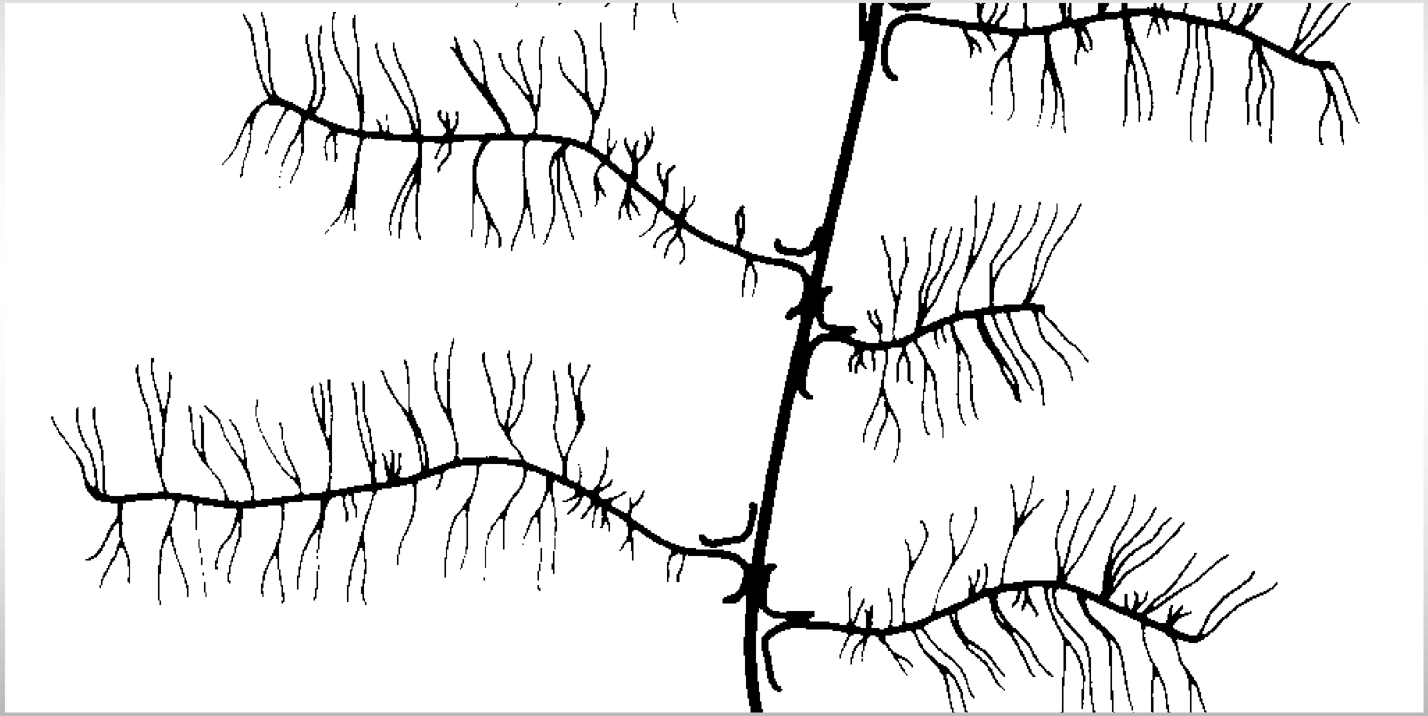
Typ:
Inner damping



Typ:
Memory effect

Poroelastic

→ Solid structure with pores: Cartilage, Fibrous Tissue, Bone



Hyaloron

Properties for Articular Cartilage

Properties	Values
Density	1300 kg/m^3
Water content	75%
Compressive strength	5 MPa
Tensile strenght	20 MPa
E Modulus, eq.	0,8 MPa
E Modulus, initial	3 MPa
Shear Modulus	3,5 MPa
Friction	0,005
Rupture strain	6...8%
Permeability	0,8 10^{-14} m^4/Ns

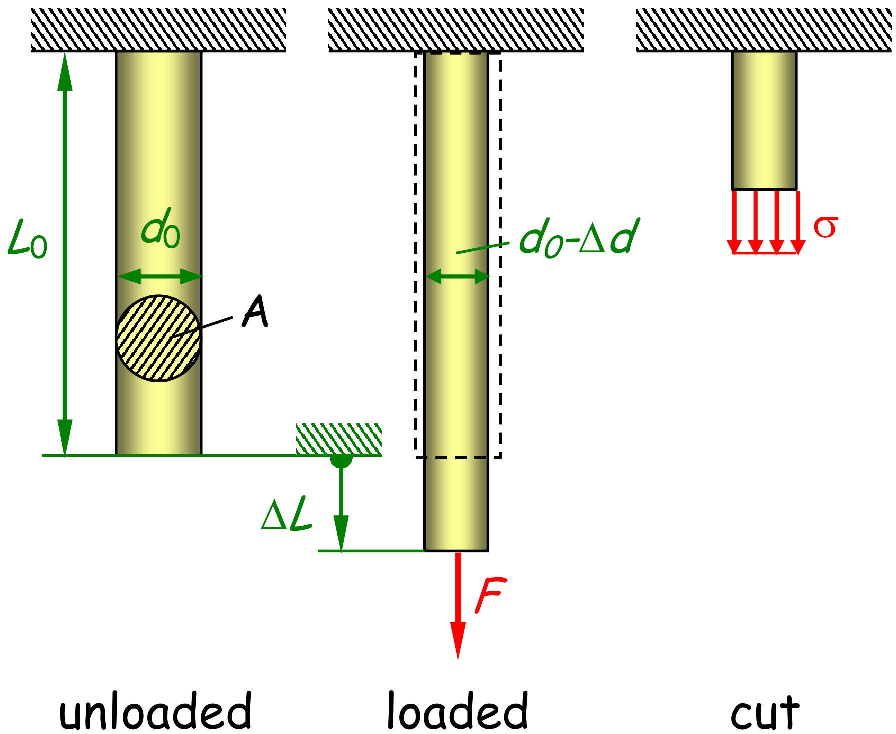
When is simplification allowed?

- Non linear: Small strains → „linear“
- Plastic: Approximation: „Ideal elastic plastic“
- Viscoelastic: Quasi-static Deformations → „elastic“
- Anisotropic: → „Worst-case scenario“

4 Simple Load Cases

1 Tension, Compression

Tensile bar (tensional rigidity *EA*)



Device stiffness

$$F = \frac{EA}{L_0}\Delta L, \quad k = \frac{EA}{L_0}$$

Axial Stress

$$\sigma = \frac{F}{A}$$

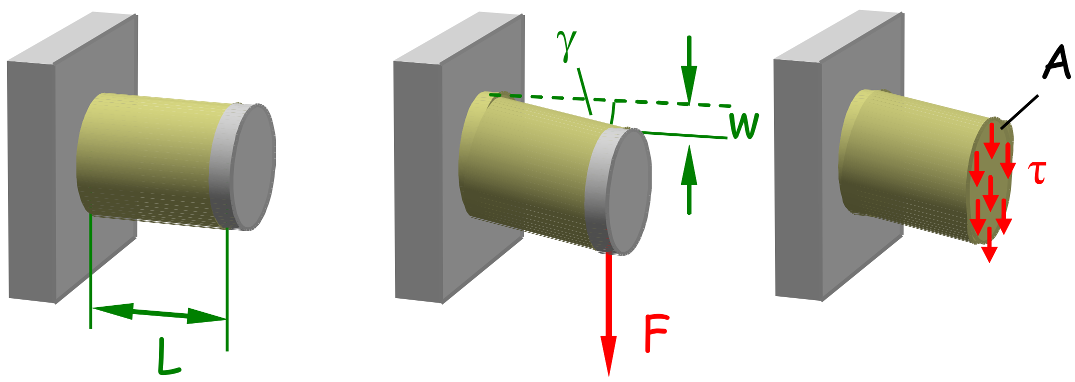
Strain

$$\varepsilon = \frac{\Delta L}{L_0}$$

$$\varepsilon_q = \frac{\Delta d}{d_0} = -\nu\varepsilon$$

2 Shear

Shear bolt (shear rigidity *GA*)



Device stiffness

$$F = \frac{GA}{L} w, \quad k = \frac{GA}{L}$$

Shear stress

$$\tau = \frac{F}{A}$$

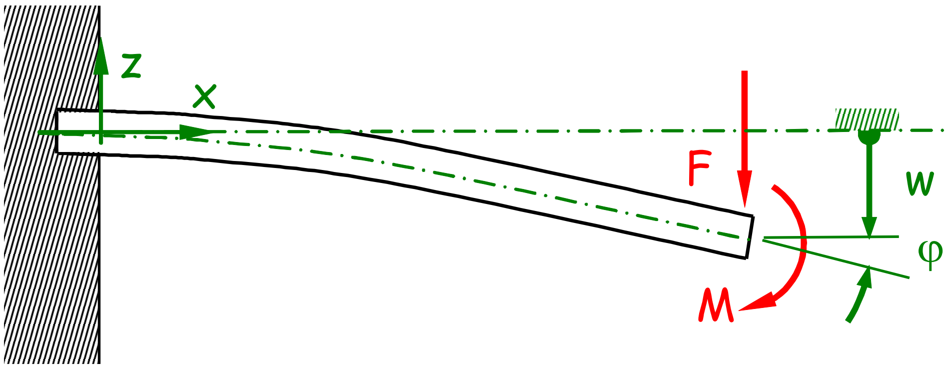
Shear strain

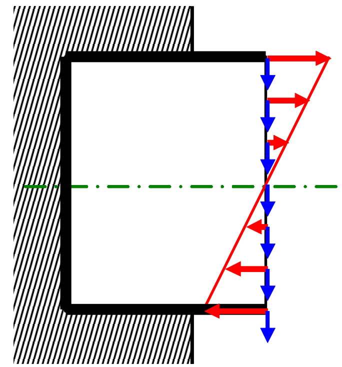
$$\varepsilon_{xy} = \frac{1}{2}\gamma$$

3 Bending (Cantilever Beam)

Beam (bending rigidity EI_a, Length L)

Cut





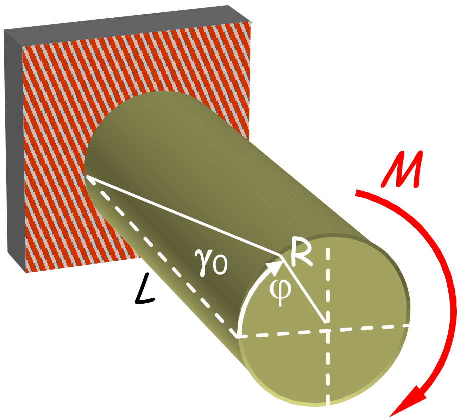
Cantilever formula:

$$\boxed{\begin{aligned} w &= \frac{L^3}{EI_a}F + \frac{L^2}{EI_a}M, \\ \varphi &= \frac{L^2}{EI_a}F + \frac{L}{EI_a}M. \end{aligned}}$$

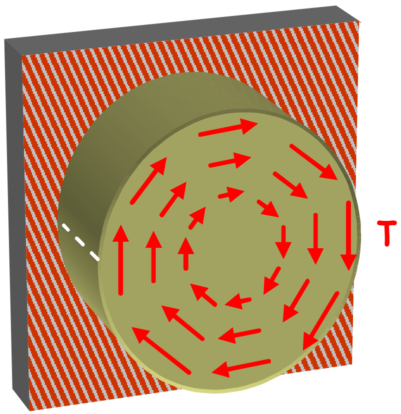
3 Torsion

Torsional rod (torsional rigidity GI_T)

$$M = \frac{GI_T}{L}\varphi, \quad c = \frac{GI_T}{L}$$



Length L, radius r

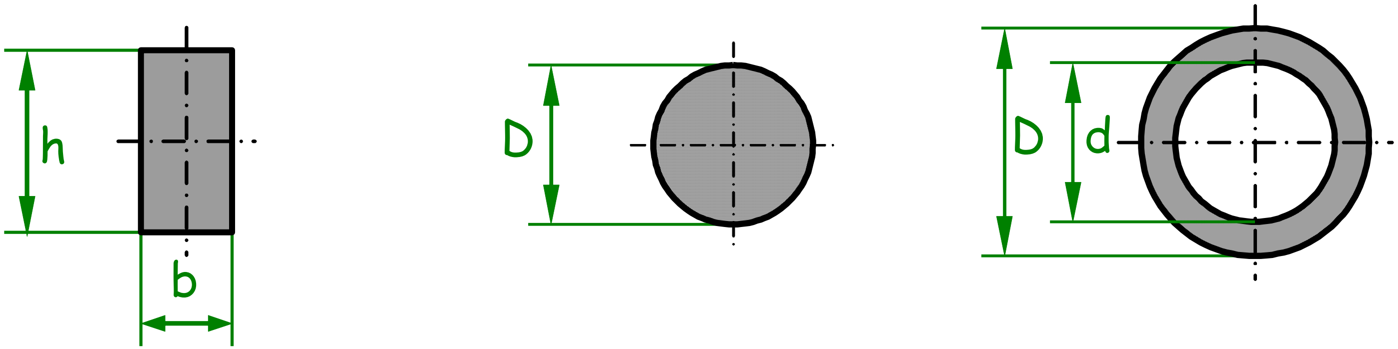


Cut

Note to Remember:

A hollow bone reaches a high stiffness and strength against bending and torsion with relatively little material.

Second Moment of Area I (SMA)
[Flächenmoment zweiten Grades]



Axial moment of area (bending)

$$I_a = \frac{b \cdot h^3}{12} \qquad I_a = \frac{\pi}{64} D^4 \qquad I_a = \frac{\pi}{64}(D^4 - d^4)$$

Polar (torsional) moment of area (torsion)

$$I_T = I_P = \frac{\pi}{32} D^4 \qquad I_T = I_p = \frac{\pi}{32}(D^4 - d^4)$$