Computational Biomechanics 2018

Lecture 02:
Introduction, Basic Mechanics 2

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Contents

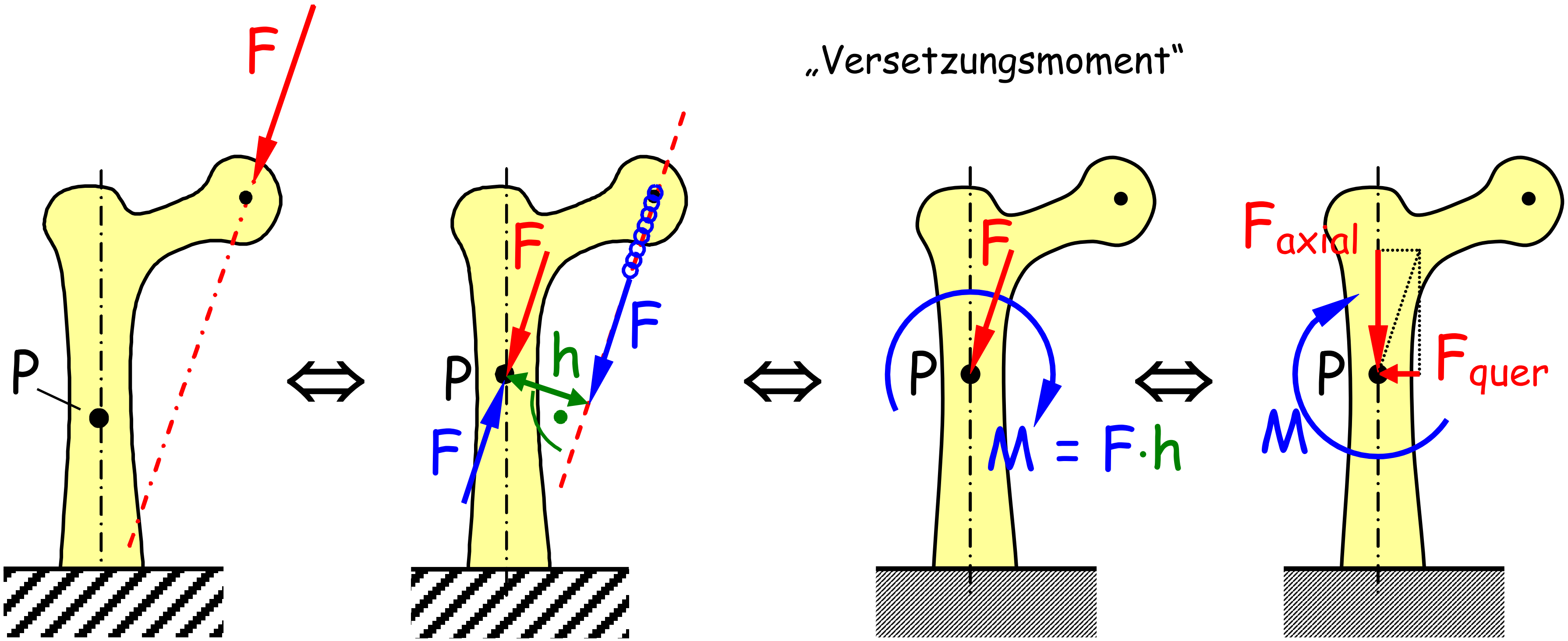
Detailed Schedule Summer 2018

No	Day	Date	Topics of Lecture and Laboratory	Lecturer
01	Mo	16 Apr	Lec: Intro to Biomechanics; Mechanical Basics 1 Lab: Intro to Ansys WB, Simple Bone Model	Ulli
02	Mo	23 Apr	Lec: Mechanical Basics 2 Lab: Loadcases, Stresses and Strains	Ulli
03	Mo	30 Apr	- Homework -	Ulli
04	Mo	07 May	Lec: Material Properties of Biol. Tissues, Intro FEA Lab: Trabecular Bone Structural Model	Ulli
05	Mo	14 May	Lec: Forward Dynamics Lab: Forward Dyn., Multi Body Model with ADAMS	Lucas
--	Mo	21 May	- Pentecost -	---
06	Mo	28 May	Lec: Inverse Dynamics, Muscuoloskelettal Modells Lab: Inverse Dyn. Model with ANYBODY	Lucas

Mechanical Basics 2

Moment of a Force about a Point [Versetzungsmoment]

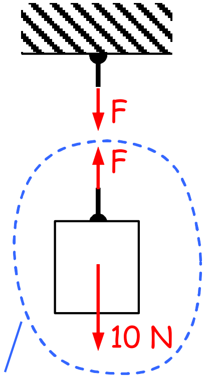
„Versetzungsmoment"



Note to Remember:

Moment = Force times lever-arm

Static Equilibrium



Free-body diagram (FBD)

Important:

Free-body diagram (FBD) first, then equilibrium!

3 equations of equilibrium for each FBD in 2D:

$$\text{Sum of all forces in x-direction: } \quad F_{1,x} + F_{2,x} + \ldots \overset{!}{=} 0,$$

$$\text{Sum of all forces in y-direction: } \quad F_{1,y} + F_{2,y} + \ldots \overset{!}{=} 0,$$

$$\text{Sum of all moments w.resp. to P: } \quad M_{1,z}^{P} + M_{2,z}^{P} + \ldots \overset{!}{=} 0.$$

- Force EEs can be substituted by moment EEs
- 3 moment reference points should not lie on one line

6 equlibrium equations for one FBD in 3D:

$$\text{Summe aller Kräfte in x-Richtung:} \quad \sum_i F_{ix} \overset{!}{=} 0,$$

$$\text{Summe aller Kräfte in y-Richtung:} \quad \sum_i F_{iy} \overset{!}{=} 0,$$

$$\text{Summe aller Kräfte in z-Richtung:} \quad \sum_i F_{iz} \overset{!}{=} 0,$$

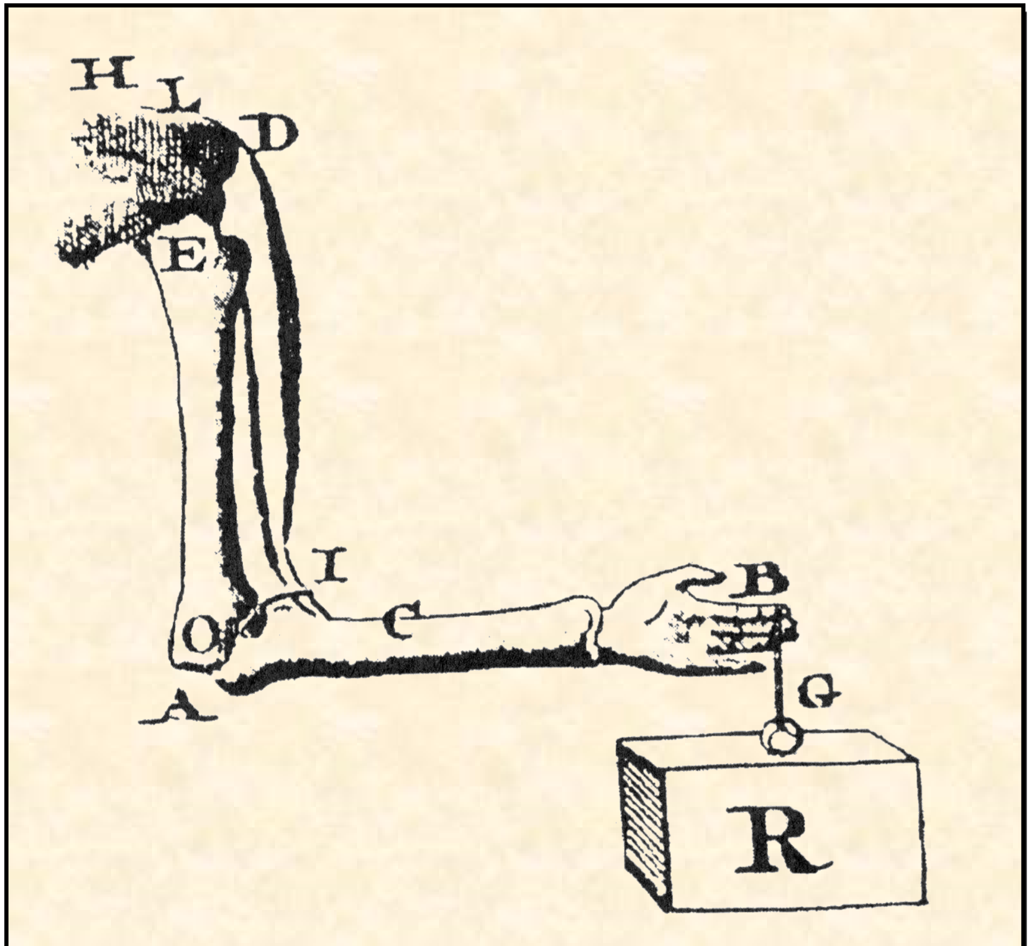
$$\text{Summe aller Momente um x-Achse bezüglich Punkt P:} \quad \sum_i M_{ix}^{P} \overset{!}{=} 0.$$

$$\text{Summe aller Momente um y-Achse bezüglich Punkt Q:} \quad \sum_i M_{iy}^{Q} \overset{!}{=} 0.$$

$$\text{Summe aller Momente um z-Achse bezüglich Punkt R:} \quad \sum_i M_{iz}^{R} \overset{!}{=} 0.$$

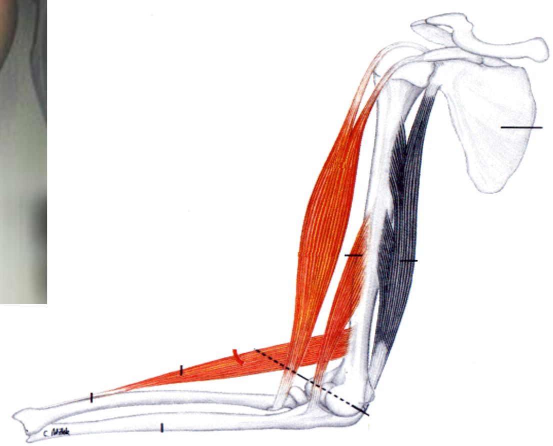
- Force EEs can be substituted by moment EEs
- Max. 2 moment axis parallel to each other
- Determinant of coef. matrix not zero

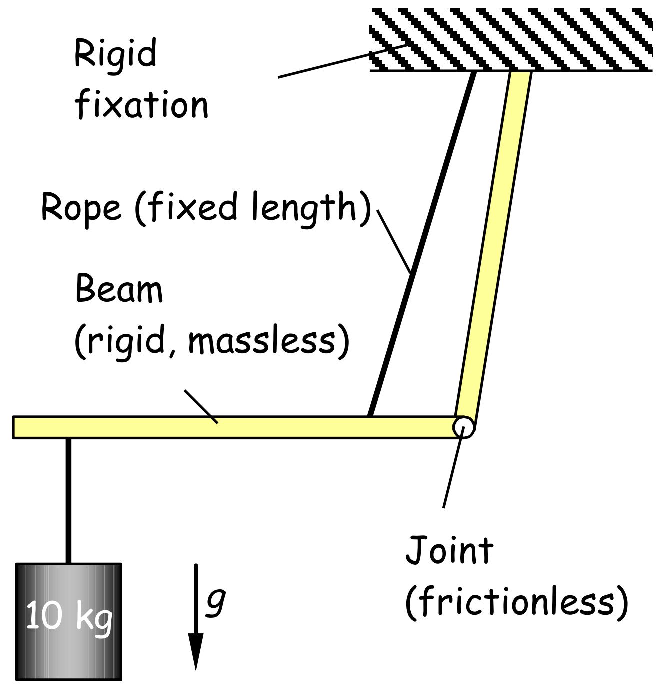
Classical Example: „Biceps Force“



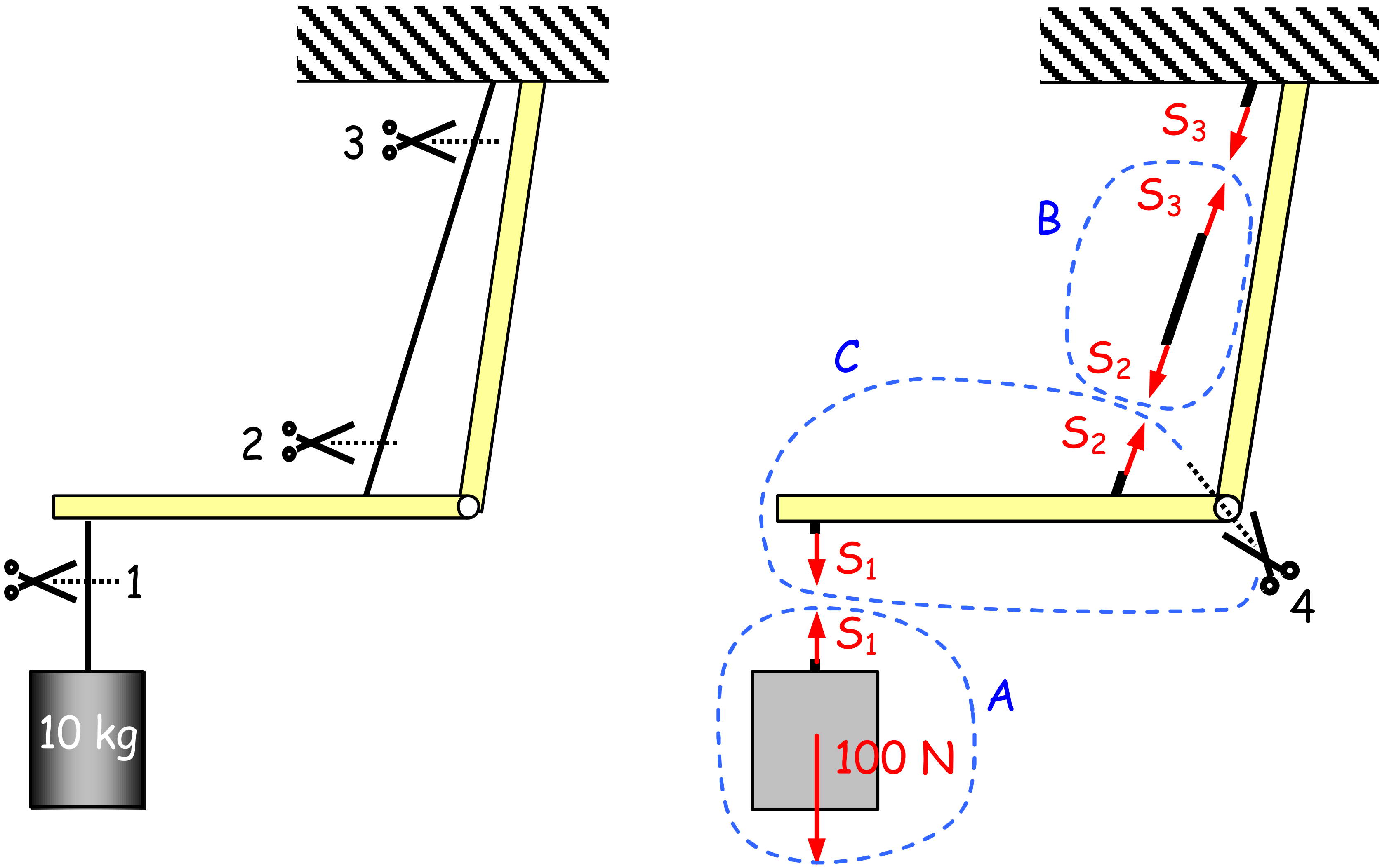
From:
„De Motu Animalium“
G.A. BORELLI
(1608-1679)

Step 1: Model building

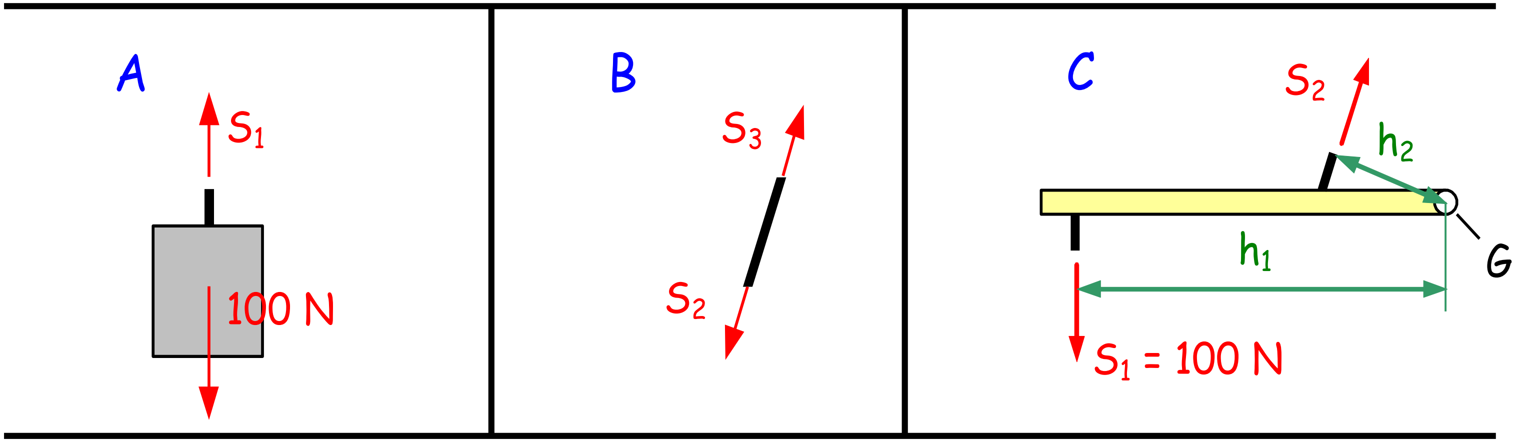




Schritt 2: Schneiden und Freikörperbilder



More to Step 2: Cutting and Free-Body Diagrams



Step 3 and 4: Equilibrium and Solving the Equations

Sum of all forces in vertical direction = 0	Sum of all forces in "rope" direction = 0	Sum of all moments with respect to Point G = 0
$100\,\text{N} + (-S_1) \overset{!}{=} 0$ $\Rightarrow \underline{\underline{S_1 = 100\,\text{N}}}$	$S_2 + (-S_3) \overset{!}{=} 0$ $\Rightarrow S_3 = S_2$	$-S_1 \cdot h_1 + S_2 \cdot h_2 \overset{!}{=} 0$ $-100\,\text{N} \cdot 35\,\text{cm} + S_2 \cdot 5\,\text{cm} \overset{!}{=} 0$ $\Rightarrow S_2 = 100\,\text{N} \cdot \frac{35\,\text{cm}}{5\,\text{cm}} = \underline{\underline{700\,\text{N}}}$

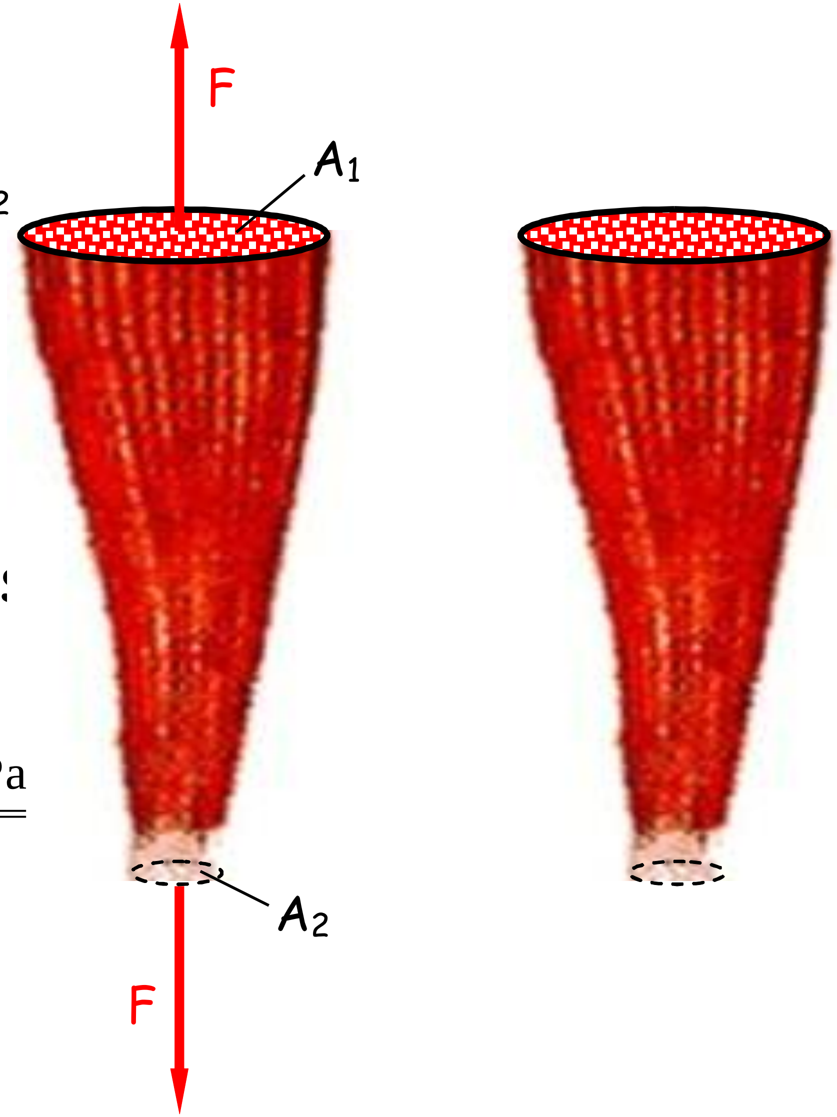
Units of Stress

Pascal: 1 Pa = 1 N/m²

Mega-Pascal: 1 MPa = 1 N/mm²

3.2 Example:

"Tensile stress in Muscle:



$$\sigma_1 = \frac{F}{A_1} = \frac{700\,\mathrm{N}}{7000\,\mathrm{mm}^2} = 0{,}1\frac{\mathrm{N}}{\mathrm{mm}^2} = \underline{\underline{0{,}1\,\mathrm{MPa}}}$$

$$\sigma_2 = \frac{F}{A_2} = \frac{700\,\mathrm{N}}{70\,\mathrm{mm}^2} = 10\frac{\mathrm{N}}{\mathrm{mm}^2} = \underline{\underline{10\,\mathrm{MPa}}}$$

General (3D) Stress State: Stress Tensor

... in one point of the body: **How much numbers do we need?**

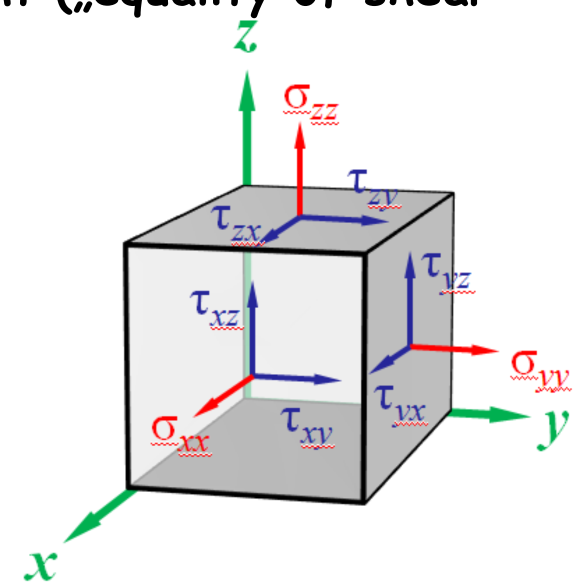
- 3 stress components in one cut (normal str., 2x shear str.)

  times
- 3 cuts (e.g. frontal, sagittal, transversal)

  results in
- 9 stress components for the full stress state in the point.
- But only 6 components are linear independent („equality of shear stresses“)

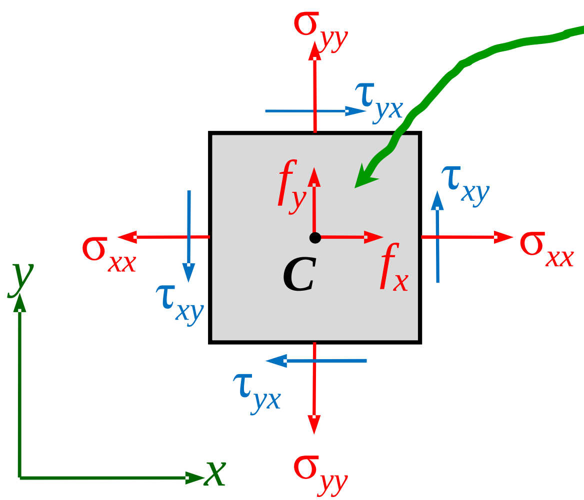
„Stress Tensor“

$$\underline{\underline{\sigma}} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$



Symmetry of the Stress Tensor

Boltzmann Continua: Only volume forces (f_x und f_y), no volume moments assumed
→ „Equality of corresponding shear stresses"
[“Gleichheit der einander zugeordneten Schubspannungen”]

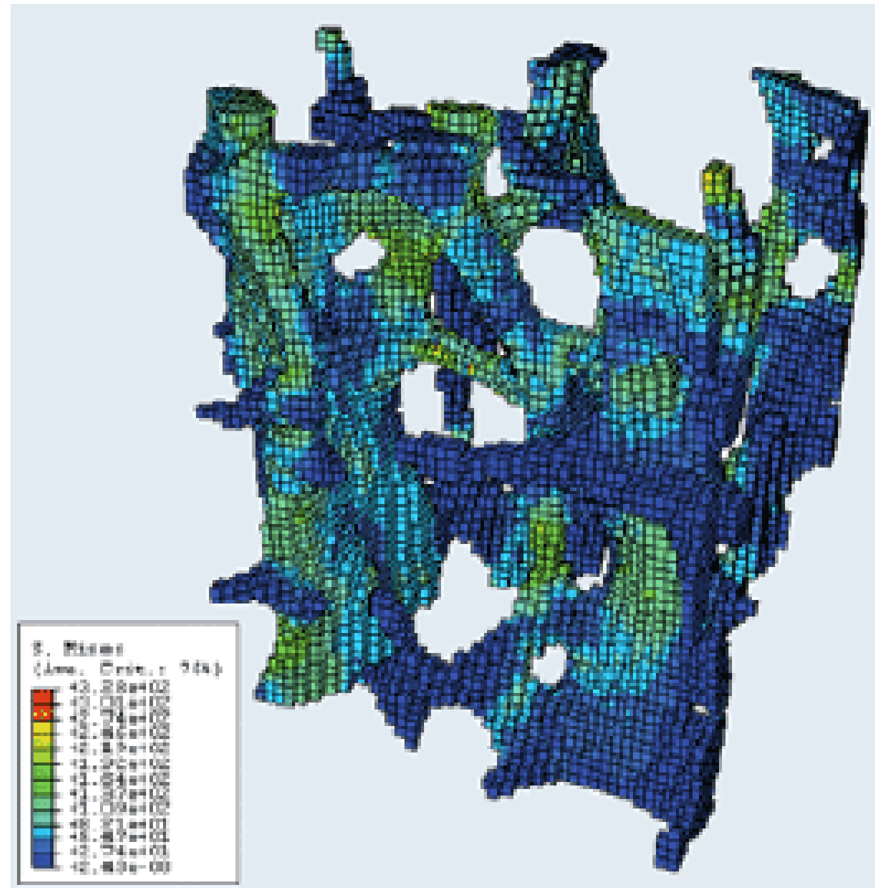


$$\underline{\underline{\sigma}} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ . & \sigma_{yy} & \tau_{yz} \\ \text{sym} & . & \sigma_{zz} \end{bmatrix}$$

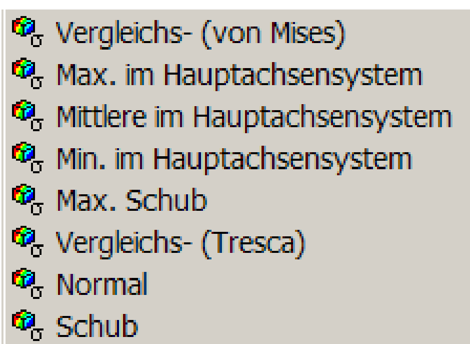
$$\sum M^{(C)} = 2 \cdot \underbrace{\tau_{xy} \Delta y \Delta z}_{\text{Kraft}} \cdot \underbrace{\frac{1}{2} \Delta x}_{\text{Hebelarm}} - 2 \cdot \underbrace{\tau_{yx} \Delta x \Delta z}_{\text{Kraft}} \cdot \underbrace{\frac{1}{2} \Delta y}_{\text{Hebelarm}} = 0.$$

Problem:

- How to produce nice Pictures?
- Which component should I use?
- Do I need 6 pictures at the same time?



So called „Invariants“ are „smart mixtures“ of the components



$$\sigma_{Mises} = \sqrt{\sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2 - \sigma_{xx}\,\sigma_{yy} - \sigma_{xx}\,\sigma_{zz} - \sigma_{yy}\,\sigma_{zz} + 3\,\tau_{xy}^2 + 3\,\tau_{xz}^2 + 3\,\tau_{yz}^2}$$

Strains

- Global, (external) strains

$$\varepsilon := \frac{\text{Change in length}}{\text{Original length}} = \frac{\Delta L}{L_0}$$

- Local, (internal) strains

Units of Strain

without a unit

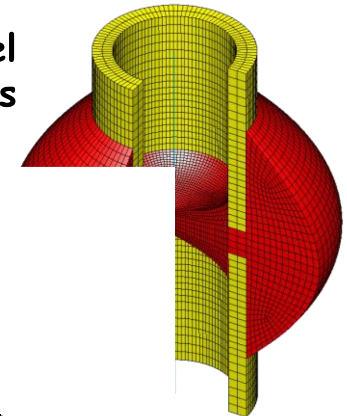
1

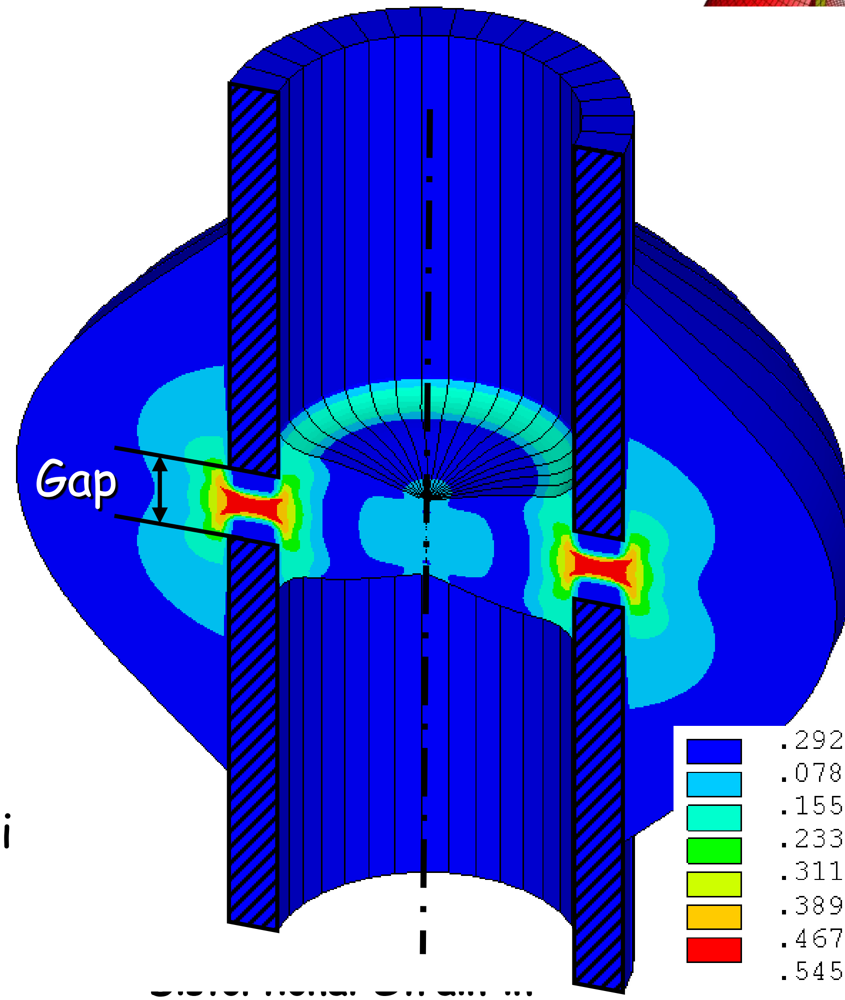
1/100 = %

1/1.000.000 = με (micro strai

= 0,1 %

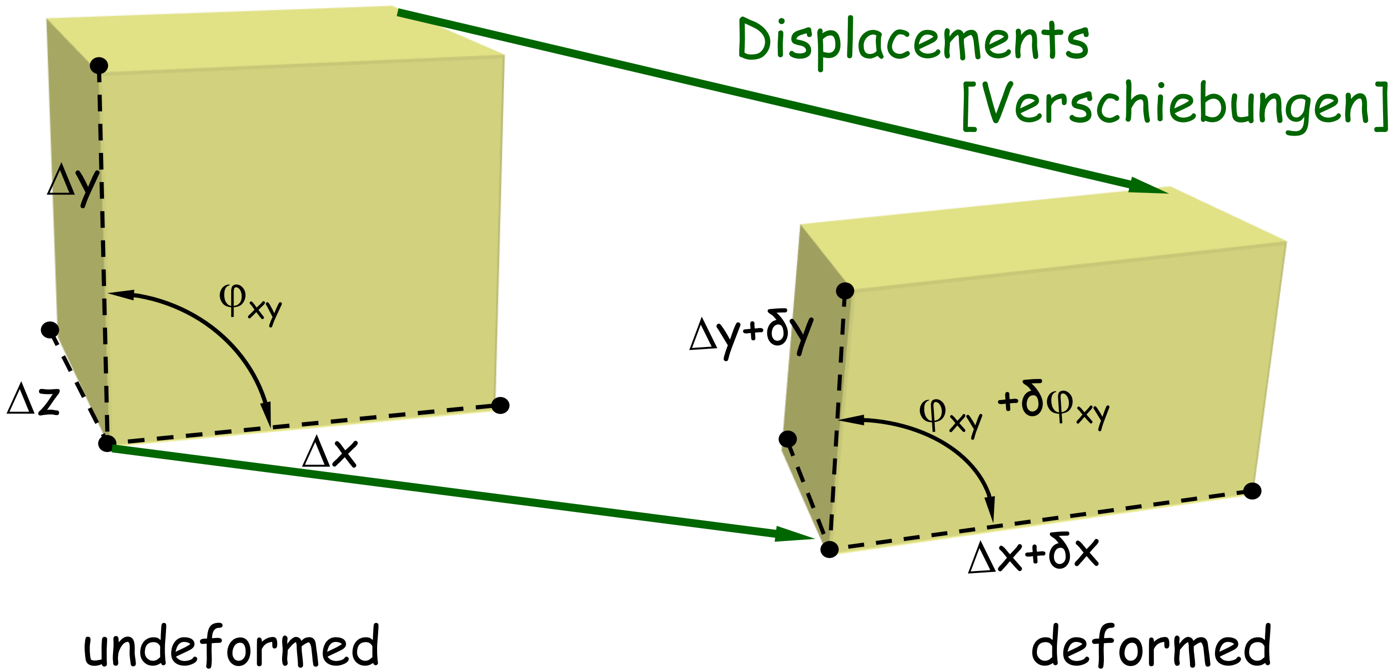
Finite element model of the fracture callus





a fracture callus

3D Local Strain State: Strain Tensor



3D Local Strain State: Strain Tensor

Definition:

$$\varepsilon_{xx} = \lim_{x_0 \to 0} \frac{\Delta x}{x_0}, \quad \varepsilon_{yy} = \lim_{y_0 \to 0} \frac{\Delta y}{y_0}, \quad \varepsilon_{zz} = \lim_{z_0 \to 0} \frac{\Delta z}{z_0}$$

$$\varepsilon_{xy} = \frac{1}{2} \cdot \Delta \gamma, \quad \varepsilon_{xz} = \frac{1}{2} \cdot \Delta \beta, \quad \varepsilon_{yz} = \frac{1}{2} \cdot \Delta \alpha$$

Universal Strain Definition:

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \quad i, j = \{x, y, z\}$$

$$\underline{\underline{\varepsilon}} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{zz} \end{bmatrix}$$

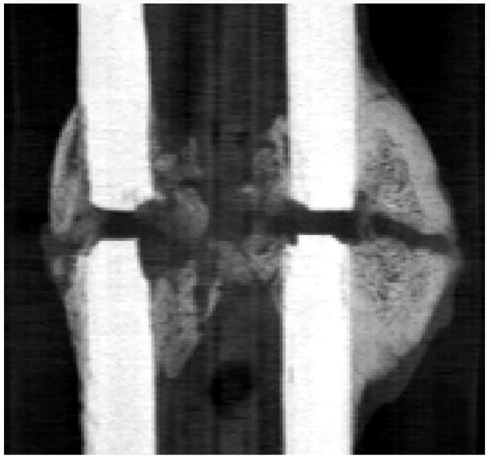
Strain Tensor

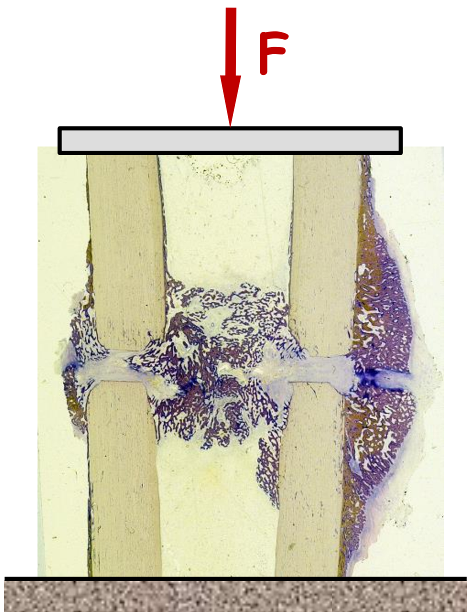
Note to Remember:

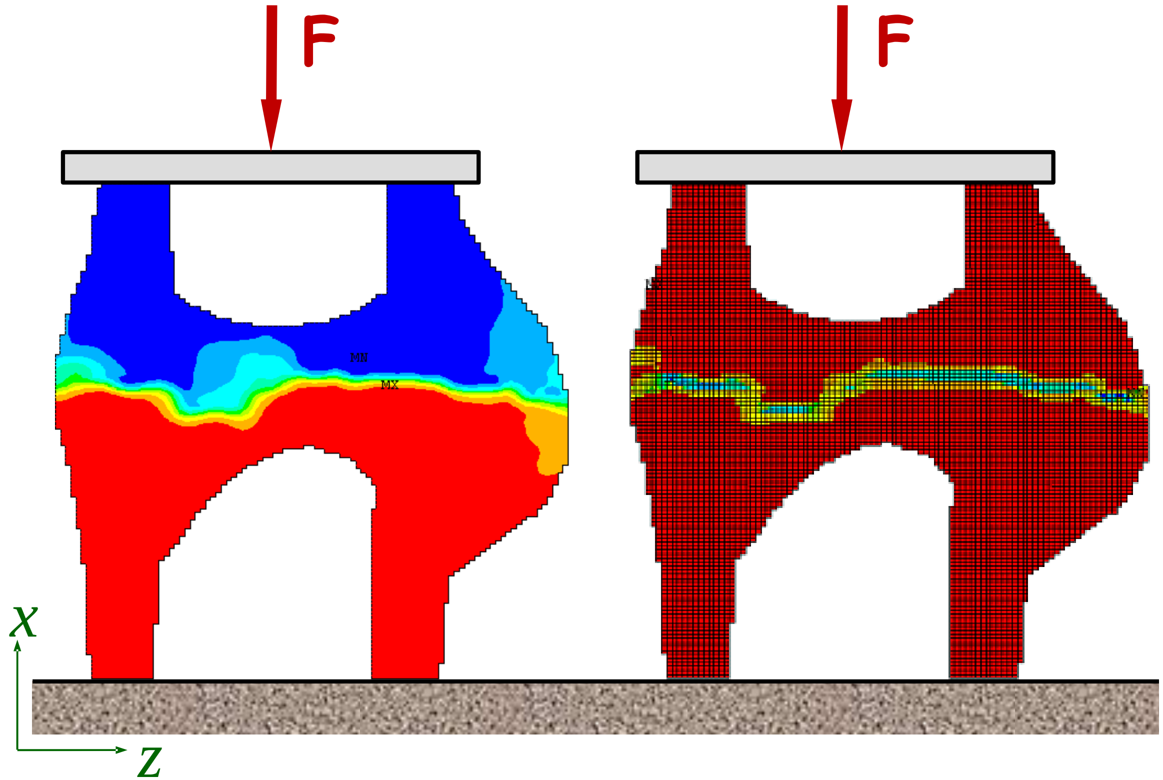
Strain is relative change in length (and shape)

Similar to stress tensor 6 independent components

Displacement vs. Strain







Displacement u_x

Strain, eps_xx

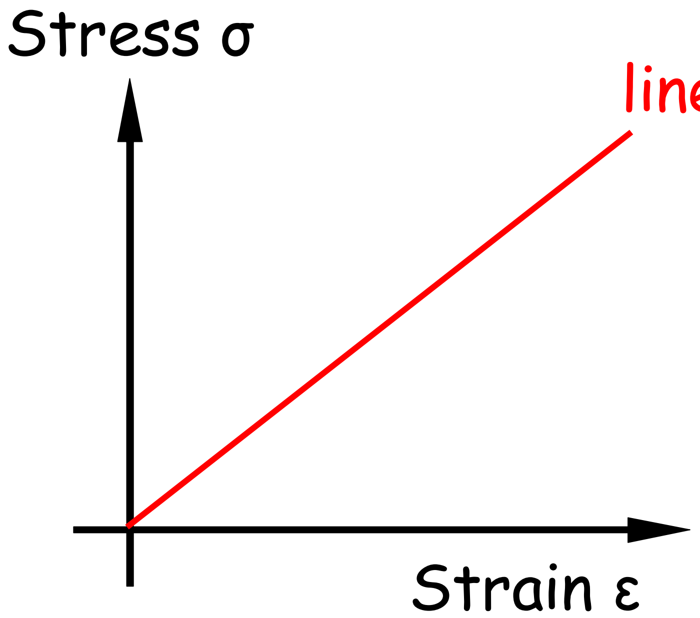
Isotropic vs. Anisotropic (linear elastic)

Linear stress-strain relation

$$\sigma = E \cdot \varepsilon$$

$$\underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{E}}}} \cdot \underline{\underline{\varepsilon}} \qquad \text{(81 Param.)}$$

$$\underline{\sigma} = \underline{\underline{E}} \cdot \underline{\varepsilon} \qquad \text{(36 Param.)}$$



- Full 3^4 material properties tensor of 4^{th} order (81 Param.)
- Equality of shear stresses (Boltzmann Continua) and strains: (36 Param.)
- Reciprocity Theorem from Maxwell → fully anisotropic: (21 Param.)
- Orthotropic: (9 Param.)
- Transverse isotropic: (5 Param.)
- Full isotropic: (2 Param.)

Isotropic vs. Anisotropic (linear elastic)

$$\underline{\sigma} = \underline{\underline{E}} \cdot \underline{\varepsilon}$$

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \frac{E}{(1+\nu)\cdot(1-2\nu)} \cdot \begin{bmatrix} (1-\nu) & \nu & \nu & 0 & 0 & 0 \\ & (1-\nu) & \nu & 0 & 0 & 0 \\ & & (1-\nu) & 0 & 0 & 0 \\ & & & \frac{(1-2\nu)}{2} & 0 & 0 \\ & & & & \frac{(1-2\nu)}{2} & 0 \\ \text{sym} & & & & & \frac{(1-2\nu)}{2} \end{bmatrix} \cdot \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix}$$

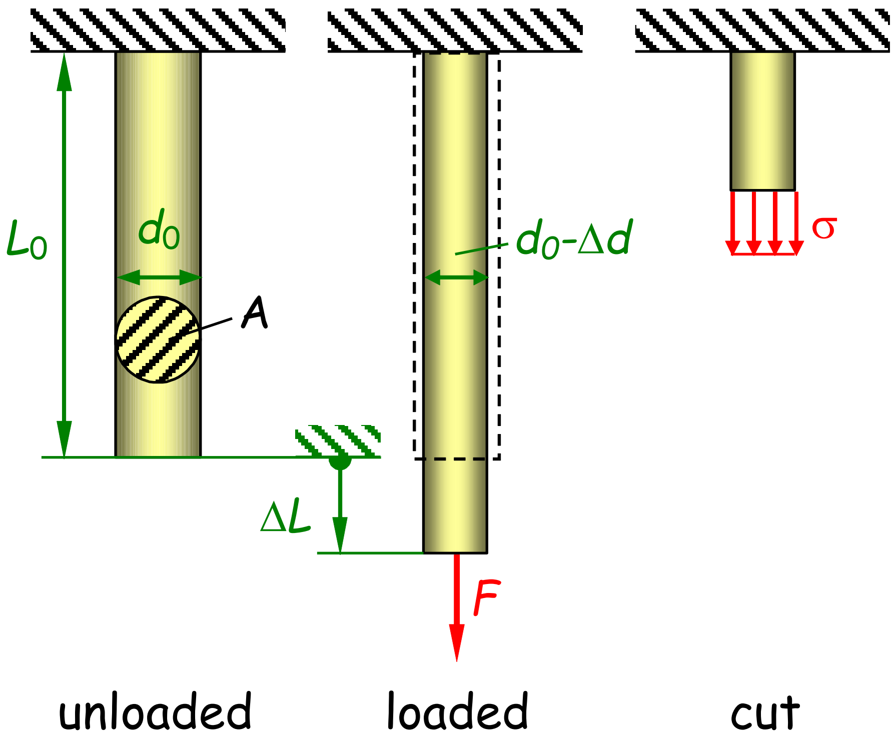
E	-	Young's modulus
ν	-	Poisson's ratio (0 ... 0.5)
G	-	Shear modulus
K	-	Bulk modulus
μ, λ	-	Lame Constants

← 2 of these

Four Simple Load Cases

A Tension, Compression

Tensile bar (tensional rigidity *EA*)



Global behavior

Device stiffness

$$F = \frac{EA}{L_0}\Delta L, \quad k = \frac{EA}{L_0}$$

Local behavior

Axial Stress

$$\sigma = \frac{F}{A}$$

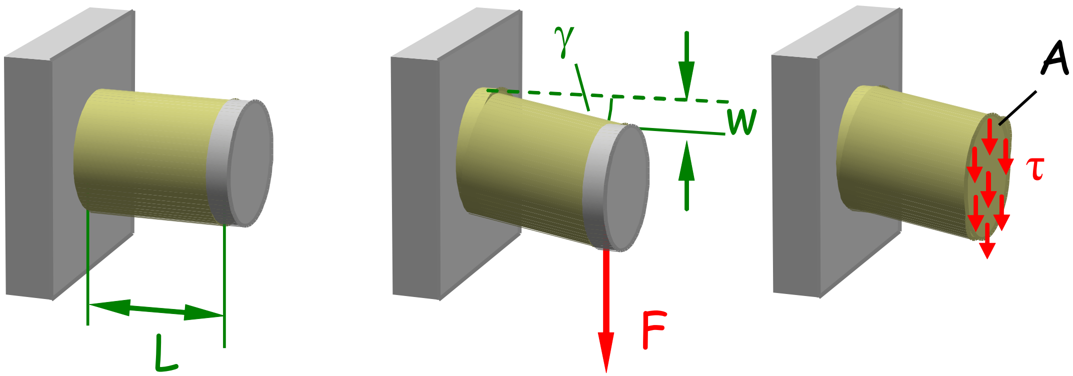
Strain

$$\varepsilon = \frac{\Delta L}{L_0}$$

$$\varepsilon_q = \frac{\Delta d}{d_0} = -\nu\varepsilon$$

B Shear

Shear bolt (shear rigidity *GA*)



Global behavior

Device stiffness

$$F = \frac{GA}{L} w, \quad k = \frac{GA}{L}$$

Local behavior

Shear stress

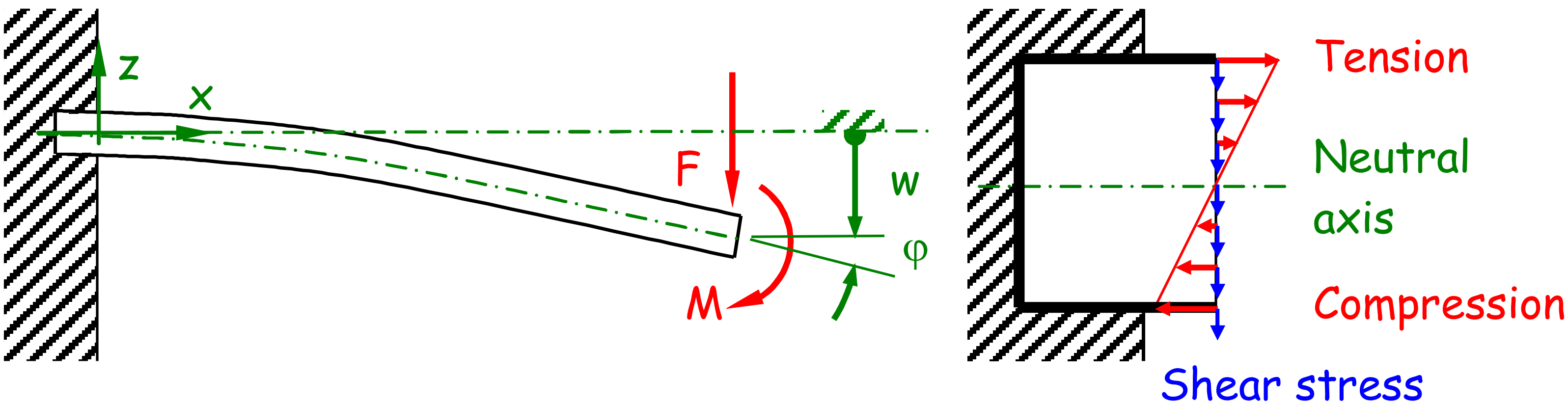
$$\tau = \frac{F}{A}$$

Shear strain

$$\varepsilon_{xy} = \frac{1}{2}\gamma$$

C Bending (Cantilever Beam)

Beam (bending rigidity EI_a, Length L) Cut



Global behavior, compliance

→ Cantilever formula [Kragbalkenformel]

$$w = \frac{L^3}{3EI_a} F + \frac{L^2}{2EI_a} M$$

$$\varphi = \frac{L^2}{2EI_a} F + \frac{L}{EI_a} M$$

Local behavior

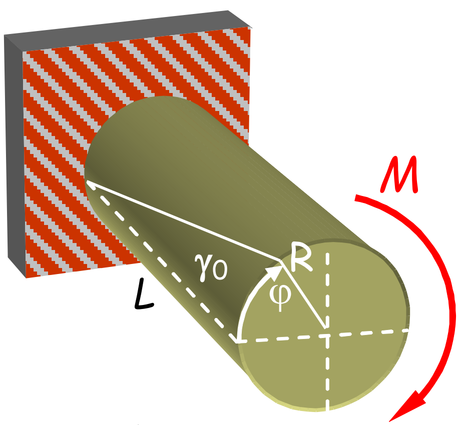
→ Stresses in transverse cut

$$\sigma_{xx}(x,z) = \frac{M + F(x-l)}{I_a} z$$

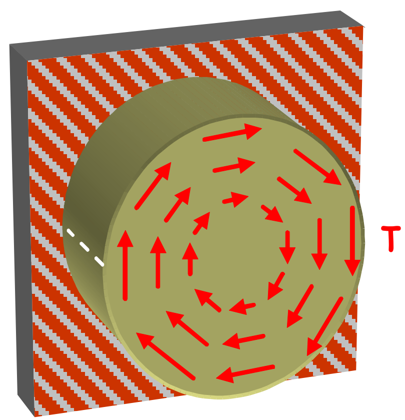
$$\sigma_{xz}(x,z) = \frac{F}{A}$$

D Torsion

Torsional rod (torsional rigidity GI_T)



Length L, radius r



Cut

Global behavior, stiffness

$$M = \frac{GI_T}{L}\varphi, \quad c = \frac{GI_T}{L}$$

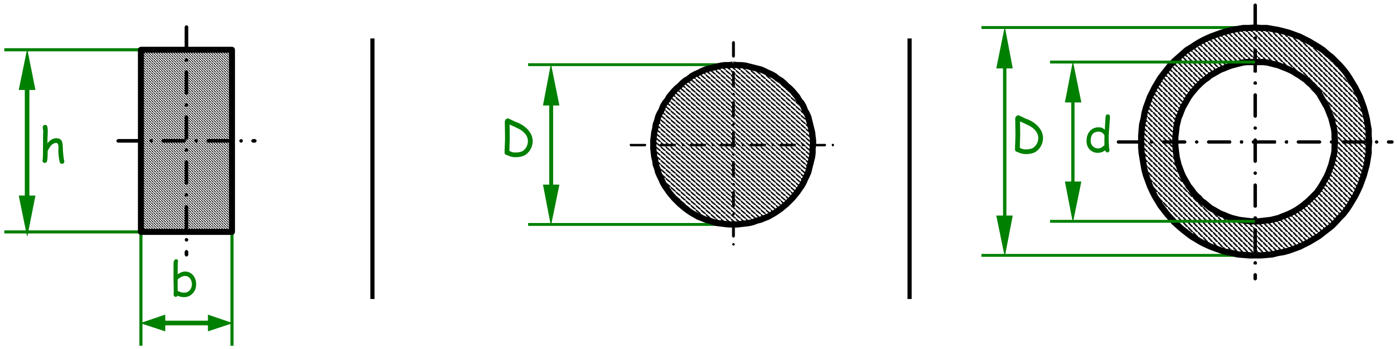
Local behaviour

Stresses in transverse cut

$$\tau = \frac{M}{I_T}\rho$$

$$\rho = \text{Distance from Center}$$

Second Moment of Area I (SMA)
[Flächenmoment zweiten Grades]



Axial (second) moment of area [axiales Flächenmoment]

$$I_a = \frac{b \cdot h^3}{12}$$

$$I_a = \frac{\pi}{64} D^4$$

$$I_a = \frac{\pi}{64}(D^4 - d^4)$$

Polar/torsional (second) moment of area [polares Flächenmoment]

$$I_T = I_P = \frac{\pi}{32} D^4$$

$$I_T = I_p = \frac{\pi}{32}(D^4 - d^4)$$

Note to Remember:

A hollow bone reaches a high stiffness and strength against bending and torsion with relatively little material.

Experiment:
Alu-Stäbe biegen