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Affine Decompositions of Parametric Stochastic Processes

for Application within Reduced Basis Methods

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Introduction & Problem Description

EIM: empirical interpolation method

POIM: proper orthogonal (emp.) interpolation method

LSEIM: least square empirical interpolation method

Introduction

Definitions

- $\mu \in \mathcal{P}$ deterministic parameter
- $\omega \in \Omega$ stochastic event, $(\Omega, \mathcal{F}, P)$ probability space
- D spatial domain
- X appropriate function space

Stochastic Process

- parameter dependent spatial stochastic process

$$c : D \times (\mathcal{P} \times \Omega) \to \mathbb{R}, \qquad (x; \mu, \omega) \mapsto c(x; \mu, \omega)$$

- For each $(\mu, \omega) \in \mathcal{P} \times \Omega$, we obtain a trajectory

$$c(\mu, \omega) \in X$$

Problem Description

Operator

- $\mathcal{L}(c(\mu,\omega))$ spatial differential operator
- $\mathcal{L}$ linear w.r.t. trajectories $c(\mu,\omega)$, i.e. $\mathcal{L}\big(c_1 + \lambda c_2\big) = \mathcal{L}(c_1) + \lambda\mathcal{L}(c_2)$

PDE

- Find solutions $u(\mu,\omega) \in X(\mathcal{D})$ such that

$$\mathcal{L}\big(c(\mu,\omega)\big)[u(\mu,\omega)] = 0$$

RBM Approach

We need affine decompositions of $\mathcal{L}$ into

- terms only (μ,ω)-dependent
- terms only x -dependent

$\Big\} \Leftrightarrow$ affine decomposition of $c(\mu,\omega)$

Example: Smoothed Wiener Process

Wiener Process

- $W(0; \omega) = 0$
- $W(x; \omega) - W(y; \omega) \sim N(0, x - y)$
- Karhunen-Loève expansion is known for fixed interval $[a, b]$
- slow error decay: $\mathcal{O}(1/M)$

Smoothed Wiener Process

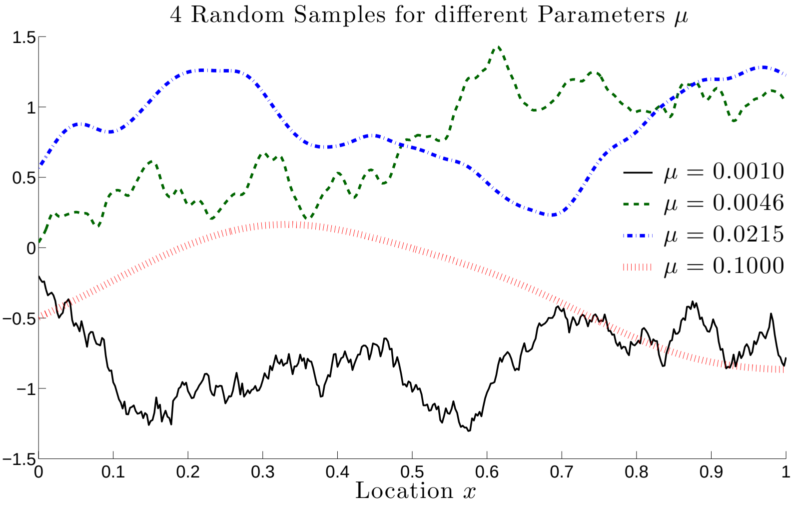
- parameter dependent Gaussian filter, $\mu \in [10^{-3}, 10^{-1}]$

$$F(x, y; \mu) = \frac{1}{\sqrt{2\pi}\mu} \exp\left(-\frac{1}{2}\frac{(x-y)^2}{\mu^2}\right)$$

- smoothed wiener process $c : [0, 1] \to \mathbb{R}$

$$c(x; \mu, \omega) = \int_{x-\delta}^{x+\delta} F(x, y; \mu) W(y; \omega) dy$$

Example: Smoothed Wiener Process



EIM: empirical interpolation method

offline

for $m = 1$ **to** M

$$(\mu, \omega) = \arg\max_{\mu,\omega} \left\| c(\mu,\omega) - c_m^{\text{EIM}}(\mu,\omega) \right\|_\infty$$

$$r_m = c(\mu,\omega) - c_m^{\text{EIM}}(\mu,\omega)$$

$$t_m = \arg\max_x |r_m(x)|$$

$$Q_m = \{Q_{m-1}, \ r_m / r_m(t_m)\}$$

end

Output:

- Interpolation Points:
  $\underline{t} = (t_1, ..., t_M)$
- Basis $Q = (q_1, ..., q_M)$ s.t.
  $q_m(t_m) = 1$
  $q_m(t_n) = 0$ for $n < m$
- Matrix $B \in \mathbb{R}^{M \times M}$
  $B_{nm} = q_m(t_n)$
  B lower triangular

online For new parameter (μ, ω)

- evaluate $\underline{c}(\mu,\omega) = c(\underline{t}; \mu, \omega)$
- solve $B \cdot \underline{\theta}(\mu,\omega) = \underline{c}(\mu,\omega)$
- return $\underline{\theta}(\mu,\omega)$

evaluate $c_M^{\text{EIM}}(\mu,\omega) := \sum_{m=1}^{M} \theta_m(\mu,\omega) q_m(x)$

Complexity:

- $\mathcal{O}(M)$
- $\mathcal{O}(M^2)$
- $\mathcal{O}(M\mathcal{N})$

» average L^2-error over training set

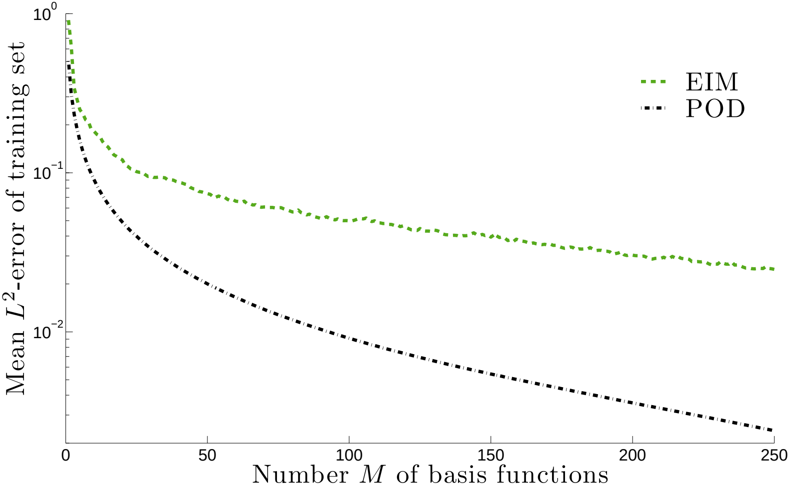


Figure: 3000 Samples for 30 values of μ, Discretization $\mathcal{N} = 400$

POIM: proper orthogonal (emp.) interpolation method

Problem of EIM

Algorithm based on L^∞ error

- $\Rightarrow$ selects rather fluctuating snapshots
- $\Rightarrow$ basis is not smooth
- $\Rightarrow$ "singularity" as interpolation point
- $\Rightarrow$ bad error decay rate

Idea

Use POD modes as basis

- $\Rightarrow$ "optimal" L^2-basis
- $\Rightarrow$ leading basis functions are smooth
- $\Rightarrow$ similar to Discrete EIM [Chaturantabut, Sorensen 2009]

POIM: proper orthogonal (emp.) interpolation method

Differences to EIM

offline

for $m = 1$ **to** M

$$c \leftarrow m^{th} \text{ POD mode}$$
$$r_m = c - c_m^{\text{EIM}}$$
$$t_m = \arg\max_x |r_m(x)|$$
$$Q_m = \{Q_{m-1},\ r_m / r_m(t_m)\}$$

end

- Interpolation Points:
  $\underline{t} = (t_1, \ldots, t_M)$
- Basis $Q = (q_1, \ldots, q_M)$ s.t.
  $q_m(t_m) = 1$
  $q_m(t_n) = 0$ for $n < m$
- Matrix $B \in \mathbb{R}^{M \times M}$
  $B_{nm} = q_m(t_n)$
  B lower triangular

online

For new parameter (μ, ω)

- evaluate $\underline{c}(\mu, \omega) = c(\underline{t}; \mu, \omega)$
- solve $B \cdot \underline{\theta}(\mu, \omega) = \underline{c}(\mu, \omega)$
- return $\underline{\theta}(\mu, \omega)$

Complexity:

- $\mathcal{O}(M)$
- $\mathcal{O}(M^2)$

However: we can show that $c_M^{\text{POIM}}(\mu, \omega) = c_M^{\text{DEIM}}(\mu, \omega)$

POIM: proper orthogonal (emp.) interpolation method

Differences to EIM / Additional differences to DEIM

offline

for $m = 1$ **to** M

$$c \leftarrow m^{th} \text{ POD mode}$$
$$r_m = c - c_m^{\text{EIM}}$$
$$t_m = \arg\max_x |r_m(x)|$$
$$Q_m = \{Q_{m-1}, \ c\}$$

end

online

For new parameter (μ, ω)

- evaluate $\underline{c}(\mu, \omega) = c(\underline{t}; \mu, \omega)$
- solve $B \cdot \underline{\theta}(\mu, \omega) = \underline{c}(\mu, \omega)$
- return $\underline{\theta}(\mu, \omega)$

- Interpolation Points:
  $\underline{t} = (t_1, ..., t_M)$
- Basis $Q = (q_1, ..., q_M)$ s.t.
  $q_m(t_m) \neq 1$
  $q_m(t_n) \neq 0$ for $n < m$
- Matrix $B \in \mathbb{R}^{M \times M}$
  $B_{nm} = q_m(t_n)$
  B full

Complexity:

- $\mathcal{O}(M)$
- $\mathcal{O}(M^3)$

However: we can show that $c_M^{\text{POIM}}(\mu, \omega) = c_M^{\text{DEIM}}(\mu, \omega)$

» average L^2-error over training set

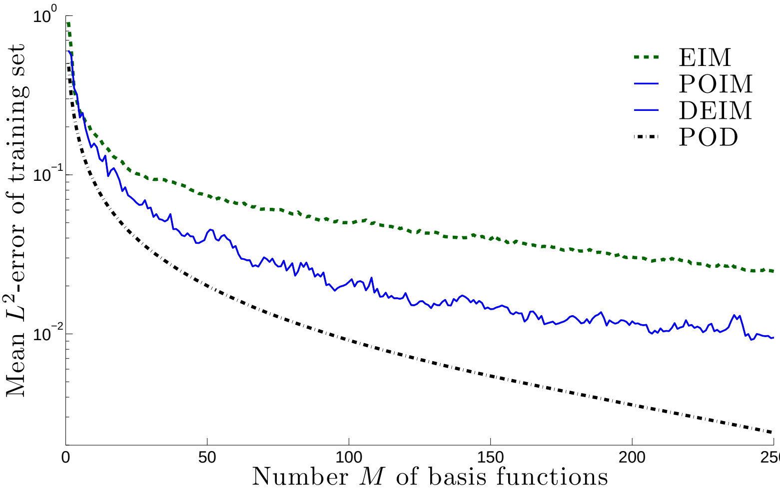


Figure: 3000 Samples for 30 values of μ, Discretization $\mathcal{N} = 400$

LSEIM: least square empirical interpolation method

Problem of POIM

Optimal Basis, but still slow error decay

$\Rightarrow$ coefficients $\underline{\theta}$ are not sufficiently exact

Idea

Use more than one *interpolation* point per basis

$\Rightarrow$ matrix B will be non-quadratic

$\Rightarrow$ matrix B will be non-triangular

$\Rightarrow$ solve least square problems for θ

LSEIM: least square empirical interpolation method

Differences to EIM/POIM

offline

for $m = 1$ **to** M

$c \leftarrow m^{th}$ POD mode

$r_m = c - c_m^{\text{LSEIM}}$

$t_{2m-1} = \arg\min_x \; r_m(x)$

$t_{2m} = \arg\max_x \; r_m(x)$

$Q_m = \{Q_{m-1}, \; r_m / \max_x |r_m(x)|\}$

end

- Interpolation Points:
  $\underline{t} = (t_1, ..., t_{2M})$
- Basis $Q = (q_1, ..., q_M)$
  $q_m(t_m) = 1$
  $q_m(t_n) \neq 0$
- Matrix $B \in \mathbb{R}^{2M \times M}$
  $B_{nm} = q_m(t_n)$
  B full

online

For new parameter (μ, ω)

- do once QR-decomposition of B
- evaluate $\underline{c}(\mu, \omega) = c(\underline{t}; \mu, \omega)$
- solve $\|B \cdot \underline{\theta}(\mu, \omega) - \underline{c}(\mu, \omega)\|_2^2 \rightarrow \min$
- return $\underline{\theta}(\mu, \omega)$

Complexity:

- $\mathcal{O}(M^3)$
- $\mathcal{O}(2M)$
- $\mathcal{O}(2M^2)$

» average L^2-error over training set

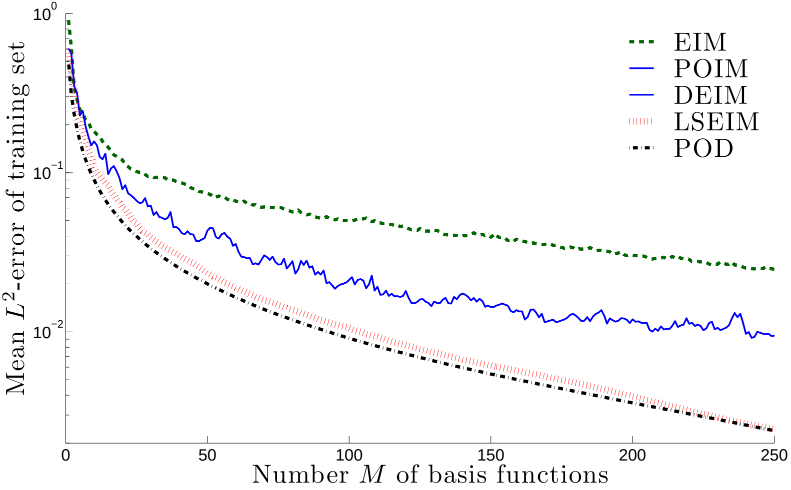


Figure: 3000 Samples for 30 values of μ, Discretization $\mathcal{N} = 400$

Conclusions

- + POIM = DEIM
- + we can decompose noisy data
- + LSEIM close to L^2-“optimal”
- – more interpolation points
- – little more effort

Questions... ?