Reduced Basis Methods for PDEs with Stochastic Influences

Workshop on RBM, Ulm University

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Problem Description

Error and Effectivity Bounds

Example

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Problem Formulation

Given

- a deterministic parameter set $\mathcal{D}$,
- the probability space $(\Omega, \mathcal{B}, P)$,
- the symmetric coercive bilinear form $a(w, v; \mu, \omega)$ and
- the linear forms $f(v; \mu, \omega)$ and $\ell(v; \mu)$

Variational Formulation

For $\mu \in \mathcal{D}, \omega \in \Omega,$ find $u(\mu, \omega) \in X$ s.t.

$$a(u(\mu, \omega), v;\ \mu, \omega) \quad = \quad f(v; \mu, \omega) \quad \forall v \in X$$

Output of Interest

$$\begin{aligned} s(\mu, \omega) &= \ell\left(u(\mu, \omega);\ \mu\right) \\ \mathbb{V}(\mu, \omega) &= \mathbb{E}\left[s^2(\mu, \cdot)\right] - \mathbb{E}^2\left[s(\mu, \cdot)\right] \end{aligned}$$

Affine Decomposition

Karhunen-Loève (KL) Expansion

$$a(w, v; \mu, \omega) = \sum_{q=1}^{Q^a} \Theta_q^a(\mu) \left(a_{0q}(w, v) + \sum_{k=1}^{\bar{K}} \sqrt{\lambda_{kq}^a}\, \xi_{kq}^a(\omega)\, a_{kq}(w, v) \right)$$

$$f(v; \mu, \omega) = \sum_{q=1}^{Q^f} \Theta_q^f(\mu) \left(f_{0q}(v) + \sum_{k=1}^{\bar{K}} \sqrt{\lambda_{kq}^f}\, \xi_{kq}^f(\omega)\, f_{kq}(v) \right)$$

- $\bar{K} \in \mathbb{N} \cup \{\infty\}$
- $\xi_{kq}^\circ(\omega)$ zero mean, unit variance
- λ_{kq}° decreasing exponentially

RB System

- Truncate KL series at some $K \ll \bar{K}$
- Truncated bilinear and linear forms $a^K(w, v; \mu, \omega)$, $f^K(v; \mu, \omega)$

- deterministic parameters $\mu \in \mathcal{D}$
- stochastic parameters $\{\xi^a_{kq}, \xi^f_{kq}\}_{k,q=1,\dots}$

- RB subspaces: X^N

RB Variational Problem

For $\mu \in \mathcal{D}, \omega \in \Omega$, find $u^{NK}, p^{NK}, y^{NK}, z^{NK} \in X^N$ s.t.

$$
\begin{aligned}
a^K(u^{NK}, v; \mu, \omega) &= f^K(v; \mu, \omega) \quad \forall v \in X^N \\
a^K(v, p^{NK}; \mu, \omega) &= -\ell(v; \mu) \quad \forall v \in X^N \\
a^K(v, y^{NK}; \mu, \omega) &= -2s^{NK}(\mu, \omega) \cdot \ell(v; \mu) \quad \forall v \in X^N \\
a^K(v, z^{NK}; \mu, \omega) &= -2\mathbb{E}^{NK}(\mu) \cdot \ell(v; \mu) \quad \forall v \in X^N
\end{aligned}
$$

Outputs

Primal residual

$$r^K(v;\mu,\omega) \quad = \quad f^K(v;\mu,\omega) - a^K(u^{NK}, v;\mu,\omega)$$

Linear RB outputs

$$\begin{aligned} s^{NK}(\mu,\omega) \quad &:= \quad \ell(u^{NK}) - r^K(p^{NK}) \\ \mathbb{E}^{NK}(\mu) \quad &:= \quad \mathbb{E}\left[s^{NK}(\mu,\cdot)\right] \end{aligned}$$

Quadratic RB outputs

$$\begin{aligned} s^{2,NK}(\mu,\omega) \quad &:= \quad \left(\ell(u^{NK})\right)^2 - \left(r^K(p^{NK})\right)^2 - r^K(y^{NK}) \\ &:= \quad \left(s^{NK}\right)^2 + 2s^{NK} r^K(p^{NK}) - r^K(y^{NK}) \\ \mathbb{E}^{2,NK}(\mu) \quad &:= \quad \left(\mathbb{E}^{NK}\right)^2 + 2\mathbb{E}^{NK}\mathbb{E}\left[r^K(p^{NK})\right] - \mathbb{E}\left[r^K(z^{NK})\right] \\ \mathbb{V}^{NK}(\mu) \quad &:= \quad \mathbb{E}\left[s^{2,NK}(\mu,\cdot)\right] \ - \ \mathbb{E}^{2,NK}(\mu) \end{aligned}$$

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KL Truncation Error

- determine K_{max} s.t. the additional KL error is negligible
- replace ξ°_{kq} by some upper bound or quantile ξ_{UB}

$$\delta^f_{KL}(v_0;\mu) := \sum_{q=1}^{Q^f} \Theta^f_q(\mu) \sum_{k=K+1}^{K_{max}} \sqrt{\lambda^f_{kq}} \cdot \xi_{UB} \cdot |f_{kq}(v_0)|,$$

$$\delta^a_{KL}(v_0;\mu) := \sum_{q=1}^{Q^a} \Theta^a_q(\mu) \sum_{k=K+1}^{K_{max}} \sqrt{\lambda^a_{kq}} \cdot \xi_{UB} \cdot \left|a_{kq}(u^{NK}, v_0)\right|,$$

- for $(\mathcal{A}^u_{kq}, v)_X = a_{kq}(u^{NK}, v)$ for all $v \in X$
- and $(\mathcal{F}_{kq}, v)_X = f_{kq}(v)$ for all $v \in X$

$$\Delta^f_{KL}(\mu,\omega) = \frac{1}{\alpha_{LB}} \left\| \sum_{q=1}^{Q^f} \Theta^f_q(\mu) \sum_{k=K+1}^{K_{max}} \sqrt{\lambda^f_{kq}} \cdot \xi_{UB} \cdot \mathcal{F}_{kq} \right\|_X$$

$$\Delta^{a,u}_{KL}(\mu,\omega) = \frac{1}{\alpha_{LB}} \left\| \sum_{q=1}^{Q^a} \Theta^a_q(\mu) \sum_{k=K+1}^{K_{max}} \sqrt{\lambda^a_{kq}} \cdot \xi_{UB} \cdot \mathcal{A}^u_{kq} \right\|_X$$

Error and Effectivity

Error Upper Bounds

$$\begin{aligned}
\|u - u^{NK}\|_X &\leq \Delta^u &&:= \Delta^u_{RB} + \Delta^{a,u}_{KL} + \Delta^f_{KL} \\
\|p - p^{NK}\|_X &\leq \Delta^p &&:= \Delta^u_{RB} + \Delta^{a,p}_{KL} \\
\|y - y^{NK}\|_X &\leq \Delta^y &&:= \Delta^u_{RB} + \Delta^{a,y}_{KL} \\
\|z - z^{NK}\|_X &\leq \Delta^z &&:= \Delta^u_{RB} + \Delta^{a,z}_{KL}
\end{aligned}$$

Effectivity Upper Bounds

If $\Delta_{RB} > \Delta^{a,u}_{KL} + \Delta^f_{KL}$ we have

$$\frac{\Delta^u}{\|u - u^{NK}\|_X} \leq \eta^u := \frac{\gamma_{UB}}{\alpha_{LB}} \left(\frac{\Delta^u_{RB} + (\Delta^{a,u}_{KL} + \Delta^f_{KL})}{\Delta^u_{RB} - (\Delta^{a,u}_{KL} + \Delta^f_{KL})} \right)$$

and analogously for the other solutions.

Linear Output Error

The linear output error bound is given by

$$|s - s^{NK}| \leq \Delta^s := \alpha_{LB}\Delta^u\Delta^p + \delta^a_{KL}(p^{NK}) + \delta^f_{KL}(p^{NK})$$

- all error parts Δ_{RB} and Δ_{KL} appear in products with other error parts

$\Rightarrow$ only small N necessary

- δ_{KL} is more precise than Δ_{KL} and decreases fast in K

- since δ_{KL} and Δ_{KL} do not directly depend on N, we can determine an appropriate value for K *a-priori*:
  - use the “initial reduced basis” in the Greedy algorithm
  - test KL-errors for a test parameter sample
  - choose K s.t. KL error is smaller than some tolerance

Quadratic Output Error

Output Error

$$s^2 - s^{2,NK} = s^2 - \left(s^{NK}\right)^2 - 2s^{NK} r^K(p^{NK}) + r^K(y^{NK})$$

Consider the first part:

$$s^2 - \left(s^{NK}\right)^2 = \underbrace{\left(s - s^{NK}\right)^2}_{\leq (\Delta^s)^2} + 2s^{NK}\left(s - s^{NK}\right)$$

It remains

$$\begin{aligned} 2s^{NK}\left(s - s^{NK}\right) &= 2s^{NK}\left(\ell(u) - \ell(u^{NK}) + r^K(p^{NK})\right) \\ &= -a^K(u, y^K) + a^K(u^{NK}, y^K) + 2s^{NK} r^K(p^{NK}) \end{aligned}$$

$$\Rightarrow \quad s^2 - s^{2,NK} = \left(s - s^{NK}\right)^2 - a^K(e^{NK}, y^K) + r^K(y^{NK})$$

with $e^{NK} := u - u^{NK}$

Quadratic Output Error

Linear output error bound

$$|s - s^{NK}| \leq \Delta^s := \alpha_{LB}\Delta^u\Delta^p + \delta^a_{KL}(p^{NK}) + \delta^f_{KL}(p^{NK})$$

Quadratic output error bound

$$\begin{aligned} |s^2 - s^{2,NK}| \leq \Delta^{s^2} := & (\Delta^s)^2 \\ & + \alpha_{LB}\Delta^u\Delta^y + \delta^a_{KL}(y^{NK}) + \delta^f_{KL}(y^{NK}) \end{aligned}$$

- Δ^s is already small and $(\Delta^s)^2$ therefore almost negligible

$\Rightarrow$ Δ^{s^2} will probably be of the same order than Δ^s

Variance Error

It remains to find error bounds for

$$\mathbb{E}^2 - \mathbb{E}^{2,NK}$$

Analogously to Δ^{s^s}, we obtain

$$\begin{aligned} |\mathbb{E}^2 - \mathbb{E}^{2,NK}| \le \Delta^{\mathbb{E}^2} \quad &:= \quad \left(\Delta^{\mathbb{E}}\right)^2 \\ &+ \quad \mathbb{E}\left[\alpha_{LB}\Delta^u\Delta^z\right] \; + \; \mathbb{E}\left[\delta^a_{KL}(z^{NK}) + \delta^f_{KL}(z^{NK})\right] \end{aligned}$$

and the error bound for the variance

$$|\mathbb{V} - \mathbb{V}^{NK}| \le \Delta^{\mathbb{V}} \quad := \quad \mathbb{E}\left[\Delta^{s^2}\right] + \Delta^{\mathbb{E}^2}$$

Improvement of the Variance Error

Reconsider the additional dual problems:

$$\begin{aligned} a^K(v, y^{NK}; \mu, \omega) &= -2\, s^{NK}(\mu,\omega) \cdot \ell(v;\mu) \quad \forall v \in X^N \\ a^K(v, z^{NK}; \mu, \omega) &= -2\, \mathbb{E}^{NK}(\mu) \quad \cdot \ell(v;\mu) \quad \forall v \in X^N \end{aligned}$$

For small $\mathbb{V}$ we get $s^{NK} \approx \mathbb{E}^{NK}$ and hence, $y^{NK} \approx z^{NK}$.

With little more effort we get the better variance error bound

$$\begin{aligned} |\mathbb{V} - \mathbb{V}^{NK}| \le \tilde{\Delta}^{\mathbb{V}} &:= \mathbb{E}\Big[(\Delta^s)^2 \Big] + \left(\Delta^{\mathbb{E}}\right)^2 \\ &+ \mathbb{E}\Big[\alpha_{LB} \Delta^u \Delta^{y-z} \Big] \\ &+ \mathbb{E}\Big[\delta^a_{KL}(y^{NK} - z^{NK}) + \delta^f_{KL}(y^{NK} - z^{NK}) \Big] \end{aligned}$$

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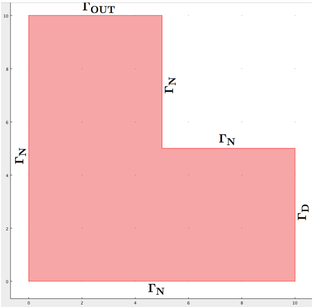
For an “L-Shape” L, we have the following PDE:

$$\begin{cases} -\nabla \cdot \Big(\kappa(x;\mu,\omega)\nabla u(x;\mu,\omega)\Big) & = & f(x;\omega) & \forall x \in L \\ u(x;\mu,\omega) & = & 0 & \forall x \in \Gamma_D \\ \vec{n} \cdot \Big(\kappa(x;\mu,\omega)\nabla u(x;\mu,\omega)\Big) & = & 0 & \forall x \in \Gamma_N \\ \vec{n} \cdot \Big(\kappa(x;\mu,\omega)\nabla u(x;\mu,\omega)\Big) & = & \ell(x) & \forall x \in \Gamma_{OUT} \end{cases}$$



- deterministic parameter domain
  $\mathcal{D} = [0.1, 10]$
- random process
  $\kappa(x;\mu,\omega) := \Theta_1(\mu)\kappa_1(x) + \Theta_2(\mu)\kappa_2(x;\omega)$
- Output
  $\ell(x) \equiv 1$ constant
  $s(\mu,\omega) := \int_{\Gamma_{OUT}} \ell(x) \cdot u(x;\mu,\omega)dx$
- Karhunen-Loève Expansion:
  $K^\kappa = 17,\ K^f = 20,\ K^\kappa_{max} = 23,\ K^f_{max} = 24$

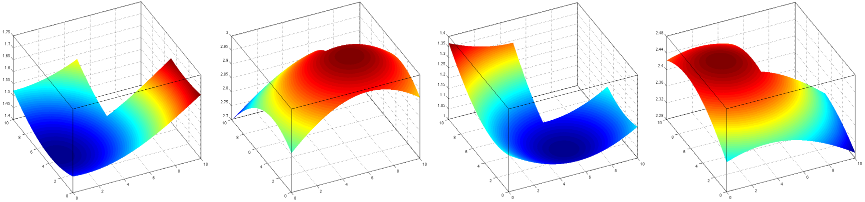


Figure: Four Random Realizations of $\kappa_2(x;\omega)$

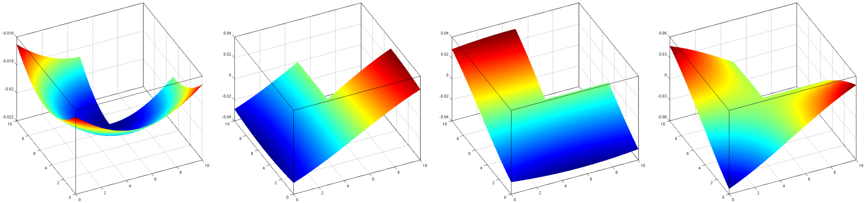


Figure: First 4 KL Expansion Terms of $\kappa_2(x;\omega)$

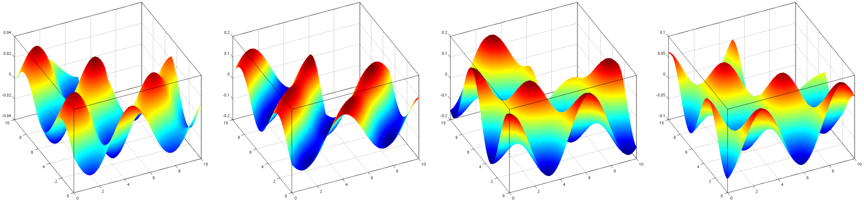


Figure: Four Random Realizations of $f(x;\omega)$

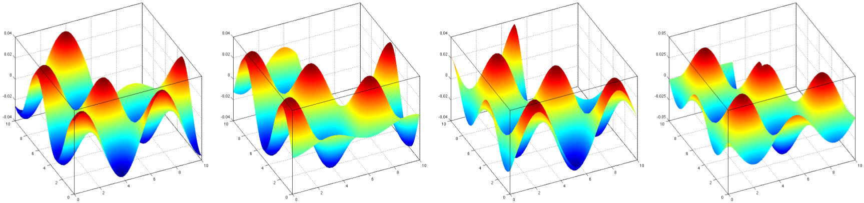
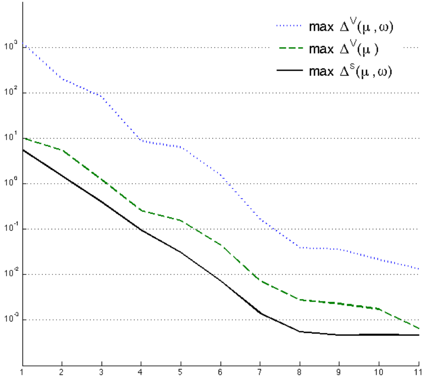
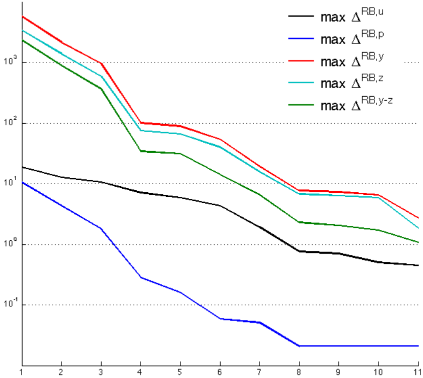


Figure: First 4 KL Expansion Terms of $f(x;\omega)$

Convergence of Error Bounds



(a) Noncompliant Output and Variance Error Convergence



(b) RB Solution Error Convergence

Effectivity

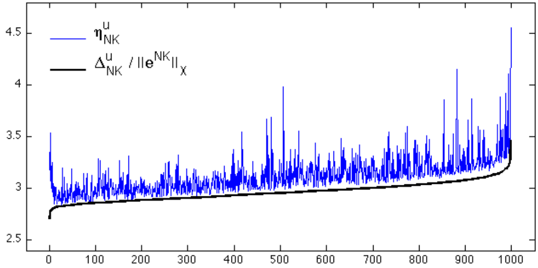


Figure: Effectivities for $\mu = 10$ and 1000 random realizations

μ	$\frac{\Delta^u_{RB}+(\Delta^{a,u}_{KL}+\Delta^f_{KL})}{\Delta^u_{RB}-(\Delta^{a,u}_{KL}+\Delta^f_{KL})}$	$\frac{\Delta^p_{RB}+\Delta^{a,p}_{KL}}{\Delta^p_{RB}-\Delta^{a,p}_{KL}}$	η^u_{NK}	$\eta^p_{\tilde{N}K}$
0.1	1.000058	1.001645	1.5176	1.5201
1.0	1.000087	1.001751	1.1171	1.1190
10	1.000226	1.001766	3.1325	3.1373

Table: Sample Means of effectivities for 1000 realizations

Online Costs

$\mathcal{N}$	$(N_u\ N_p\ N_y\ N_z)$	RB: ω/hour	Full: ω/hour	Factor
3965	(11 8 11 11)	617699	61387	10.06
15609	(11 8 11 11)	617699	13652	45.25
61937	(11 8 11 11)	617699	3147	196.3

Table: Realizations / Hour

Reference:
3.06 GHz Intel Core 2 Duo
4 GB 1067 MHz DDR3
Mac OS X Version 10.6.5
Matlab 7.8.0 (R2009a)

Main References

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  PhD thesis, Ulm University, coming soon.